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PHYSICS

FUNDAMENTAL PRINCIPLES FOR STUDENTS OF SCIENCE AND ENGINEERING

Volume I

PHYSICS

FUNDAMENTAL PRINCIPLES FOR STUDENTS OF SCIENCE AND ENGINEERING

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IN TWO VOLUMES

Volume I

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PREFACE

This book has been written for use as a text in an introductory course in physics such as that normally taken by sophomore students of science and engineering in the Ohio State University. The typical student enrolled in this course has completed a year's work in chemistry and in mathematics, has had high-school physics, and is concurrently taking a course in the calculus.

In writing the book, we have also kept in mind the needs of advanced students and of professional scientists and engineers for a reference treatment of the fundamentals of physics that is both lucid and accurate.

We attempt to give a *rigorous* introductory treatment of the *fundamental principles of physics* in which full advantage is taken of the student's background training. The 'lack of brevity' in the presentation is intentional, since we feel that *full discussion of basic ideas* is essential to the student at the early stage of his career. This full discussion is especially important in view of the fact that we have adopted an *operational* approach to physics and have attempted to give careful operational definitions of the various physical quantities.

Our *primary purpose* is to lead the student to an *understanding* of the fundamental principles and of the way in which they are employed in logical deductions and analyses. Discussions of the practical applications of the principles are included to assist in the acquisition of this understanding. Much material normally termed 'modern physics' is introduced at appropriate points throughout the book since this material also assists in the understanding of the principles.

We use mathematics only as a tool to further our primary purpose. In no case do we attempt to *teach* mathematics for its own sake in this course. We omit all mathematical detail that is not directly contributory to the understanding of the large body of physical ideas that we *are* attempting to teach.

Plane trigonometry is used extensively from the beginning, but calculus is introduced gradually as the student can be expected to have acquired a working knowledge of the various techniques from a calculus course taken concurrently. Statics is treated before kinematics and dynamics in order to give the student time to acquire a proficiency in the elementary differential calculus that he will need to use in the study of kinematics and dynamics. The first simple integral occurs on p. 384, but the integral calculus is not used extensively before PART V, ELECTRICITY AND MAGNETISM.

In keeping with the primary purpose, we have avoided dissipating the student's energies by requiring him to learn an unnecessary number of systems of units. We have employed only two systems of mechanical units and only one system of electromagnetic units in the text. We use the *gravitational* BRITISH-ENGINEERING *system* whenever English units are involved, because this system has come into widest use among mechanical and aeronautical engineers in this country. Where international metric units are involved, we employ the *absolute* METER-KILOGRAM-SECOND *system* because this system has long been preferred by electrical engineers and now seems destined to replace the older CENTIMETER-GRAM-SECOND systems in scientific work. The student is thus trained in the use of *one gravitational* and *one absolute* system of mechanical units and in the use of *one* system of electromagnetic units. In the appendices various other systems of units are discussed, and extensive tables of conversion factors are given. We attempt to teach the proper use of conversion factors by illustrative examples throughout the text. Instructors who have long used the CGS systems of units will perhaps experience some minor 'discomforts' in becoming accustomed to the MKS system, but it is believed that they will quickly recognize the practical as well as the pedagogical advantages of employing this system in the handling of electrical and magnetic quantities.

Illustrative examples are used in the text (1) to illustrate the basic principles, (2) to demonstrate desirable techniques to be used in applying the principles, and (3) to emphasize that for each physical quantity both a *number* and a *unit* must be substituted in an equation. We believe that it is important for the student to acquire a 'feeling' for the *dimensions* of physical quantities. To this end, we carefully point out the way in which the various units are all derived from a small number of fundamental units, and emphasize the fact that valid equations must 'balance' dimensionally. An insistence on explicit substitution of units in all equations should quickly give the student a familiarity with the basic ideas of *dimensional analysis*, even though this subject is not given a formal treatment.

In order to make this book as valuable as possible for reference and to give the instructor some freedom of choice, we have included somewhat more material than can be adequately covered in a one-year course. Certain sections are marked as omittable; others are capable of being omitted or studied only in part. We have also included many more problems than are normally worked in a one-year course. The problem sets, which are at the ends of the sections, are carefully 'graded,' and range from the simplest type of substitution problems to problems of some complexity, derivations of special formulas, and proofs of theorems, where the student must do considerable logical analysis. Answers are given to the odd-numbered problems. We have attempted to give the instructor, or the advanced student using the book for review of fundamental principles, a wide selection of problems from which to choose. With this wide selection of problems, the book should prove adequate for the three-semester or four-quarter type of physics course that is coming into vogue in many engineering schools.

Finally, we should remark that our aim has been to give the student an introduction to physics in which *nothing will have to be 'unlearned'* when he takes up more advanced work. In accordance with this aim, we have generally used the notations and symbols that are standard in more advanced treatments.

We wish to express our appreciation to Dean Alpheus Smith, Professor Harald Nielsen, and our other colleagues at the Ohio State University for their interest, encouragement, and suggestions. Our thanks go to Professor J. W. Buchta of the University of Minnesota for his careful reading and helpful detailed criticism of the manuscript. We also wish to express our appreciation to Professor Clyde Kearns and his associates in the department of Engineering Drawing at the Ohio State University for their excellent work in preparing the figures, and to Mr W. C. Hendrix of the Department of Terrestrial Magnetism, Carnegie Institution of Washington for drafting the magnetic charts in Chap. 36. Finally, we wish to acknowledge our indebtedness to the many individuals and firms who have supplied illustrative material, as mentioned in the legends of the appropriate figures, and to the staff of Prentice-Hall, Inc., for their excellent editorial work and their cooperation in acceding to all the wishes of the authors in matters of style and typography.

During the period in which this book has been going through the press, one of the authors (G.S.) has been on leave of absence from the Ohio State University for military work with the Operations Research Office of the Johns Hopkins University, in Washington.

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Washington, D. C.,
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VOLUME I

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PHYSICS

FUNDAMENTAL PRINCIPLES FOR STUDENTS OF SCIENCE AND ENGINEERING

Volume I

CHAPTER 1

INTRODUCTION

It has long been customary to begin textbooks by giving a 'definition' of the field to be covered. This practice has certain merits, since it provides the student with a general idea of the subject matter to be included. However, there are frequently many difficulties involved in defining a subject in terms that have the same clear and definite meaning for everyone, and there are additional difficulties in formulating a definition for a field like *physics*, which is not static but is continually changing as new discoveries are made.

However, it does seem desirable to attempt to give a definition. Let us first present the definition formulated by the professional lexicographer. The definition given several years ago in one of the standard dictionaries* is presented below:

> "**Physics.**—1. Originally, that branch of knowledge treating of the material world and its phenomena; natural philosophy;—later excluding in turn various branches of natural science, as biology, astronomy, chemistry, and geology.
>
> 2. The science which deals with those phenomena of inanimate matter involving no changes in chemical composition or, more specifically, with the most general and fundamental of such phenomena, namely motion; the science of matter and motion. *Physics* is usually held to comprise the closely related sciences of mechanics, heat, electricity, light, and sound, and the branches of sciences devoted to the study of radiations (X rays, gamma rays, cosmic rays) and of atomic structure."

This definition at first glance appears to be quite satisfactory and, in some respects, it is as good a definition as one can present. However, if we look more closely at the definition, we find many terms that would need further elaboration before they denoted and connoted the same things to all of us. Some of these terms are: 'science,' 'natural philosophy,'

---

* *Webster's New International Dictionary of the English Language*, 2d ed. (G. & C. Merriam Company, 1934).

'biology,' 'astronomy,' 'chemistry,' 'geology,' 'motion,' and 'matter'—not to mention the various branches of physics listed. Furthermore, the restriction of physics to "those phenomena of inanimate matter involving no changes in chemical composition" no longer holds; the publication of the JOURNAL OF CHEMICAL PHYSICS, which made its first appearance in 1934, indicates that many 'chemical' phenomena do fall within the field of interest of the physicist; much valuable work is also being done at present by physicists who are studying 'biological' phenomena. Thus, there appears to be much in the dictionary definition that is subject to valid criticism.

However, this particular definition of physics is not bad in all respects, since it does serve to give us a *general* idea of what physics is. Criticisms like those we have made could also be made of almost any definition that might be proposed. There is a new approach toward formulation of definitions which is called the *operational approach.* Since we shall use this approach in defining the various physical quantities, it might be well to introduce it at this point. From the operational point of view, we define a physical quantity when we describe a set of *operations* used to measure the quantity; for example, we might define the area of a surface by describing the set of operations involved in measuring the area. From some points of view, an operational definition begs the question, but a good case can be made for its use in physics.*

The following operational definition of physics has been proposed by one of our colleagues in the department of philosophy:

Physics is what the physicists do.

This definition, proposed somewhat facetiously, is certainly correct, even though it would not serve the purposes of the lexicographer.

What *do* the physicists do? One of the most widely known activities of present-day physicists is the study of the structure and properties of the atomic nucleus; one phase of this work has found application in the atomic bomb and the nuclear reactor being developed for power production, and has opened the way to the so-called 'atomic age.' Another widely publicized activity of the physicist is the study of the 'cosmic radiation' that enters the earth's atmosphere from outer space; this 'radiation' includes *particles* with enormous energies; the study of the properties of these particles is still just beginning. Physicists also study the structure and properties of molecules and crystals; these studies have found application in the field of chemical analysis; in the development of the recently publicized 'atomic clock'; in new developments in metallurgy; and in the development of the 'transistors' which may serve to replace in part the thermionic vacuum tubes used in communication systems. Physicists are also employed in the development of communication systems such as television and 'microwave links' for telephonic communication. During World War II, physicists were engaged in the development of radar, of underwater-sound detection systems, of magnetic and acoustic mines, of proximity fuzes, of rockets and guns, and of many other military devices. Other activities of physicists include the study of fluid dynamics involved in the design of airplanes for operation at supersonic speeds; recent work on jet propulsion and on guided missiles might also be mentioned. We have listed only a *few* of the things physicists do and have made no mention at all of the physicists' work on low-temperature phenomena, on photosynthesis, on physiological optics, on transmission of 'signals' along nerve fibers, on X rays, on ultrasonics, or on many other important fields of research. However, the activities and 'operations' listed may serve to give some idea of the scope of modern physics.

---

* See Bridgeman, *Nature of Physical Theory* (Princeton University Press, 1936).

Now that we have 'defined' the field of physics by this somewhat unusual operational definition, the student may be somewhat surprised to find that several chapters that follow deal with such prosaic subjects as the motion of particles, rigid bodies, weightless springs, and various other 'simple systems' that apparently have no relation to nuclear reactors, rockets, television systems, or to the other spectacular phases of modern engineering physics. Therefore, it may be well to point out at the very beginning that this text will treat primarily the *fundamental principles* of physics and engineering. These *principles* are the same for simple processes involving simple bodies as for the more complex processes and the complicated mechanisms encountered in some of the present activities of the modern physicist or engineer. By limiting our early discussions to simplified problems and idealized situations, we can introduce the basic principles most effectively. Only by mastering the fundamental principles as they are applied to simple situations can one hope to learn to understand more complicated phenomena. Where they are appropriate, devices and processes utilized in the more recent developments of physics and engineering will be used as illustrative material and as the basis for problems to which basic principles can be applied.

PART I

MECHANICS

CHAPTER 2

SCALAR AND VECTOR QUANTITIES; DISPLACEMENT, FORCE, TORQUE

Many quantities encountered in the study of physics and engineering have associated with them a *direction* as well as a *magnitude,* and have a common method of composition or 'addition' that is quite different from the method of addition of dollars or gallons. Such quantities are called *vector quantities* or simply *vectors.* Familiar examples of vector quantities are displacement, velocity, acceleration, and force.* *Vector quantities* must be carefully distinguished from *scalar quantities* such as temperature and volume which are completely specified by a magnitude, and whose addition merely involves the addition of the magnitudes. Since much of physics is concerned with the study of vector quantities, we shall devote a large part of this chapter to the definition of certain vector quantities and to methods of handling them in analytical discussions.

1. MEASUREMENT OF LENGTHS AND ANGLES

The description of the position and orientation of an object in space is one of the first steps in the analysis of its motion. Such a description is made in terms of lengths of lines and of angles between lines. The determination of the length of a line is one of the fundamental measurements in physical science and engineering. Everyone is familiar with such instruments as calibrated sticks, tapes, and calipers for the measurement of the distance between two points and with instruments for the measurement of angles. The precise definition of the units used for the calibration of such instruments forms a starting point for the study of physics.

The standard unit of length used for the calibration of most scientific instruments is the *meter,* which is the distance at the temperature of melting ice between two fine lines ruled on a platinum-iridium bar kept at the International Bureau of Weights and Measures in Sèvres, near Paris. This bureau was set up by international treaty on May 20, 1875, to prepare, measure, and preserve the standards of the international metric system. The meter was originally intended to be one ten-millionth of the distance between the earth's equator and the North Pole measured along the meridian of Paris, but the original surveys were not accurate and this relation is only approximately satisfied. *The meter is actually defined as the distance between the lines on the international prototype meter bar when it is at the temperature of melting ice.* An accurate replica of this bar is kept in the National Bureau of Standards in Washington and constitutes the secondary standard used for calibrations in the United States.

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* **Formal definitions of these quantities will be given later.**

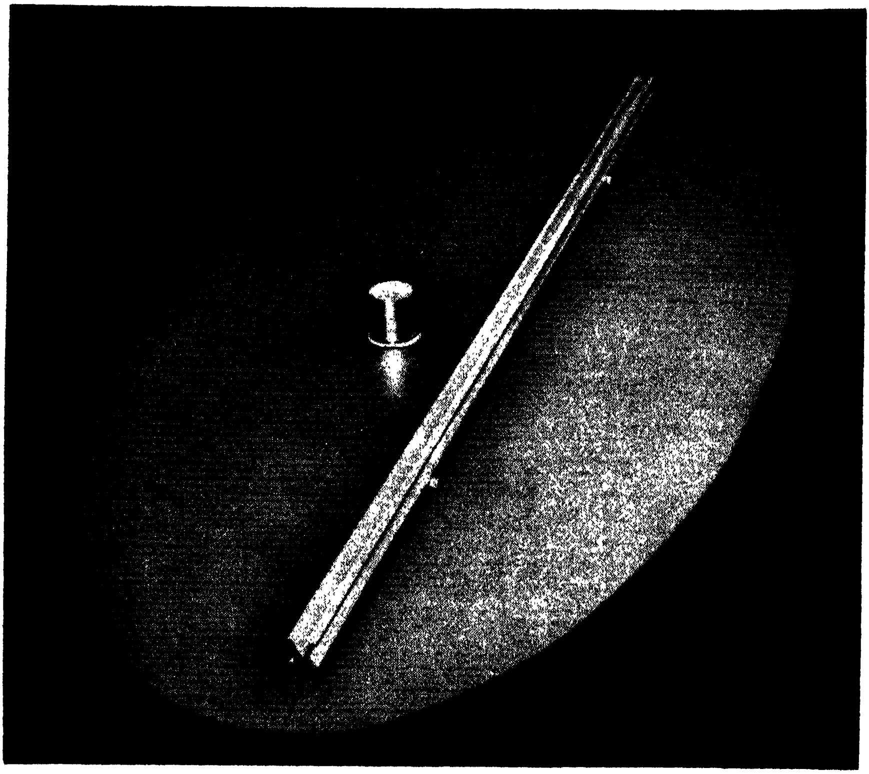

Fig. 1. Standard kilogram and standard meter kept at the National Bureau of Standards in Washington, D.C. These are the national standards of mass and length, and have been carefully compared with the international standards kept at Sèvres, France. (Photograph courtesy of the National Bureau of Standards.)

The meter is one of the three fundamental units in the international metric MKS (meter-kilogram-second) system of units in which the following prefixes are used to indicate multiples and submultiples:

deka-	for 10	deci-	for 0.1
hecto-	for 100	centi- (c)	for 0.01
kilo- (k)	for 1000	milli- (m)	for 0.001
mega- (M)	for 1,000,000	micro- (μ)	for 0.000001.

In abbreviations of the names of units, these prefixes are abbreviated as indicated in parentheses in this list. Those prefixes for which no abbreviations are shown are seldom used in scientific work.

The commonly used metric units of length and their abbreviations are the kilometer (km), which is 1000 m; the meter (m); the centimeter (cm), which is 0.01 m; the millimeter (mm), which is 0.001 m, the 'micrometer' or micron (μ), which is 10^{-6} m; and the millimicron (mμ), which is 10^{-9} m.

Another standard unit of length used in most commercial transactions and in much engineering practice in English-speaking countries is the foot (ft) which is 12 inches (in) or one-third of a yard (yd). The yard is legally defined in the United States as $3600/3937$ meter. This definition leads to the following relations between commonly used English and metric units of length:

$$\begin{array}{lllll} 1\text{ in} = & 2.5400 & \text{cm} = 0.025400 & \text{m} \\ 1\text{ m} = & 39.37 & \text{in} = 3.281 & \text{ft} \\ 1\text{ ft} = & 30.48 & \text{cm} = 0.3048 & \text{m} \\ 1\text{ km} = & 3281 & \text{ft} = 0.6214 & \text{mi.} \end{array}$$

Measurement of angles. The angle between two lines is readily defined if any circle is drawn with its center at the intersection of the two lines as shown in Fig. 2. The magnitude of the angle is proportional to the fraction of the circumference of the circle lying between the two lines. By plane geometry, this fraction is independent of the radius of the circle.

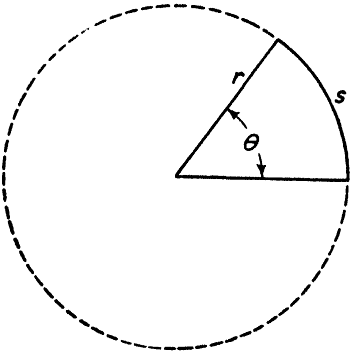

Fig. 2. Definition of an angle θ.

The *degree* (°) is the angular unit commonly used for calibration of instruments designed for measuring angles; it is arbitrarily chosen as the angle subtended at the center of the circle by $1/360$ of the circumference.

The *radian* (rad) is a more fundamental angular unit, which is defined as the angle subtended at the center of the circle by a portion of the circumference equal in length to the radius of the circle, that is, by the fraction $1/2\pi$ of the circumference. The angle θ in Fig. 2 is given in radians by the relation

$$\theta = s/r$$

if the arc length s on the circumference and the radius r are both measured in the same units of length. It follows that $\theta = 1$ rad when $s = r$, and $\theta = 2\pi$ rad when s is one circumference.

The angle subtended by the full circumference is commonly called 1 *revolution* (rev). Thus,

$$1 \text{ rev} = 360^\circ = 2\pi \text{ rad},$$
$$1 \text{ rad} = 57\overset{\circ}{.}3 = (1/2\pi) \text{ rev},$$
$$1^\circ = (2\pi/360) \text{ rad} = 0.01745 \text{ rad} = \tfrac{1}{360} \text{ rev}.$$

These angular units are shown in Fig. 3.

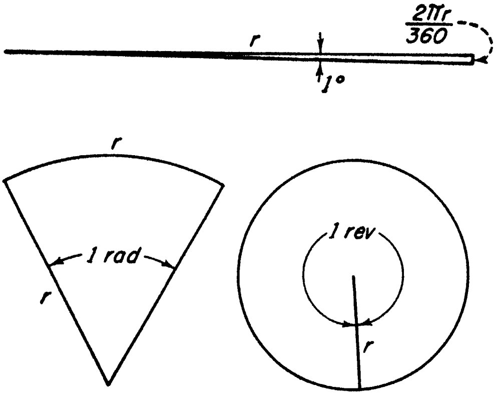

Fig. 3. The angular units: 1 degree, 1 radian, 1 revolution.

PROBLEMS

1. The angstrom is defined by 1 angstrom $= 10^{-8}$ cm. If green light has a wavelength of 5900 angstroms, express its wavelength in mμ. Ans: 590 mμ.

2. The wavelength of red light is 640 mμ. Express this in inches.

3. Express 5 ft 10 in in meters. Ans: 1.778 m.

4. Express 50 mi in km.

5. Express $17\overset{\circ}{.}3$ in rad. Ans: 0.302 rad.

6. Express 0.45 rad in degrees.

2. METHODS OF SPECIFYING POSITIONS

It is necessary in many physical problems to specify precisely the positions of points in space. In the analysis of motion of a bullet, for example, the first essential step is the choice of a proper method of specifying the positions of successive points on the path of the bullet. The specification of position must be made in such a manner that it is easily understood and fits readily into quantitative discussions and computations. Coordinate systems like those familiar from the study of algebra or analytical geometry offer the best method of specifying positions for analytical purposes.

A coordinate system has a fixed reference point called the origin O and fixed reference lines in terms of which the positions of points can be specified. A rectangular xy-coordinate system like that in Fig. 4(a) can be used, for example, to specify the position of the ball on a football field if one takes the coordinate origin O at the southwest corner of the playing field, the x-axis along the south goal line, and the y-axis along the west side of the field. When the ball is on the ground at the 40-yd line and 30 yd from the west side of the field, its position P is simply given by the statement that

$$x = 30 \text{ yd}, \quad y = 40 \text{ yd}.$$

A third coordinate is needed for specifying positions when motion occurs in three dimensions. If the football is in the air, its position can be given in terms of an xyz-coordinate system whose z-axis points vertically upward from the southwest corner of the field. When the ball is 5 yd above a point on the ground which is on the 40-yd line and 30 yd from the west side of the field, its position is simply given by the statement that

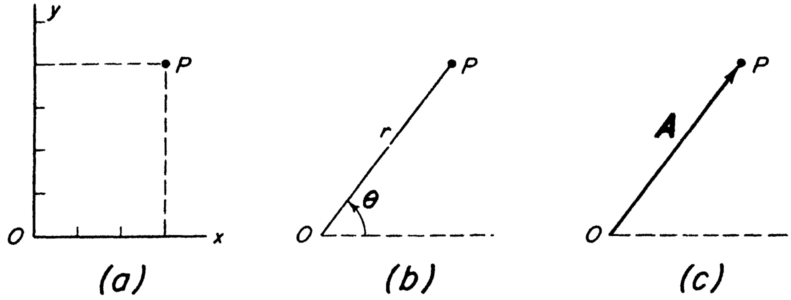

Fig. 4. (a) Rectangular coordinates of a point P. (b) Plane polar coordinates of the point P. (c) Displacement vector from O to P.

$$x = 30 \text{ yd}, \quad y = 40 \text{ yd}, \quad z = 5 \text{ yd}.$$

It is possible to use a rectangular xyz-coordinate system for specifying the positions of objects in any situation that we shall encounter. For example, such a system could be used for stating the instantaneous position of an airplane relative to an airport, the position of a light bulb relative to one corner of the floor of a room, and so on.

The position of a point in a plane can also be specified in terms of a plane polar coordinate system in which one gives the angular orientation and length of the line drawn from the coordinate origin to the point under consideration. The position of a point P is given by the length r and the angle θ shown in Fig. 4(b). The polar coordinates r and θ are related to the rectangular coordinates x and y by the equations

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta,$$

or by

$$r = \sqrt{x^2+y^2} \quad \text{and} \quad \theta = \arctan(y/x).$$

Thus, the position P of the football on the ground at the 40-yd line and 30 yd from the west side of the field is given in terms of plane polar coordinates by the statement: $r = 50$ yd and $\theta = \arctan(\frac{4}{3}) = 53\overset{\circ}{.}13$.

PROBLEMS

1. What are the plane polar coordinates r and θ of a point having the following rectangular coordinates?

(a) $x = 3$ cm, $y = 4$ cm.
(b) $x = -1$ m, $y = 6$ m.
(c) $x = 10$ ft, $y = -3$ ft.

Ans: (a) 5 cm, $53\overset{\circ}{.}1$; (b) 6.08 m, $99\overset{\circ}{.}5$; (c) 10.44 ft, $-16\overset{\circ}{.}7$.

2. What are the plane polar coordinates r and θ of a point having the following rectangular coordinates?

(a) $x = 7$ in, $y = 3$ in.
(b) $x = 7$ ft, $y = -8$ ft.
(c) $x = -4$ m, $y = -4$ m.

3. What are the rectangular coordinates of a point having the following plane polar coordinates?

(a) $r = 5$ m, $\theta = 60°$.
(b) $r = 7$ ft, $\theta = 195°$.

Ans: (a) $x = 2.50$ m, $y = 4.33$ m; (b) $x = -6.76$ ft, $y = -1.81$ ft.

4. What are the rectangular coordinates of a point having the following plane polar coordinates?

(a) $r = 16$ m, $\theta = 72°$.
(b) $r = 4$ ft, $\theta = 300°$.

3. VECTORS

If an object, such as a football, moves from O to P in Fig. 4(c) along the straight line OP, it is said to undergo a rectilinear *displacement*. The displacement or change of position is represented by the arrow-tipped line segment $\boldsymbol{A}$ drawn from O to P. The directed line segment $\boldsymbol{A}$ has two essential properties:

(1) *magnitude*, as indicated by the length of the line segment;

(2) *direction*, as indicated by the angular orientation of the line and the placement of the arrowhead on a particular end of the line.

Rectilinear displacement is an example of the type of physical quantity called a *vector quantity*, or simply a *vector*. A vector requires for its specification both a magnitude and a direction. The usefulness of the vector concept lies in the fact that those physical quantities that are called vectors have a common method of composition or 'addition,' which is identical with the method of addition of displacements that we shall discuss in Sec. 5.

> A quantity that is specified by a magnitude and a direction, and that has a rule for composition identical with the rule for addition of rectilinear displacements, is a vector quantity.*

The following mechanical quantities, which will be defined later, are all vectors: *velocity*, *acceleration*, *momentum*, *force*, *weight*, *torque*, *angular velocity*, *angular acceleration*. It is very important to become thoroughly familiar with the methods of handling vectors because of their frequent occurrence in mechanics and other parts of physics and engineering.

A quantity that has no direction associated with it and that can be completely specified by giving merely a magnitude is called a *scalar quantity* or simply a *scalar*. Examples of scalars are energy, mass, volume, temperature, time, length of a string, and size of an angle. *A scalar is completely specified by the statement of a numeral and a unit*—for example, a volume of *ten gallons*. On the other hand, *a vector requires for its specification a numeral, a unit, and a direction*—for example, a displacement of *two feet eastward.*

A vector quantity is represented on a diagram by an arrow-tipped straight line segment having a length proportional to the magnitude of the vector and pointing in the proper direction. It is convenient to call two vectors equal if they have the same magnitude and direction, regardless of where they are located on the diagram. Thus, a displacement of 100 mi northeastward from the Chicago airport and a displacement of 100 mi northeastward from the Washington airport are represented by equal displacement vectors; one even goes so far as to say that they are represented by the *same* vector, namely, a vector 100 mi long pointing northeastward. By this convention the three vectors in Fig. 5 are considered equal, or one can say that the three arrows represent the same vector.

A vector is usually denoted in print by boldface type. For example, the symbol ***A*** denotes the displacement vector in Fig. 4(c). A vector can be denoted in handwriting by underscoring the letter or by putting an arrow over it; for example, by $\underline{A}$ or $\overrightarrow{A}$.

The *magnitude* of a vector is indicated in ordinary type; thus, A denotes the magnitude of the vector ***A***. The magnitude of a vector is always taken as a *positive* quantity. Thus, if the displacement

---

* Almost all physical quantities that are specified by a magnitude and a direction do compose like displacements and are vector quantities. But there are certain exceptions. For example, the rotation of a sphere about an axis through its center is completely specified by the magnitude of the angle of rotation and the direction of the axis, but two successive rotations about different axes do not give the resultant rotation that would be obtained from the law of vector addition; hence such a rotation is not a vector quantity.

$$\boldsymbol{A} = 100 \text{ mi northeastward,}$$

its magnitude

$$A = 100 \text{ mi.}$$

A negative sign before the symbol indicating a vector merely changes the sense of the direction; that is, it interchanges the head (arrow tip) and tail without changing the length or orientation of the line segment. This convention is illustrated in Fig. 6. A numerical factor before the symbol indicating a vector changes the magnitude of the vector by that factor; thus, $2\boldsymbol{A}$ indicates a vector whose length is twice that of $\boldsymbol{A}$ but whose

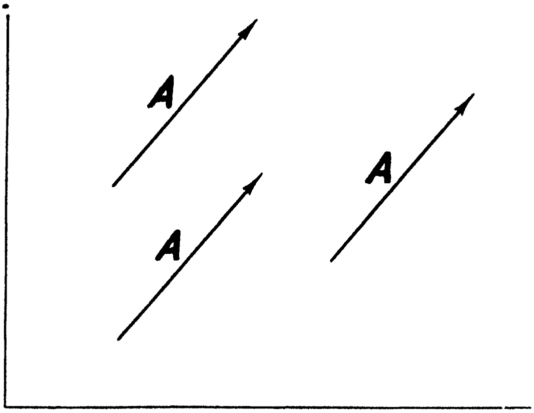

Fig. 5. Equal vectors.

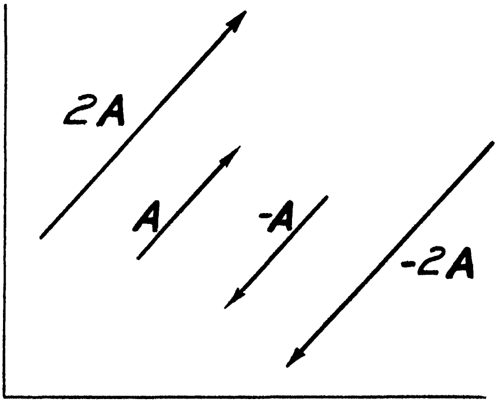

Fig. 6. The vectors $-\boldsymbol{A}$, $2\boldsymbol{A}$, $-2\boldsymbol{A}$.

direction is the same. It follows that $-2A$ indicates a vector parallel to A with the opposite sense of direction and twice the magnitude. Thus, if

$$A = 100 \text{ mi northeastward,}$$

then

$$-A = 100 \text{ mi southwestward,}$$
$$2A = 200 \text{ mi northeastward,}$$
$$-2A = 200 \text{ mi southwestward.}$$

The vectors A and $-A$ have the same magnitude $A = 100$ mi.

It is at once evident from comparison of Figs. 4(b) and 4(c) that the vector representing a displacement from O to P can be specified in terms

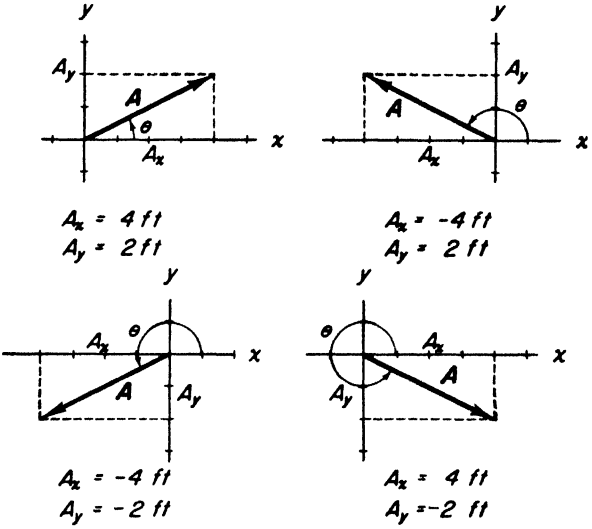

Fig. 7. Illustrating the signs attached to components A_x and A_y of a displacement vector.

of the polar coordinates of P relative to O. The polar coordinate r is the magnitude A of the vector A, and the angle θ defines the direction of A in the plane of the diagram.

The displacement vector A could also be specified by giving rectangular coordinates of P relative to O. In this method of specifying a vector A, the projections of the vector on the x- and y-axes or on lines parallel to these axes are called the *rectangular components* of the vector A. These components are denoted by the symbols A_x and A_y in Fig. 7. The components A_x and A_y are scalar quantities that are given a + or − sign according to whether the x- and y-coordinates of the head of the arrow are positive or negative when the tail of the arrow is placed at the origin.

It is seen that if the angle θ is always measured counterclockwise from the $+x$-axis as in Fig. 7, the components are correctly given by the relations

$$A_x = A\cos\theta, \quad A_y = A\sin\theta,$$

where A, the magnitude of the vector $\mathbf{A}$, is a positive number.

The rectangular components A_x and A_y of the displacement vector $\mathbf{A}$ in Fig. 4(c) are just the rectangular coordinates of the point P.

The discussion above has been confined to vectors lying in a given plane. If we desire to represent a vector which may be pointing in any arbitrary direction in space, we must choose a rectangular xyz-coordinate system and represent the vector by three components, which are its projections, taken with proper sign, on the three coordinate axes. This is illustrated in Fig. 8. From analytical geometry, it follows that if $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosines of the vector direction, then

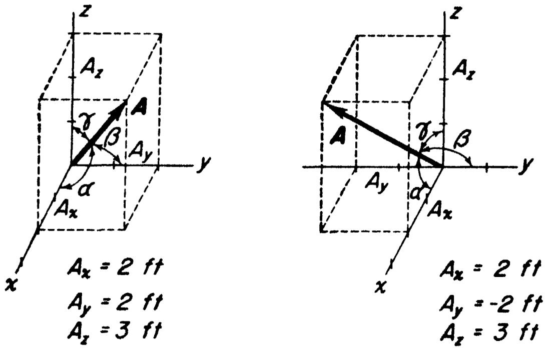

Fig. 8. The three components of a vector in three dimensions.

$$A_x = A\cos\alpha, \quad A_y = A\cos\beta, \quad A_z = A\cos\gamma,$$

where A is the magnitude of $\mathbf{A}$.

It is evident that a vector can be uniquely constructed, both as to magnitude and direction, if its rectangular components are given.

To summarize the above discussion of vector quantities, we note that we have given three methods of representing a vector:

(1) *in print* by a letter symbol in boldface type;

(2) *on a diagram* by an arrow-tipped segment of a straight line;

(3) *analytically*, by a statement of the rectangular components, or by a statement of the magnitude of the vector and the angles it makes with the coordinate axes.

It is important to become thoroughly familiar with all these ways of representing vectors. We have introduced the following notation, which is that used in most scientific and technical literature: If

$\boldsymbol{A}$ represents a vector quantity, then

A represents its magnitude (a scalar quantity, always positive);

A_x, A_y, A_z represent its rectangular components (scalar quantities, positive or negative).

One must always remember that a vector quantity is completely specified only if sufficient information is given to determine both its *magnitude* and its *direction.*

PROBLEMS

1. A displacement vector $\boldsymbol{A}$ has magnitude 6 ft and makes an angle of 150° with the x-axis. Find its rectangular components. Ans: $A_x = -5.20$ ft; $A_y = 3.00$ ft.

2. A displacement vector $\boldsymbol{B}$ has magnitude 11 m and makes an angle of 240° with the x-axis. Find its rectangular components.

3. A vector has components $C_x = 4$ m, $C_y = -7$ m. Find the magnitude and direction of the vector. Ans: $C = 8.06$ m, $\theta = 299^\circ.7$.

4. A vector has components $D_x = -4$ ft, $D_y = -2$ ft. Find the magnitude and direction of the vector.

5. A vector E has magnitude $E = 10$ ft and makes angles of 45°, 60°, and 120° with the x-, y-, and z-axes, respectively. Find its rectangular components and make a sketch showing the orientation of the vector.
Ans: $E_x = 7.07$ ft; $E_y = 5.00$ ft; $E_z = -5.00$ ft.

6. A vector $\boldsymbol{F}$ has magnitude $F = 15$ ft and makes angles of 120°, 120°, and 45° with the x-, y-, and z-axes, respectively. Find its rectangular components and make a sketch showing the orientation of the vector.

4. DISPLACEMENTS

In the previous section on vectors, we used as an example the rectilinear displacement of a particle. Here we shall discuss more general types of displacements of an object of any type.

An object is said to be *displaced* when its position in space is changed. The change in position is called its *displacement.* Two general kinds of displacement can occur. One is *translation* or *linear displacement,* in which all points of the body move along paths of the same length and a given line in the body keeps the same angular orientation in space, as shown in Fig. 9. The other kind is *rotation* or *angular displacement,* in which all points of the body move along concentric circles around a given axis as shown in Fig. 10. Any general motion of a body can be described as a combination of translation and rotation. For example, an automobile wheel rolling along a road rotates around its own axle, which has at the same time translational motion parallel to the road. The automobile wheel has pure rotational motion when it is spinning with the automobile standing still, and it has pure translational motion when the car is skidding straight ahead with its brakes locked. It is apparent that the description of general motions can become quite complicated; for the present we shall confine our attention to translations in one plane.

It is sufficient to consider the motion of a single point of a body in describing its linear displacement in pure translational motion because all the parts of a body follow exactly similar paths. For example, the center dashed line in Fig. 9 can be used alone to represent the linear displacement of the body. The simplest kind of displacement occurs in *rectilinear*

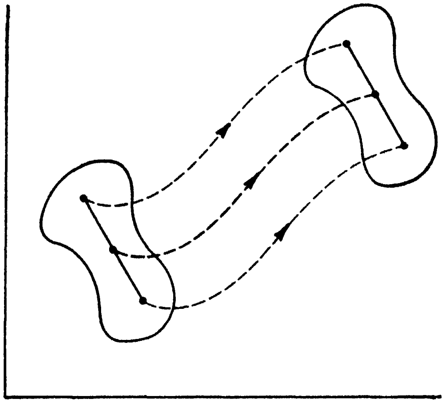

Fig. 9. Linear displacement in translational motion.

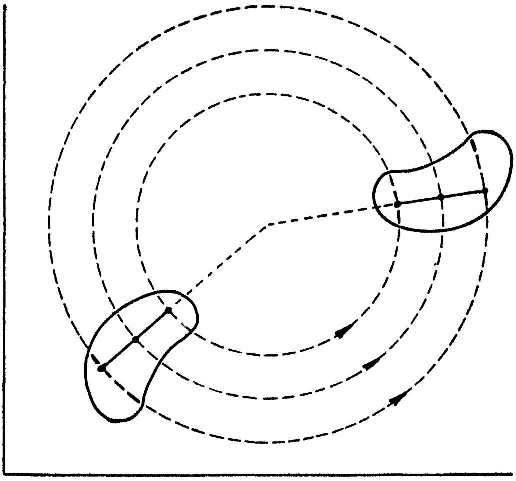

Fig. 10. Displacement in pure rotational motion.

motion, which is motion along a straight line. Any linear displacement can be approximately described in terms of successive rectilinear displacements as shown in Fig. 11. If the successive straight segments are taken short enough, the rectilinear description can be made as accurate as one pleases. It is sufficient to consider for the present only rectilinear displacements.

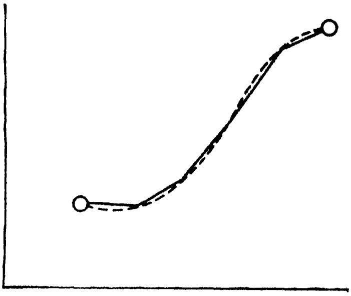

Fig. 11. Approximate description of a general linear displacement as a succession of rectilinear displacements.

A rectilinear displacement is a change of position along a segment of a straight line. As we have seen, it can be represented by a vector. A rectilinear displacement from point O to point P_1 in Fig. 12, for example, is represented on the diagram by the directed line segment ***A***.

An object moving between an initial point O and a final point P does not necessarily follow a single straight path. Instead it ordinarily must, because of obstructions, follow a connected series of straight-line segments if not a smooth curved path. For example, a taxicab going from a house in the suburbs to the railroad station in a city must ordinarily follow an indirect or roundabout path because of the manner in which the streets are laid out. The broken line $OP_1P_2P_3P_4P$ in Fig. 12 could represent the path followed by a taxicab in going from the initial point O to the final point P.

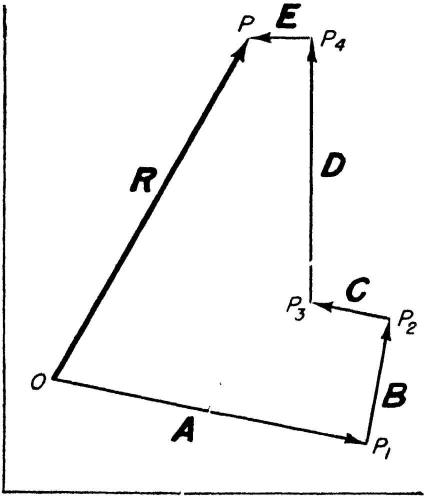

Fig. 12. Linear displacement of a body starting from O.

The term *resultant displacement* denotes the single straight-line displacement from the initial position of the body to its final position, irrespective of the actual path followed. In other words, the *resultant displacement* is the *vector sum* or net result of the successive intermediate displacements which the body actually undergoes. In Fig. 12 the directed line segment ***R*** represents the resultant displacement or the sum of the separate displacements ***A, B, C, D,*** and ***E***. The *resultant* or *sum* of two or more vectors is defined as that single vector which represents their combined or net effect. The vector $\boldsymbol{R}$ in Fig. 12 is called the *vector sum* of $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{D}$, $\boldsymbol{E}$, and we write

$$\boldsymbol{R} = \boldsymbol{A} \mathbf{+} \boldsymbol{B} \mathbf{+} \boldsymbol{C} \mathbf{+} \boldsymbol{D} \mathbf{+} \boldsymbol{E}.$$

The $\mathbf{+}$ signs here are made heavy to indicate that this is not ordinary addition but a special kind of *vector addition* that we shall study in the next section.

5. ADDITION OF VECTORS

The addition of vectors is the process of finding their *sum* or *resultant*. No matter whether the vectors represent displacements or any other type of vector quantity, the sum is defined just as the resultant displacement was defined in connection with Fig. 12. The rule is: to add vectors $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{D}$ shown at the left of Fig. 13, start at the origin and place the vectors tail to head in any order, as at the right of Fig. 13. The sum $\boldsymbol{R}$ is then the single vector leading from the origin to the head of the last vector in the series. That the vector $\boldsymbol{R}$ obtained by this procedure is independent of the *order* in which the vectors are placed tail to head will become clear from the analysis of the next paragraph.

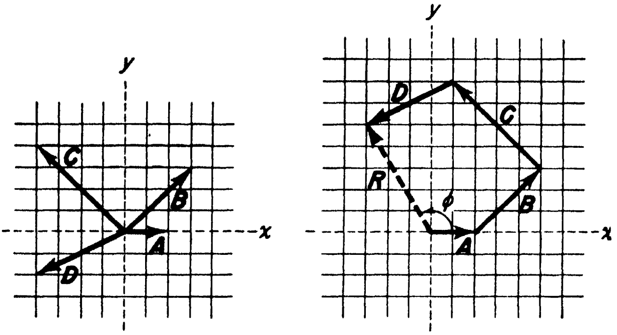

Fig. 13. Addition of vectors

$$\begin{array}{llll}
A_x = 2 & A_y = 0 & \tan\phi = R_y/R_x = -5/3 \\
B_x = 3 & B_y = 3 & \phi = 121\overset{\circ}{.}0 \\
C_x = -4 & C_y = 4 & \\
\underline{D_x = -4} & \underline{D_y = -2} & R = R_y/\sin\phi = 5/0.857 \\
R_x = -3 & R_y = 5 & R = 5.83.
\end{array}$$

The above rule is directly applicable to the graphical addition of vectors, but it may also be easily adapted to the analytical addition by expressing all vectors in terms of their components, as in the legend of Fig. 13. It is easy to show that the components of the resultant are just the sums of the components of the vectors which are added. Thus, if

$$\boldsymbol{R}=\boldsymbol{A}+\boldsymbol{B}+\boldsymbol{C}+\boldsymbol{D}, \tag{1}$$

we can show that and

$$\left.\begin{aligned} R_x&=A_x+B_x+C_x+D_x \\ R_y&=A_y+B_y+C_y+D_y. \end{aligned}\right\} \tag{2}$$

To prove this statement, consider the diagram at the right of Fig. 13. Since $A_x=2$, the head of $\boldsymbol{A}$ is 2 units to the right of the y-axis. Since $B_x=3$, the head of $\boldsymbol{B}$ is 3 units further to the right, or $2+3=5$ units to the right of the y-axis. Since $C_x=-4$, the head of $\boldsymbol{C}$ is back 4 units, or $2+3-4=1$ unit to the right of the y-axis; and since $D_x=-4$, the head of $\boldsymbol{D}$ is back 4 more units, or $2+3-4-4=-3$ units to the right of the y-axis (this means 3 units to the left). But the position of the head of $\boldsymbol{D}$ is the same as the position of the head of $\boldsymbol{R}$, and the position of the head of $\boldsymbol{R}$ relative to the y-axis is the x-component R_x. Hence $R_x=2+3-4-4=-3=A_x+B_x+C_x+D_x$. A similar argument may be applied to the y-component of $\boldsymbol{R}$. Thus we see that the geometrical definition which we have given for the vector sum (1) implies the analytical relations (2) between the components. In most cases we shall wish to use analytical rather than graphical methods for the addition of vectors, employing the rules contained in the equations (2):

> The x-component of the sum of a number of vectors is the sum of the x-components of the vectors.
>
> The y-component of the sum of a number of vectors is the sum of the y-components of the vectors.

Another illustration of the same procedure is given in Fig. 14. After we have found the components R_x and R_y, we can find the magnitude R and direction ϕ from simple trigonometry. The most convenient procedure is first to compute the angle ϕ from

$$\phi=\arctan(R_y/R_x) \tag{3}$$

and then the magnitude R from either

or

$$\left.\begin{aligned} R&=R_y/\sin\phi \\ R&=R_x/\cos\phi. \end{aligned}\right\} \tag{4}$$

The proper quadrant in which to put the angle may be determined from a rough sketch showing R_x, R_y, $\boldsymbol{R}$. The magnitude R may also be determined directly from the Pythagorean relation

$$R=\sqrt{R_x^2+R_y^2}. \tag{5}$$

For the addition of just two vectors, the following rule is frequently given:

> Lay off the vectors from a common origin and complete the parallelogram. The resultant is the directed diagonal from the origin to the opposite corner of the parallelogram.

This procedure is illustrated in Fig. 15(b). Comparison with Fig. 15(a) shows that this rule is equivalent to the one we have previously given. The triangle in Fig. 15(a) is just half the parallelogram of Fig. 15(b).

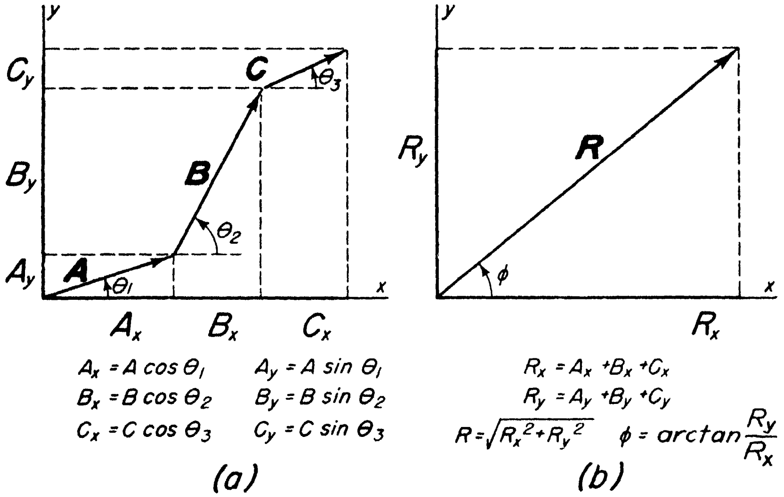

Fig. 14. Addition of three coplanar vectors. (a) Determination of the rectangular components of the given vectors. (b) Determination of the resultant R from its rectangular components.

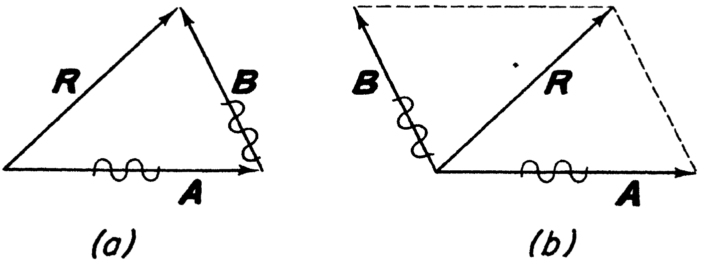

Fig. 15. Addition of two vectors A and B by (a) laying them off end to end, (b) completing the parallelogram. The vectors A and B are shown 'crossed-out' to emphasize that the resultant R completely *replaces* them in physical effect.

If we wish to add vectors which are not coplanar, we must work in three-dimensional space. One can imagine the vectors as represented by straight wires or rods placed end to end just as we placed our directed lines in two dimensions. The resultant is, by definition, the vector leading directly from the tail of the first vector to the head of the last in the series. The addition can be handled analytically by choosing an *xyz*-coordinate system and determining the three components of each of the vectors, as in Fig. 8. By an extension of our previous argument we see that if the original vectors are ***A, B, C,*** · · ·, and their resultant is

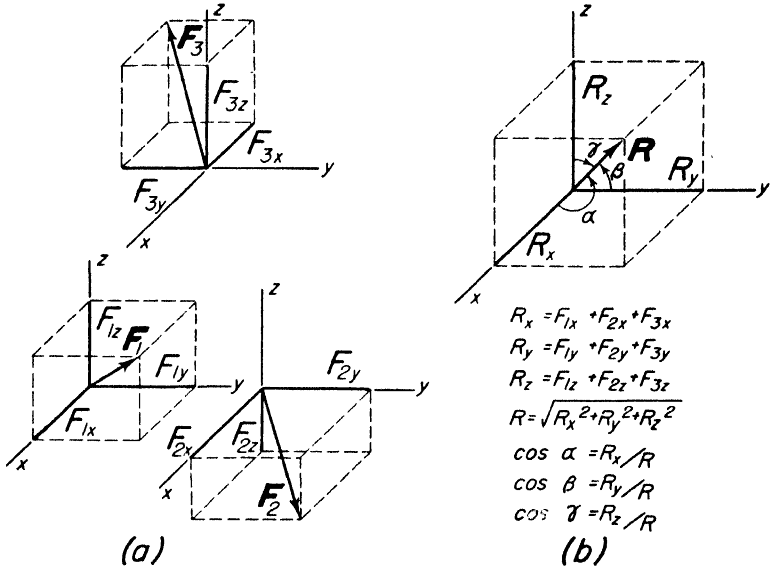

Fig. 16. Addition of three non-coplanar vectors $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, $\boldsymbol{F}_3$. (a) Determination of the rectangular components of the given vectors. (b) Determination of the resultant ***R*** from its rectangular components.

$$\boldsymbol{R}=\boldsymbol{A}+\boldsymbol{B}+\boldsymbol{C}+\cdots,$$

the components of ***R*** are given by the relations

$$\left.\begin{aligned} R_x&=A_x+B_x+C_x+\cdots,\\ R_y&=A_y+B_y+C_y+\cdots,\\ R_z&=A_z+B_z+C_z+\cdots. \end{aligned}\right\}\tag{6}$$

The magnitude of the resultant is

$$R=\sqrt{R_x^2+R_y^2+R_z^2}.\tag{7}$$

The orientation of ***R*** in space is given by the angles α, β, and γ which it makes with the coordinate axes as in Fig. 8. These angles are given by the trigonometric relations

$$\cos\alpha=(R_x/R),\quad \cos\beta=(R_y/R),\quad \cos\gamma=(R_z/R).\tag{8}$$

These cosines are called the direction cosines of the vector ***R***.

These considerations are illustrated by Fig. 16 for the case of the addition of three vectors $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, and $\boldsymbol{F}_3$.

PROBLEMS

1. Find the resultant of a displacement of 30 m at an angle of 30° with the x-axis, one of 80 m at an angle of 60°, and one of 100 m at an angle of 225°.
Ans: A displacement of 14.4 m at an angle of 109°.2 with the x-axis.

2. Find the resultant of a displacement of 50 ft at an angle of 0° with the x-axis, one of 100 ft at an angle of 120°, and one of 120 ft at an angle of 315°.

3. Add the following vectors: 1 ft E, 2 ft NE, 3 ft N, 4 ft NW, 5 ft W, 6 ft SW, 7 ft S, and 8 ft SE. Ans: 10.5 ft, 22°.5 west of south.

4. Add the following vectors: 1 m N, 2 m W, 3 m S, 4 m E, 5 m NE, 6 m NW, 7 m SW, and 8 m SE.

5. Find the resultant $\boldsymbol{R}$ in Fig. 12, if the individual displacements have the following magnitudes and make the following angles, measured counterclockwise from the horizontal x-axis: $\boldsymbol{A}$, 74 m, 350°; $\boldsymbol{B}$, 30 m, 80°; $\boldsymbol{C}$, 29 m, 170°; $\boldsymbol{D}$, 63 m, 90°; $\boldsymbol{E}$, 15 m, 180°. Ans: 91.5 m, 67°.8.

6. Find the resultant $\boldsymbol{R}$ in Fig. 14 if $A = 4.0$ ft, $B = 5.7$ ft, $C = 2.7$ ft, $\theta_1 = 20°$, $\theta_2 = 63°$, $\theta_3 = 25°$.

7. Add the vectors $\boldsymbol{E}$ and $\boldsymbol{F}$ of Probs. 5 and 6, p. 17. Ans: A vector of magnitude 6.15 ft, making angles of 94°.0, 114°.0, 24°.3 with the x-, y-, z-axes.

8. To the vector $\boldsymbol{E}$ of Prob. 5, p. 17, add a vector $\boldsymbol{G}$ of magnitude 8 ft, lying in the xy-plane and making angles of 45° with the x- and y-axes.

6. RESOLUTION OF A VECTOR

The process of resolution of a vector is the inverse of the process of vector addition in that it *replaces* a vector by two or more vectors which have the given vector as their resultant. The most useful type of resolution is that in which the vector is replaced by its vector projections on a set of mutually perpendicular axes. If attention is confined to a plane we use two such axes, but if we are working in three dimensions we must use three. These vector projections are known as *rectangular component vectors.* Figure 17 shows a vector $\boldsymbol{A}$ resolved in two dimensions into two component vectors $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$.

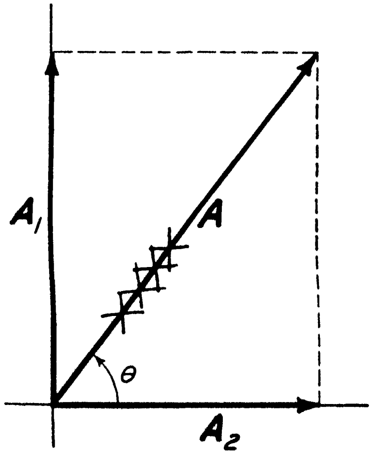

Fig. 17. Resolution of a vector $\boldsymbol{A}$ into two rectangular component vectors in a plane.

It must be emphasized that the component vectors can be used on a diagram only to *replace* the original vector, *not to supplement it.* For this reason the original vector is shown scratched out in Fig. 17 after it has been replaced by its component vectors.

Resolution of a given vector A into a set of rectangular component vectors in three dimensions is illustrated in Fig. 18. Three mutually perpendicular intersecting axes are chosen, with their point of intersection at the tail end of the given vector. The component vector along one of the chosen axes is simply the vector projection of A on that axis. In physical problems the vector A may be *replaced* by the three component vectors A_1, A_2, A_3, whose *sum* is A.

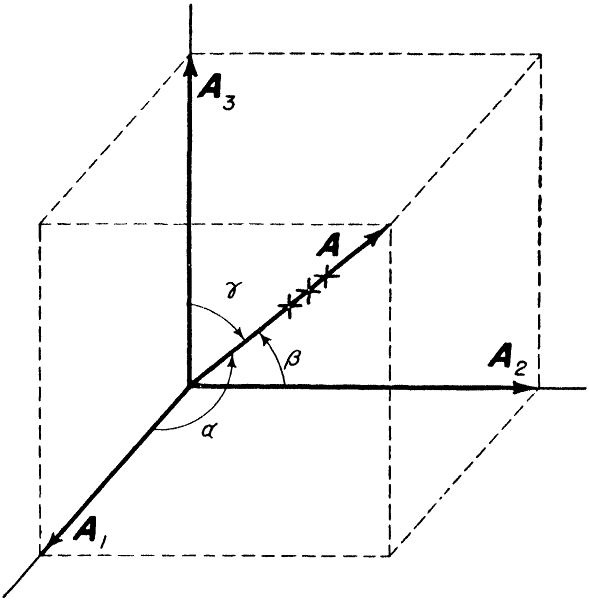

Fig. 18. Resolution of a vector A into three rectangular component vectors in three dimensions.

7. FORCES

The word *force* is popularly used to denote an influence which tends to produce a change in the present state of affairs. For example, the force of public opinion tends to produce a change in the behavior of individuals. The word *force* is used in physical science and engineering with a restricted technical meaning to denote an explicitly definable vector quantity which tends to produce a change in the motion of objects. We shall use the word *force* with this restricted meaning.

Our simplest intuitive idea of a force as a push or pull resulting from muscular exertion comes from the experience that it is necessary to push or pull on an object to change its motion. For example, it is necessary to push or pull backward or forward on a moving object to slow it down or speed it up, and to push or pull sidewise on it to change its direction of motion. For the present we can think of a force simply as a push exerted by one object on another. Force will later be precisely defined in terms of its effect on motion.

Familiar origins of forces include, in addition to muscular exertions of organisms, (a) gravitation, (b) friction, (c) elasticity, (d) electricity, and (e) magnetism. The first three of these types of forces are treated in the branch of physics called mechanics and are briefly discussed below; forces associated with electricity and magnetism are treated in later chapters.

Gravitation is the universal attraction which exists between any two samples of matter. The most familiar manifestation of gravitation is the force of attraction which the earth exerts on any object in its neighborhood. The force with which an object is pulled vertically downward toward the earth (the *force of gravity*) is called the *weight* of that object. Weight is an ever-present force which must always be considered in engineering problems. The *vertical* itself is defined as the direction of the force of gravity, as is evident when one determines the vertical by means of a plumb bob. The *horizontal* is a plane normal to the vertical.

Friction is the opposition to motion which occurs whenever one object either moves or merely tends to move along the surface of another. Familiar manifestations of friction include the forces of opposition encountered in sliding objects along a level floor, in the bearings of moving parts of machinery, and in the brakes of automobiles. Forces of friction must be taken into account in mechanics problems whenever there is tendency of one object to move along the surface of another. A frictional resistance also occurs when a body moves through a fluid medium such as air or water.

Elasticity is the tendency of a sample of matter to resist an attempted change in its size or shape and to return to its original size or shape when the attempt ceases. Familiar manifestations of elasticity include the force exerted by a stretched coil spring and the force exerted by a compressed gas on the walls of its container. All substances exhibit this property to a greater or less degree, and it must be taken into account whenever changes occur in the size or shape of objects. Forces of this type include the force with which the floor pushes up on a man. This is merely one example of the forces which occur generally between objects to resist the tendency of one body to sink into another and thus change its shape.

It is well known from experience that the application of a force to a body does not always result in a change in its motion. For example, the body may be fastened so that it cannot move, but an applied force can still produce an effect on the body. The common effects which result from the application of a force to an object include (a) the balancing or counteracting of another force which is acting on the body, (b) a change in the motion of the body, and (c) a change in the size or shape of the body. One or more of these effects is always produced when forces are applied to objects. The balancing of forces is treated in that branch of mechanics called *statics.* The effects of forces on motions form the subject matter of *dynamics.* The effects of force on the shape and size of objects form the basis of the study of *elasticity* and the strength of materials.

8. MEASUREMENT OF FORCE

A measuring instrument for the quantitative comparison of magnitudes of forces could, in principle, be based on any one of the effects produced by forces. A familiar instrument for measuring forces is the spring balance, which consists, in its simplest form, of a coil spring with one end attached to a rigid case and the other end free to move a pointer along a scale on the case. The spring changes in length when a stretching force is applied to its free end but returns to its original length when the force is no longer applied. A force of given magnitude always produces the same displacement of the free end of the spring if the balance is properly constructed and handled. A larger force produces a greater displacement of the free end of the spring, a smaller force a lesser displacement.

A spring balance or other device for measuring forces must be calibrated by means of a reproducible force such as the gravitational pull of the earth on a certain specimen of material at a specified location. The unit of force used for calibration of most scientific instruments is the *kilogram-force* (kgf), which is the weight at a specified location on the earth of a standard cylinder of platinum-iridium kept at the International Bureau of Weights and Measures. The *gram-force* (gf) is 0.001 kgf.

Many commercial and engineering instruments used for measuring forces in English-speaking countries are calibrated in terms of the *pound-force* (lbf), which was originally the weight at a specified location of a standard cylinder of platinum kept at the Standards Office, Westminster, London. The pound-force is now defined by the United States government as 453.5924277 gf. The ounce-force (ozf) is $\frac{1}{16}$ lbf. The ton-force is 2000 lbf.

It is necessary to specify a standard location in defining the unit force in this manner because the weight of a given object near the earth varies somewhat with altitude and latitude. The force units lbf, gf, and kgf are called *gravitational units* because they are defined in terms of the earth's gravitational pull on a standard sample of matter at a specified location on the earth's surface. From the above definitions, we see that

$$1\ \text{lbf} = 453.6\ \text{gf} = 0.4536\ \text{kgf},$$

or

$$1\ \text{kgf} = 1000\ \text{gf} = 2.205\ \text{lbf}.$$

The *newton* (nt) is the fundamental unit of force in the metric MKS system of units which is used in much scientific work and in electrical engineering. The value of the newton is such that 1 lbf = 4.448 newtons and 1 kgf = 9.807 newtons. The newton is called an *absolute unit* of force because its value is defined in terms of the effect of forces on the motion of an object quite independently of the position of the object in the universe or of gravitational pull on it. We shall have to postpone the formal definition of the newton until Chap. 6.

9. TREATMENT OF FORCES AS VECTORS

In a tug of war it is important that each contestant not only pull with all his might on the rope but that he pull in the proper direction to aid his own team. This example of applied forces emphasizes the fact that the effect produced by a force depends on both its magnitude and its direction. In the analysis of the effects produced by forces it is always necessary to specify both the magnitude and the direction of each force. A force is a vector quantity, and the methods of handling vectors discussed earlier are applicable to it.

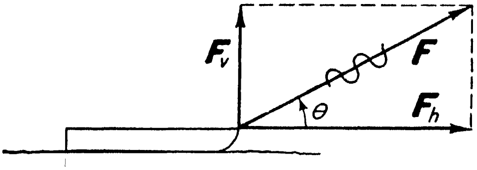

Fig. 19. Resolution of a force into horizontal and vertical component forces.

A force vector is denoted in print by ***F***, ***f***, or some other suitable letter in boldface type, and it is represented on a diagram by an arrow whose length is proportional to the magnitude of the force and whose direction corresponds to that of the force.

The resolution of a force vector into rectangular component vectors is illustrated in Figs. 19 and 20. It must be emphasized again that the component forces can be used only to take the place of the original force, not to supplement it. For this reason the original force vector is shown crossed out in these figures in favor of the two mutually perpendicular forces that replace it. The meaning of these component forces is easy to understand in a specific example such as that in which a sled is pulled along the ground. The horizontal force $\boldsymbol{F}_h$ in Fig. 19 is the effective force pulling the sled along the ground, and the vertical force $\boldsymbol{F}_v$ is the effective force tending to lift the sled off the ground. The resolution of a force into rectangular component forces in the above manner is frequently very useful because the two mutually perpendicular forces can be treated independently of each other so far as their physical effects are concerned.

Figure 20 illustrates the resolution of a vertical force **W** into forces parallel and normal to an inclined plane. This case is of very common occurrence because the weight of an object is a force **W** directed vertically downward. It is frequently useful, in analyzing the effect of a body's weight on its motion along an inclined plane, to resolve **W** into component vectors in directions parallel and normal to the inclined plane. The magnitude of the tangential force is $W_T = W \sin\theta$ and that of the normal force is $W_N = W \cos\theta$ if θ is the angle between the inclined plane and the horizontal and W is the magnitude of the weight. The tangential force $\boldsymbol{W}_T$ is the effective force that tends to pull the body downward along the incline, and $\boldsymbol{W}_N$ is the effective force that pulls the body normally against the inclined plane.

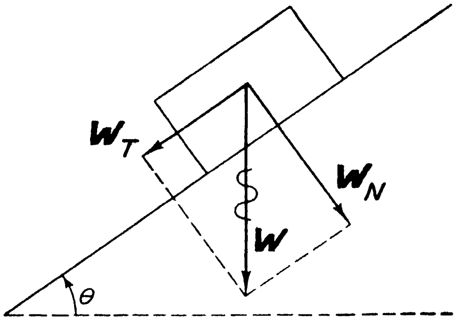

Fig. 20. Resolution of a vertical force into component forces parallel and normal to an inclined plane.

The addition of a set of force vectors (the determination of their resultant) is carried out by the same methods as in the case of displacement vectors. The resultant (vector sum) of a set of forces plays an important role in the laws of mechanics as we shall formulate them in succeeding chapters. It turns out that if several forces act at the same point of a body, their net effect is in all respects identical with the effect of a single force which is their vector sum. If several forces act on a body but not all at the same point, their net effect on the *translational* motion of the body is the same as the effect of a single force which is their vector sum; however, their effect on the *rotational* motion of the body involves the summation of the *torques* which they exert. We consider torque in the next two sections.

PROBLEMS

1. Find the resultant of the three forces $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, $\boldsymbol{F}_3$ of Fig. 21 if they have magnitudes of 37 lbf, 25 lbf, and 30 lbf, respectively, and make angles of 30°, 60°, and 135° with the horizontal x-axis. Ans: 65.6 lbf, 69°2.

2. What would be the resultant of three forces, such as those in Fig. 21, if $F_1 = 4$ kgf, $F_2 = 2$ kgf, $F_3 = 3$ kgf, and $\theta_1 = 25°$, $\theta_2 = 65°$, $\theta_3 = 140°$?

3. Find the resultant $\boldsymbol{R}$ of the three forces shown in Fig. 16 if $\boldsymbol{F}_1$ has magnitude 15 gf and makes angles of 54°.7 with the x-, y-, and z-axes; $\boldsymbol{F}_2$ has magnitude 15 gf and makes angles of 48°.4, 48°.4, and 110° with the x-, y-, and z-axes; and $\boldsymbol{F}_3$ has magnitude 15 gf and makes angles of 120°, 120°, and 45°.

Ans: 21.2 gf, $\alpha = 58°.3$, $\beta = 58°.3$, $\gamma = 48°.1$.

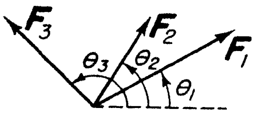

Fig. 21.

4. Find the components R_x, R_y, and R_z of the resultant of three forces such as those of Fig. 16 if $\boldsymbol{F}_1$ has magnitude 10 lbf and direction cosines 0.624, 0.600, 0.500; $\boldsymbol{F}_2$ has magnitude 12 lbf and makes an angle of 115° with the z-axis and equal acute angles with the x- and y-axes; and $\boldsymbol{F}_3$ has magnitude 13 lbf and makes angles of 115° with the x- and y-axes and an acute angle with the z-axis.

10. TORQUE ABOUT AN AXIS

It is common experience that the effectiveness of a force in changing the rotational motion of an object depends not only on the magnitude and direction of the force but also on the distance of its line of action from the axis of rotation. For example, a child learns by experiment that he can open a swinging door most readily if he pushes perpendicularly against its surface at a point as far as possible from the hinged edge. The force required to open the door is larger if he pushes on it closer to its hinged edge or along a line making an angle of less than 90° with its surface. He also learns that no rotation around the hinged edge can be produced by a force whose line of action passes through that edge. All experience shows that a force is most effective in producing rotation around an axis if its line of action is perpendicular to that axis and if the distance from the axis to its line of action is as large as possible.

The results of all experiments show that only the projection of a force on a plane perpendicular to an axis of rotation can be effective in changing rotational motion around that axis; a force component parallel to the axis has no effect. For example, if a man pulls with a force $\boldsymbol{F}$ on a string attached to a doorknob in such a way that the string makes an angle θ with the horizontal plane through the knob, then only the horizontal projection of $\boldsymbol{F}$, of magnitude $F \cos\theta$, can be effective in turning the door around its hinged edge. Hence in our study of the effect of a force on rotation about an axis, we can assume that the force is applied in a plane perpendicular to the axis of rotation. If the force is not so applied, only the component force in such a plane need be considered.

We now consider the relation between the effectiveness of a force in producing rotation and the distance of its line of action from the axis of rotation. It is convenient for this purpose to define a quantity called the *lever arm* as follows:

> The *lever arm* of a force is the perpendicular distance from the axis of rotation to the line of action of the force, provided that the line of action of the force lies in a plane perpendicular to the axis of rotation.

The distances l_1, l_2, l_3 shown in Fig. 22 are, respectively, the lever arms of forces $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, $\boldsymbol{F}_3$ applied to the hinged bar, where the axis of rotation of the bar is the horizontal line through its hinge and the forces act in a vertical plane. It is important to note that the lever arm is a purely geometrical distance and is not necessarily equal to the dimensions of parts of an object.

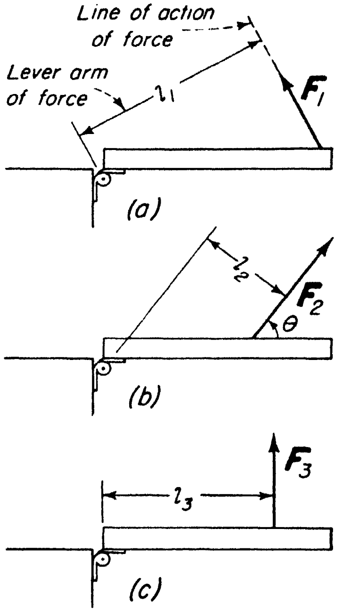

Fig. 22. Showing the line of action and the lever arm for several forces.

Many simple experiments can be devised to test the effectiveness of forces in producing changes of rotational motion. The hinged horizontal bar shown in Fig. 22 forms the basis of such an experiment. The force $\boldsymbol{F}$ applied to the bar in the vertical plane tends to turn the bar counterclockwise around a horizontal axis through the hinge. The force of gravity tends to turn the bar clockwise. The magnitude of $\boldsymbol{F}$ required to keep the bar from falling downward under the influence of gravity is measured for different points of application and different directions of the line of action of the force. If l denotes the perpendicular distance from the axis of rotation to the line along which $\boldsymbol{F}$ acts, it is found that the product Fl is a constant for the different cases shown in Fig. 22. These results show that *the effectiveness of a force in producing rotation around an axis depends on the product of the magnitude of the force and the length of its lever arm.*

The tendency of a force $\boldsymbol{F}$ to produce change of rotational motion around an axis is measured by its torque around that axis. *The torque L is defined as the product of the magnitude F of the force by its lever arm l:*

$$L = Fl. \tag{9}$$

This definition implies that a force of 10 lbf with a lever arm of 2 ft will have the same tendency to change the rotation of a given object as a force of 4 lbf with a lever arm of 5 ft. This definition fits the results of all experiments, such as the one shown in Fig. 22, on the effects of forces on rotation around a fixed axis. It is important to note that a force directed through an axis of rotation has zero lever arm, and therefore its torque around that axis is zero.

Torques are measured in units of *force*×*length*. For example, a force of 4 kgf applied with a lever arm of 3 m produces a torque whose magnitude is 4 kgf×3 m = 12 kgf·m. A force of 10 lbf applied with a lever arm of 2 ft produces a torque whose magnitude is 10 lbf×2 ft = 20 lbf·ft. Other units of torque commonly used include pound-force·inch (lbf·in), gram-force·centimeter (gf·cm), and newton·meter (nt·m). One can easily obtain the relations among different units of torque merely by using the principle that a quantity can replace one to which it is equal. For example, the relation between the units lbf·ft and kgf·m is obtained by the following procedure:

$$1 \text{ lbf·ft} = (1 \text{ lbf})\cdot(1 \text{ ft}) = (0.454 \text{ kgf})\cdot(0.305 \text{ m}) = 0.138 \text{ kgf·m}.$$

A torque is classified as *clockwise* if it tends to produce clockwise rotation or as *counterclockwise* if it tends to produce counterclockwise rotation. These designations will be useful in the next chapter, where equilibrium of forces and torques are considered. Thus, in Fig. 22(b), if $F_2 = 2.5$ kgf and $l_2 = 0.6$ m, the force $\boldsymbol{F}_2$ exerts a *counterclockwise* torque of 1.5 kgf·m. If several forces act on a body and produce torques about an axis, the *net* or *resultant* torque is obtained as the algebraic sum of the several torques, treating clockwise torques as positive, counterclockwise torques as negative, or *vice versa.*

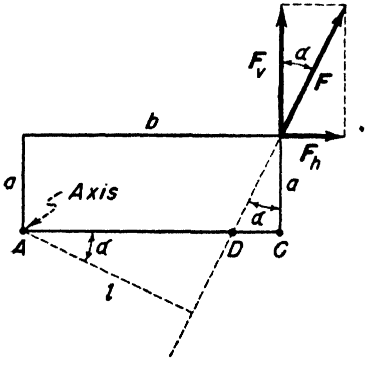

Fig. 23. The lever arm of $\boldsymbol{F}$ is l, the lever arm of $\boldsymbol{F}_v$ is b, that of $\boldsymbol{F}_h$ is a.

We shall now prove an important theorem which greatly facilitates the computation of torques. We note that if we are to compute torque about an axis, we must know not only the magnitude and direction of the force but also the position of its line of action, or alternatively, its point of application to the body. We shall now prove the following statement:

In computing the torque arising from a force $\boldsymbol{F}$*, it is permissible to resolve the force into component forces, compute the torque of each component force separately, and add them to get the torque of* $\boldsymbol{F}$*. In this procedure, the component forces must be applied at the same point of the body as* $\boldsymbol{F}$ *itself.*

Thus, in Fig. 23, the force $\boldsymbol{F}$ exerts counterclockwise torque Fl about the indicated axis. If the given data are values of F, α, a, b, it is inconvenient to compute the lever arm l. It is easier to compute the magnitudes of $\boldsymbol{F}_v$ and $\boldsymbol{F}_h$:

$$F_v = F\cos\alpha, \quad F_h = F\sin\alpha,$$

and then to compute the torques of $\boldsymbol{F}_v$ and $\boldsymbol{F}_h$, which are

$$F_v b = Fb\cos\alpha, \quad \text{counterclockwise,}$$

and

$$F_h a = Fa\sin\alpha, \quad \text{clockwise.}$$

The algebraic sum of these torques is a resultant torque

$$Fb\cos\alpha - Fa\sin\alpha, \quad \text{counterclockwise.}$$

The theorem states that the value computed in this way equals the desired torque Fl. We can prove the theorem by showing that

$$Fl = Fb\cos\alpha - Fa\sin\alpha$$

or that

$$l = b\cos\alpha - a\sin\alpha.$$

To derive this equation, note that in Fig. 23, $DC = a\tan\alpha$, and hence that $AD = b - a\tan\alpha$. Since $l = AD\cos\alpha$, we have

$$l = (b - a\tan\alpha)\cos\alpha = b\cos\alpha - a\sin\alpha,$$

the desired relation.

PROBLEMS

1. In Fig. 22(a), if $F_1 = 32$ kgf and $l_1 = 50$ cm, what is the torque of $\boldsymbol{F}_1$ in kgf·m? Ans: 16 kgf·m counterclockwise.

2. In Fig. 22(c), if $F_3 = 1$ lbf and $l_3 = 9$ in, what is the torque of $\boldsymbol{F}_3$ in ozf·in?

3. In Fig. 22(a), if the force vector has magnitude 16 lbf and makes an angle of 60° with the horizontal, and if its point of application lies 0.5 ft above and 2.5 ft to the right of the axis, find the torque. Ans: 38.6 lbf·ft counterclockwise.

4. In Fig. 22(b), if the force vector has magnitude 6 kgf with $\theta = 50°$ and if its point of application lies 0.2 m above and 0.7 m to the right of the axis, find the torque.

5. In Fig. 20, Chap. 3 (p. 60), if $F_1 = 1$ lbf, $F_2 = 2$ lbf, $F_3 = 3$ lbf, and $l_1 = 8$ ft, $l_2 = 6$ ft, $l_3 = 10$ ft, find the resultant torque about the axis. Ans: 10 lbf·ft clockwise.

6. In Fig. 20, Chap. 3 (p. 60), if $F_1 = 3$ kgf, $F_2 = 4$ kgf, $F_3 = 5$ kgf, and $l_1 = 30$ cm, $l_2 = 30$ cm, $l_3 = 50$ cm, find the resultant torque about the axis.

7. In Fig. 11, Chap. 3 (p. 51), let $F_1 = 6$ kgf, $F_2 = 4$ kgf, $F_3 = 7$ kgf, $F_4 = 10$ kgf, $l_2 = 0.3$ m, $l_3 = 0.7$ m, $l_4 = 1.0$ m, $\theta_1 = 60°$, $\theta_4 = 40°$. Find the resultant torque of these four forces about the point of application of $\boldsymbol{F}_1$. Ans: 0.328 kgf·m counterclockwise.

8. In Prob. 7, find the resultant torque of the four forces about the point of application of $\boldsymbol{F}_4$.

11. THE TORQUE VECTOR, TORQUE ABOUT A POINT*

In the discussion of torque about an axis we did not refer to torque as a vector, although torque was not really a scalar in this discussion, because we had to specify whether it was clockwise or counterclockwise.

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* Study of this section may be postponed if desired until the study of tops and gyroscopes is taken up in Chap. 13.

In discussing dynamical problems such as the motion of tops and gyroscopes, it is necessary to represent torque by a vector. In the case of torque about an axis, the torque vector is drawn along the axis in the sense indicated in Fig. 24. This sense is determined by the *right-hand rule:*

> If one encircles the axis with the fingers of the right hand to indicate the direction of the tendency to rotate, the thumb points along the axis in the direction of the torque vector.

The rule can also be stated as follows:

> If a screwdriver lined up with the axis of rotation is turned in the direction of the tendency to rotate, it drives a right-hand screw along the axis in the direction of the torque vector.

The torque about an axis, which has been discussed so far, is only a special case of torque. The more general case is that of torque about a point. The point may be a fixed point, such as the point of support of a top, or some other point of particular interest. Thus the torque of the air forces on the control surfaces of an airplane about the center of gravity* of the plane determines its response to the controls.

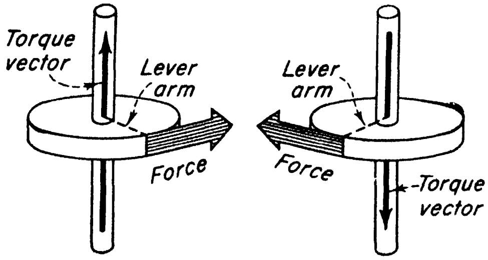

Fig. 24. Sense of the torque vector as given by the right-hand rule.

If an object is so mounted that its only possible motion is a rotation around a fixed point, a force applied to it tends to change its rotational motion around a definite axis through that point. That axis is perpendicular to the plane containing the fixed point and the line of action of the force. The axis around which the force tends to produce rotation is called the *torque axis*. As an example let us consider a rod suspended from a universal joint. If a horizontal eastward force is applied to the rod at any point below the joint, the torque axis is a horizontal north-south axis through the joint and the torque vector points southward, as in Fig. 25. This figure also shows the torque vector associated with the pull of gravity on a top.

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* Center of gravity is defined in Chap. 3.

The rules for determining the torque vector in the case of torque about a point are the following:

The lever arm *l* is the perpendicular distance from the line of action of the force to the point.

The torque vector passes through the point along a line perpendicular to the plane containing the force vector and the lever arm.

The torque magnitude is force×lever arm.

The sense of the torque direction is determined by the right-hand rule.

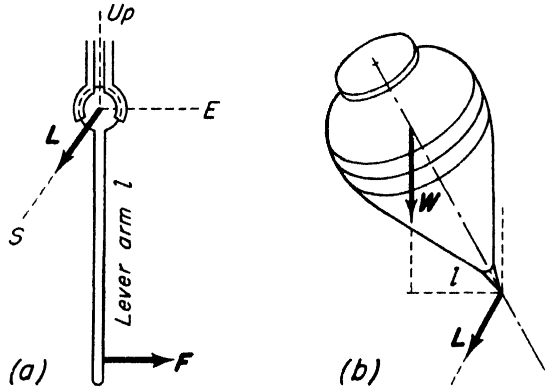


Fig. 25. (a) If an eastward force acts on a rod mounted in a universal joint, the torque vector is toward the south. (b) The torque of the force of gravity on a top is a horizontal vector normal to the vertical plane containing the top axis.

If several forces act on a body, the resultant effective torque about a point is the vector sum of the several torques due to the individual forces.

The relation between torque about an axis and torque about a point is not obvious from our definitions. Actually, however, the torques about three perpendicular axes through a point are the vector components of the torque about the point. We need not discuss these matters in further detail for purposes of this text.

CHAPTER 3

STATICS

Statics is that branch of mechanics which deals with the balance or equilibrium of forces on objects which remain at rest. The principles of statics form the basis of much of the work of mechanical, civil, and architectural engineers engaged in machine, structure, and building design. This chapter deals with cases in which the set of forces acting on a body are in translational and rotational equilibrium. These cases include not only truly static cases but also cases of translation with constant velocity and of rotation with constant angular velocity. We shall also study the laws governing the forces of friction between the surfaces of solids in contact.

1. ACTION AND REACTION; NEWTON'S THIRD LAW

When two bodies A and B are in mechanical contact, in general they exert forces *on each other* across the region of contact. It is a fundamental principle that the forces the two bodies exert on each other are always equal and opposite. That is, if the force exerted *by A on B* is represented by a certain vector, the force exerted *by B on A* will be represented by the negative of this vector—a vector of equal magnitude but oppositely directed. This statement is succinctly summarized in the third of Newton's three laws of mechanics:

Newton's third law: Action equals reaction.

Newton called the force exerted by A on B the *action*, and the *oppositely directed* force exerted by B on A the *reaction*. When the law is stated as above, it must be remembered that *action and reaction are oppositely directed forces exerted by two different bodies on each other*, and that in no case is it of importance which of the equal and opposite forces is considered the action and which the reaction.

The above law holds rigorously no matter what the state of motion of the bodies may be. Various cases are illustrated in Fig. 1.

Newton's third law also holds in cases of action at a distance such as occur with gravitational, electrical, and magnetic forces. Thus, the earth exerts a downward gravitational force on any object near its surface, for example on a flying airplane. The airplane exerts an equal and opposite upward force on the earth. Although the force of the earth on the plane is of great practical importance, that of the plane on the earth is of no detectable importance in connection with the motion of the earth. The gravitational pull of the earth on the moon keeps the moon revolving around the earth just as the pull of the sun keeps the earth revolving around the sun. In the case of the earth-moon system, however, the reaction force exerted by the moon does introduce a detectable wobble into the earth's regular motion around the sun, and the gravitational pull of the moon on the waters of the earth is the principal tide-producing force.

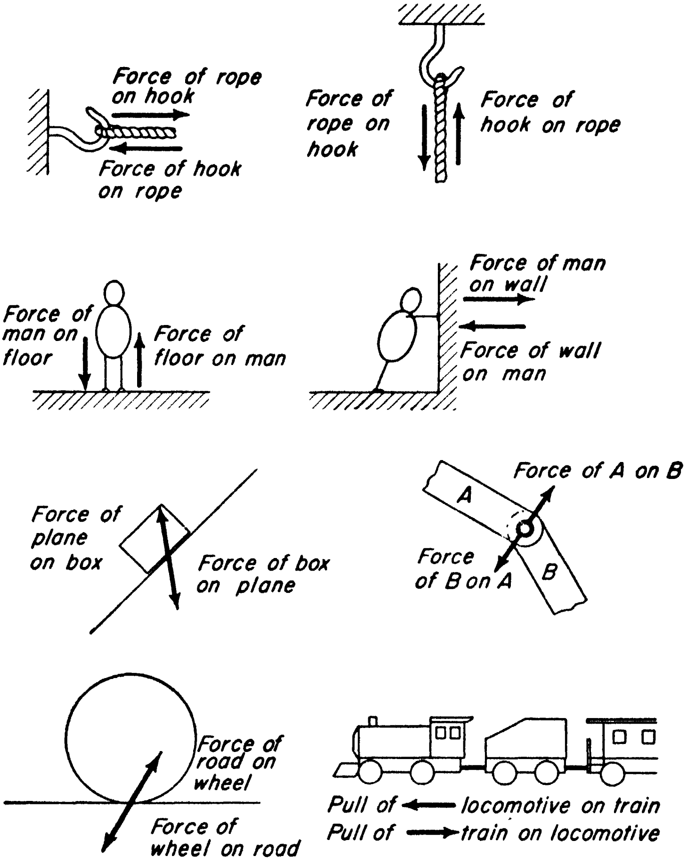


Fig. 1. Action equals reaction.

2. INERTIA; NEWTON'S FIRST LAW

Before the days of Newton it was frequently assumed that the 'natural' state of a body was one of rest and that forces or torques were required to keep the body in motion, because it was noted that all terrestrial objects come to rest when they are left alone. A box sliding on a horizontal plane or a wheel rotating on a shaft come more or less rapidly to rest. But are these objects free from forces or torques? There is a frictional force between the box and the plane acting in a direction to oppose the motion of the box, and a frictional torque in the bearings of the wheel acting in a direction to oppose the rotation of the wheel. If efforts are made to decrease this friction—for example by lubrication, by equipping the box with steel runners and making the plane out of ice, or by using ball bearings—the state of rest is achieved much less rapidly. In order to stop an automobile more rapidly than the ordinary frictional forces would stop it, we have to apply a braking torque to the wheels, just as we have to apply engine torque to speed it up. The braking torque is an additional retarding frictional torque.

The conclusion we draw from experience, as indicated in the discussion above, is that the 'natural' state of a body is one of *uniform motion.* By the term *uniform motion* we mean, in the case of a particle, motion of translation along a straight line at constant speed; in the case of a wheel mounted on a fixed axle, motion of rotation at constant angular speed (in rev/min, for example); in the case of a ball or other object rotating about an axis of symmetry, a combination of translation along a straight line at constant speed and rotation about an axis of fixed direction at constant angular speed. The 'natural' state of a free body rotating about an axis that is not an axis of symmetry is found to be somewhat more complicated; discussion of the 'natural' motion in this case will be left for more advanced treatments of mechanics.

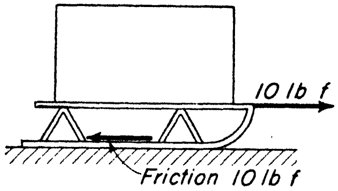
Unless a body is acted on by forces from some other body, it remains in a state of uniform motion. The natural tendency to remain in a state of uniform motion was termed *inertia* by Newton, and the law we have been discussing can be stated as follows:

> *Law of inertia (Newton's first law): Every body will remain in a state of rest or of uniform motion unless acted on by external forces.*

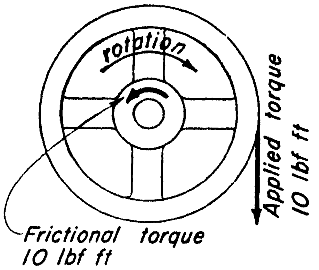
By 'external' forces, this law refers to forces exerted by some other body, whether these forces arise from *contact* with the other body (pushes, pulls, friction, air and water resistance, and the like) or are of gravitational, electrical, or magnetic nature.

According to this law, if we were out in a rocket ship in free space where there is no air and no appreciable gravitational pull, and threw a ball from our rocket, the ball would go on indefinitely with the same translational speed in a straight line and the same rotational speed about the same axis through the center of the ball that it had when it left our hand.

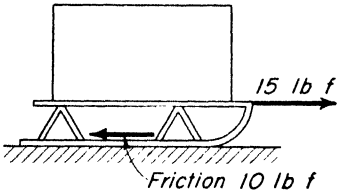
Of course it is the *resultant* force or the *resultant* torque acting on a



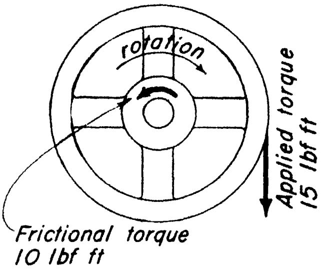
(a) Forward speed remains constant.



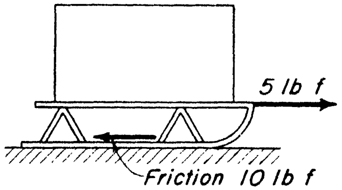
(a) Rotational speed remains constant.



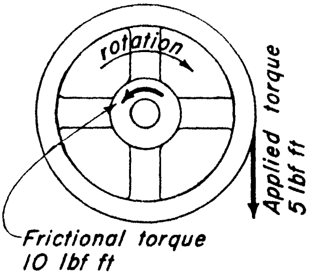
(b) Sled speeds up



(b) Rotational speed increases



(c) Sled slows down



(c) Rotational speed decreases

Fig. 2. A sled moving forward and a wheel rotating clockwise with (a) zero resultant force or torque; (b) and (c) non-zero resultant force or torque.

body that determines whether it will change its state of motion. As illustrated in Fig. 2, we can keep a box sliding over a horizontal plane at constant speed if we pull forward with just enough force to balance the backward dragging force of friction. If we pull forward with more force than this, the speed increases; if we pull with less force, the speed decreases. We can keep a wheel rotating at constant rotational speed about a fixed axis by applying just enough torque to balance the retarding frictional torque in the bearings; more torque will increase the rotational speed; less will permit the wheel to slow up. The law that gives the rate of change of speed when a resultant force or torque acts on an object (Newton's *second* law) will be studied in later chapters under the heading of dynamics. In this chapter we shall study cases in which the resultant force or torque acting on a body is *zero* and the body retains its state of rest or of uniform motion. The body is said to be *in equilibrium* under the system of applied forces.

3. EQUILIBRIUM CONDITIONS

The conditions for a body to remain in a state of rest or of uniform motion are known as the *equilibrium conditions.* The condition for *translational* equilibrium ensures that the body will have no tendency to change its translational speed or direction of motion.

> *Condition for translational equilibrium: The resultant (vector sum) of all the forces acting on the body is zero.*

The condition for *rotational* equilibrium ensures that the body will have no tendency to change its state of rotational motion and hence no tendency to change its rotational speed.

> *Condition for rotational equilibrium: The resultant torque about any axis of all the forces acting on the body is zero.*

We shall learn how to apply these conditions to increasingly complex cases in the following order:

(a) forces acting on a particle,
(b) coplanar forces acting on a body whose weight can be neglected,
(c) inclusion of the weight of the body by placing a single force at a point called the center of gravity,
(d) inclusion of forces of friction,
(e) general case of noncoplanar forces.

These cases will be taken up in succeeding sections of this chapter.

In applying the equilibrium conditions, note these three requirements:

(1) *Select a definite well-defined body to which the conditions are to be applied.*
(2) *Be sure to include in the system of forces all the forces acting on the selected body. These will be the forces exerted by all other bodies that touch the selected body, the force of gravity, and any electric or magnetic forces which may act.*
(3) *Be sure not to include in your force system any forces that are acting on bodies other than the selected body. In particular do not include forces exerted by the selected body on other bodies—include only forces exerted by other bodies on the selected body.*

Although in this chapter the 'body' to which we apply the equilibrium conditions will usually be a rigid solid body, it is not necessary that the body be of this nature. In the next chapter we shall apply the equilibrium conditions to a quantity of fluid (liquid or gas). It is only necessary that the 'body' be a well-defined quantity of matter.

4. EQUILIBRIUM OF FORCES ACTING ON A PARTICLE

By *particle* we mean a body of negligible size, so that all forces acting on it can be considered as applied at the same point. Forces applied at the same point are said to be *concurrent*. In the case of a particle, rotational equilibrium does not need to be considered, because all the forces can be considered to have zero lever arm about the center of the particle and hence to exert no torque. One type of 'particle' for which we shall consider the equilibrium of concurrent forces is a knot such as the knot in the strings of Fig. 3 or the knot in the center of a tug-of-war rope.

In the case of the tug of war, if the opposing teams exert equal but oppositely directed forces on their respective ends of the rope, the knot is in equilibrium and remains at rest. The knot would still be in equilibrium if the opposing teams were pulling with equal but opposite forces while riding on a steadily moving railroad flatcar. In this case the knot would not be at rest but would have no tendency to change its state of uniform motion.

Now consider the system of strings in Fig. 3. At the knot there are three strings. The vertical string pulls down on the knot with a force $\boldsymbol{F}_2$ equal to the weight of the suspended sphere.* The other two strings pull on the knot with forces which can be read on the spring balances. If we are given the weight and the angle θ, we have enough information to compute the other two forces from the law of translational equilibrium applied to the knot. The simplest way to express the fact that the resultant force acting on the knot is zero is to resolve $\boldsymbol{F}_1$ into vertical and horizontal component forces $\boldsymbol{F}_{1V}$ and $\boldsymbol{F}_{1H}$ and to say that

$$\left\{\begin{matrix}\text{the sum of the magnitudes of}\\ \text{the component forces acting}\\ \text{to the right}\end{matrix}\right\} = \left\{\begin{matrix}\text{the sum of the magnitudes of}\\ \text{the component forces acting}\\ \text{to the left}\end{matrix}\right\}$$

$$F_3 = F_{1H} \tag{1}$$

---

* It is instructive to consider in detail the arguments involved in drawing the conclusion that the force $\boldsymbol{F}_2$ which the vertical string exerts on the knot has the same magnitude as the weight (say W) of the sphere. By definition, the weight is the force of gravity on the sphere; therefore *gravity pulls down on the sphere with force W.* The sphere is in equilibrium; therefore *the string must pull up on the sphere with force W.* By Newton's third law (action equals reaction), *the sphere pulls down on the string with force W.* But the vertical string is in equilibrium, and so *the knot must pull up on this string with force W.* Finally, again by Newton's third law, *the string pulls down on the knot with force W.* These five steps are involved in the rigorous justification of the statement that $F_2 = W$.

and that

$$\left\{\begin{matrix}\text{the sum of the magnitudes of}\\ \text{the component forces acting}\\ \text{vertically upward}\end{matrix}\right\} = \left\{\begin{matrix}\text{the sum of the magnitudes of}\\ \text{the component forces acting}\\ \text{vertically downward}\end{matrix}\right\}$$

$$F_{1V} = F_2. \tag{2}$$

Since $F_{1V} = F_1 \cos\theta$, equation (2) becomes $F_1 \cos\theta = F_2$, or

$$F_1 = F_2/\cos\theta. \tag{3}$$

Since $F_{1H} = F_1 \sin\theta$, equation (1) becomes $F_3 = F_1 \sin\theta$. Substitution of (3) in this equation gives

$$F_3 = \frac{F_2}{\cos\theta} \sin\theta = F_2 \tan\theta. \tag{4}$$

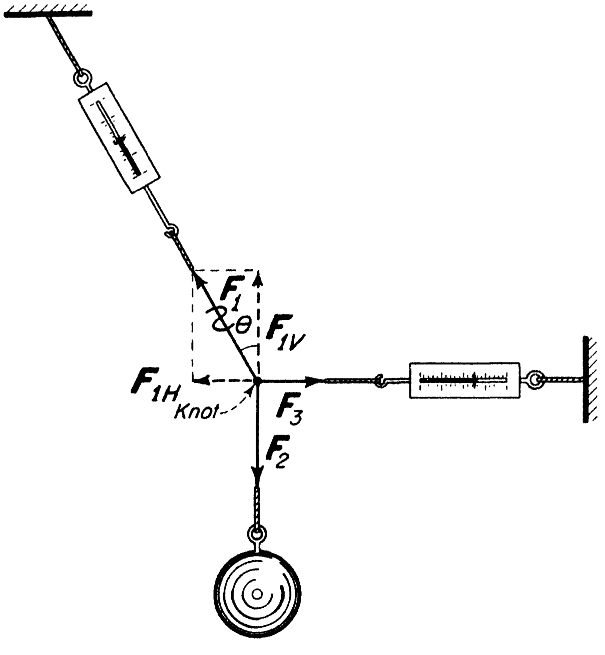


Fig. 3. A knot in equilibrium under the action of three concurrent forces.

Equations (3) and (4) determine F_1 and F_3 in terms of F_2 and θ. Thus, if the weight of the sphere is $W = F_2 = 2.40$ kgf and if the angle $\theta = 30°$, we find, from (3) and (4):

$$F_1 = 2.40/0.866 = 2.77 \text{ kgf},$$

$$F_3 = 2.40 \times 0.577 = 1.38 \text{ kgf}.$$

These values would be found to agree with the spring-balance readings in an actual experiment.

If in Fig. 3 we are given the forces F_2 and F_3 but neither F_1 nor θ, equations (1) and (2) will determine the horizontal and vertical components of F_1, and by the methods of the preceding chapter the magnitude and direction of F_1 can be found when the components are known. Suppose that we are given that

$$F_2 = 2.5 \text{ lbf}, \quad F_3 = 1.5 \text{ lbf},$$

and wish to determine F_1 and θ. From (1) and (2),

$$F_{1H} = F_3 = 1.5 \text{ lbf}, \quad F_{1V} = F_2 = 2.5 \text{ lbf}.$$

Then $\quad \theta = \arctan(F_{1H}/F_{1V}) = \arctan(1.5/2.5) = \arctan 0.6 = 31.^\circ0$

and $\quad F_1 = F_{1H}/\sin\theta = 1.5/0.515 = 2.91 \text{ lbf}.$

In a similar manner, the equations expressing the conditions for equilibrium can be written for any number of strings coming together at a knot or at a small ring, whether or not the strings are coplanar. The procedure is as follows:

(1) Choose three mutually perpendicular axes with the knot as origin; call them the x-, y-, and z-axes.
(2) Resolve all forces into component forces along these three axes.
(3) Equate the sum of the magnitudes of the component forces acting in the $+x$-direction to the sum of the magnitudes of the component forces acting in the $-x$-direction; equate similarly the components in the $+$ and $-y$-directions and in the $+$ and $-z$-directions.

This procedure will result in two equations in the coplanar case, three in the general case. Hence at most two or three unknowns can be determined in these cases. The following problems will serve for practice in carrying out this procedure.

PROBLEMS

1. In Fig. 4, if $W_1 = W_2 = W_3 = 1$ kgf, find θ_1 and θ_2. Ans: $\theta_1 = \theta_2 = 30°$.

2. In Fig. 4, show that if the three weights are equal, the three forces acting on the knot form an equilateral triangle if placed tail to head.

3. In Fig. 4, show that if $W_1 = W_2$, then $\theta_1 = \theta_2$.

4. In Fig. 4, show that if $W_1 > W_2$, then $\theta_1 > \theta_2$.

5. In Fig. 4, if $W_1 = W_2 = 10$ kgf, find θ_1 and θ_2 for the cases $W_3 = 5$ kgf, 10 kgf, 15 kgf, 20 kgf, 25 kgf. Ans: $\theta_1 = \theta_2 = 14.^\circ5$; 30°; $48.^\circ6$; 90°; no equilibrium.

6. In Fig. 4, if $W_1 = W_2$, find θ_1 and θ_2 for the cases $W_3 = \frac{1}{4}W_1$, $W_3 = \frac{3}{4}W_1$, $W_3 = \frac{5}{4}W_1$, $W_3 = \frac{7}{4}W_1$, $W_3 = \frac{9}{4}W_1$.

7. In Fig. 4, if $W_3 = 20$ lbf and $W_1 = W_2$ so $\theta_1 = \theta_2$, find the value of W_1 required to make $\theta_1 = 10°$; to make $\theta_1 = 5°$. Ans: 57.6 lbf; 115 lbf.

8. In Prob. 7, find the value of W_1 required to make $\theta_1 = 1°$.

9. In Fig. 4, if $W_2 = 2$ lbf, $W_3 = 3$ lbf, and $\theta_2 = 30°$, find W_1 and θ_1.
Ans: 2.65 lbf, 49°.1.

10. In Fig. 4, if $W_1 = 4$ lbf, $W_3 = 3$ lbf, and $\theta_1 = 45°$, find W_2 and θ_2.

11. In Fig. 4, if $W_3 = 5$ lbf, $\theta_1 = 45°$, and $\theta_2 = 30°$, find W_1 and W_2.
Ans: 4.48 lbf, 3.66 lbf.

12. In Fig. 4, if $W_3 = 5$ lbf, $\theta_1 = 60°$, and $\theta_2 = 30°$, find W_1 and W_2.

13. In Fig. 4, if $W_1 = 2$ kgf, $W_2 = 1$ kgf, and $W_3 = 2$ kgf, find θ_1 and θ_2.
Ans: 61°.1, 14°.5.

14. In Fig. 4, if W_1, W_2, and W_3 are given, show that

$$\sin\theta_1 = \frac{W_1^2 - W_2^2 + W_3^2}{2W_1W_3} \quad \text{and} \quad \sin\theta_2 = \frac{W_2^2 - W_1^2 + W_3^2}{2W_2W_3}.$$

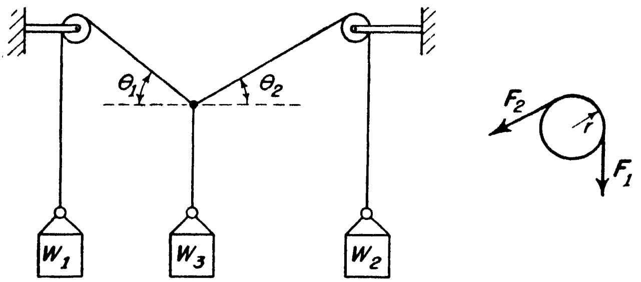


Fig. 4. The pulleys in this diagram are assumed to have frictionless bearings. When such a pulley is in equilibrium the tension in the string does not change as it passes over the pulley. To prove this, consider the little diagram at the right. One part of the string exerts clockwise torque F_1r on the pulley; the other part exerts counterclockwise torque F_2r. Since there is no frictional torque in the bearings, and the pulley is in equilibrium, these torques must balance, so $F_1 = F_2$. Hence in this setup, the forces exerted by the three strings on the knot have magnitudes W_1, W_2, W_3.

15. A weight of 150 lbf is supported by four ropes of equal length fastened at the four corners of the ceiling of a room 20 ft square. If the point of attachment of the ropes to the ring on top of the weight is 10 ft from the ceiling, find the tension in each rope. Ans: 65.0 lbf.

16. Repeat Prob. 15 for a room 10 ft × 20 ft in size.

17. A 1-ton block is suspended from a hook in the ceiling by a rope 15 ft long. Two other ropes are attached to the ring on the top of the block and are pulled by two men. One man keeps his rope horizontal and pointing due north; the other keeps his rope horizontal and pointing due east. They pull so that the block is 1 ft east and 2 ft north of the hook in the ceiling. Find the tension in each of the three ropes. Ans: 2022 lbf; 135 lbf (east); 270 lbf (north).

18. The ends of a rope 20 ft long are fastened at two adjacent corners of the ceiling of a room, the corners being 16 ft apart. At the mid-point of the rope is suspended a 100-lbf weight. Also at the mid-point of the rope is fastened another rope which is pulled horizontally in a direction normal to the wall to pull the weight 3 ft away from the wall. Find the tension in each rope.

5. EQUILIBRIUM OF COPLANAR FORCES ACTING ON A BODY WHOSE WEIGHT CAN BE NEGLECTED

Although weight is an ever-present force, in some cases the weight of the body may be very small compared to the other forces that act on the body and hence negligible to a certain approximation. Such is the case in Fig. 5, where a light, stiff meter stick suspended by vertical strings at its ends carries a heavy weight of several kgf exerting a force **F** at the 80-cm mark. It will be desirable, before considering the inclusion of the weight of the body as one of the forces, to become acquainted with the treatment of such cases.

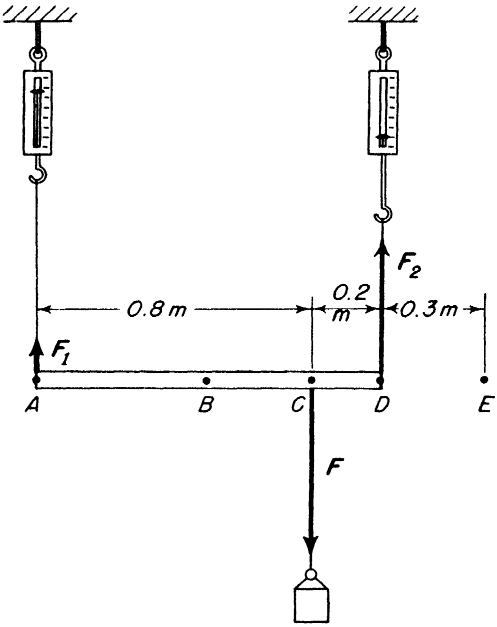


Fig. 5. A light meter stick supporting a heavy weight.

Considering the meter stick of Fig. 5 as the body for which we are going to write the equilibrium conditions, we see that there are no horizontal force components. Hence the condition for translational equilibrium reduces to

$$\text{sum of upward forces} = \text{sum of downward forces}$$

$$F_1 + F_2 = F. \qquad (5)$$

The force F is assumed known, and we want to find F_1 and F_2. Equation (5) determines the sum of F_1 and F_2, but we need another equation to determine these two forces individually. The second equation is determined by writing the condition for rotational equilibrium about any axis. Which axis we use is immaterial; no rotation about any particular axis implies no rotation about any conceivable axis. We shall verify this assertion by working the problem five times, using axes through A, B, C, D, E successively. Axes A, C, D give zero torque for one of the forces; axis B is at the center of the bar; axis E is not even on the bar, but that does not matter—the bar could rotate about an axis at E and does not. First, using A and expressing torques in kgf·m, we have

$$\left\{\begin{matrix}\text{sum of clockwise}\\ \text{torques about } A\end{matrix}\right\} = \left\{\begin{matrix}\text{sum of counterclockwise}\\ \text{torques about } A\end{matrix}\right\}$$

$$0.8F = 1F_2,$$

whence
$$\left.\begin{aligned} F_2 &= 0.8F,\\ F_1 &= 0.2F.\end{aligned}\right\} \quad (6)$$
and from (5),

The string at the end closer to the weight bears the larger share of the weight.

The same result can be obtained by using the condition (5) for translational equilibrium and the rotational condition for any other particular axis. For example, if we use B:

$$\left\{\begin{matrix}\text{sum of clockwise}\\ \text{torques about } B\end{matrix}\right\} = \left\{\begin{matrix}\text{sum of counterclockwise}\\ \text{torques about } B\end{matrix}\right\}$$

$$0.5F_1+0.3F=0.5F_2.$$

Substitution of $F_2=F-F_1$, from (5), gives

$$0.5F_1+0.3F=0.5F-0.5F_1,$$

or

$$F_1=0.2F,$$

$$F_2=0.8F,$$

as before.

Using axis C, we would get

$$\left\{\begin{matrix}\text{sum of clockwise}\\ \text{torques about } C\end{matrix}\right\} = \left\{\begin{matrix}\text{sum of counterclockwise}\\ \text{torques about } C\end{matrix}\right\}$$

$$0.8F_1=0.2F_2.$$

Substitution of $F_2=F-F_1$ gives

$$0.8F_1=0.2F-0.2F_1,$$

or

$$F_1=0.2F,$$

$$F_2=0.8F.$$

Using axis D, we would get

$$\left\{\begin{matrix}\text{sum of clockwise}\\ \text{torques about } D\end{matrix}\right\} = \left\{\begin{matrix}\text{sum of counterclockwise}\\ \text{torques about } D\end{matrix}\right\}$$

$$1F_1=0.2F,$$

and from (5),

$$F_2=0.8F.$$

Finally, if we had used axis E, the work would have gone as follows:

$$\left\{\begin{matrix}\text{sum of clockwise}\\ \text{torques about } E\end{matrix}\right\}=\left\{\begin{matrix}\text{sum of counterclockwise}\\ \text{torques about } E\end{matrix}\right\}$$

$$1.3F_1+0.3F_2=0.5F.$$

Substitution of $F_2=F-F_1$ gives

$$1.3F_1+0.3F-0.3F_1=0.5F,$$

or

$$F_1=0.2F,$$

$$F_2=0.8F,$$

as before.

We have been perhaps excessively repetitious in our effort to emphasize the fact, common to all problems of equilibrium of coplanar forces, that *it does not matter which choice of axis is used in writing the condition for rotational equilibrium, and no new information is obtained by writing this condition for more than one axis.* We also note that the simplest equation is obtained if the axis is chosen along the line of action of one of the *unknown* forces, because then this unknown force, having zero lever arm, is absent from the equation for rotational equilibrium. Such is the case for choices A and D for the axis in the above example.

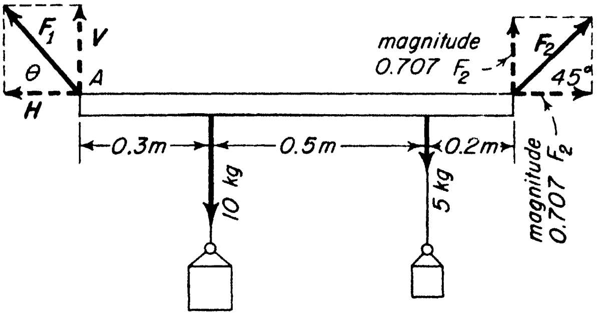


Fig. 6. A meter stick whose weight can be neglected, suspended from two strings and carrying two heavy weights.

In the more general case of coplanar forces, where the forces have horizontal as well as vertical components, the equilibrium conditions give three equations which can be solved for three unknowns. In the case of the horizontal meter stick of Fig. 6 suspended by two strings and carrying two heavy known weights, we can solve for the unknown tensions in the two strings and the unknown angle θ of the left-hand string if we know the angle of the right-hand string. We should be unable to solve for more than three unknowns because we cannot write more than three equations —namely, those for horizontal and vertical translational equilibrium and that for rotational equilibrium. We replace $\boldsymbol{F}_2$ by the two component forces of magnitudes $F_2 \sin 45° = 0.707F_2$ and $F_2 \cos 45° = 0.707F_2$. We replace $\boldsymbol{F}_1$ by its horizontal and vertical component forces. Since both the magnitude and the direction of $\boldsymbol{F}_1$ are unknown, it is simplest to give new symbols to these component forces, calling them $\boldsymbol{H}$ and $\boldsymbol{V}$. After the magnitudes H and V are determined, the values of F_1 and θ can be easily found trigonometrically.

The conditions for translational equilibrium of the bar of Fig. 6 then give the equations

$$\left\{\begin{matrix}\text{sum of magnitudes of forces}\\ \text{directed to the left}\end{matrix}\right\} = \left\{\begin{matrix}\text{sum of magnitudes of forces}\\ \text{directed to the right}\end{matrix}\right\}$$

$$H = 0.707F_2; \tag{7}$$

$$\left\{\begin{matrix}\text{sum of magnitudes of forces}\\ \text{directed upward}\end{matrix}\right\} = \left\{\begin{matrix}\text{sum of magnitudes of forces}\\ \text{directed downward}\end{matrix}\right\}$$

$$V + 0.707F_2 = 10\ \text{kgf} + 5\ \text{kgf}. \tag{8}$$

These are two equations in three unknowns, H, V, F_2. We need one further equation, to be obtained from the condition for rotational equilibrium. The best axis to use is one passing through point A, the point of application of $\boldsymbol{F}_1$, since neither $\boldsymbol{H}$ nor $\boldsymbol{V}$ exerts a torque about this axis and our equation will have just one unknown, F_2. Equating moments about A gives

$$\left\{\begin{matrix}\text{sum of counterclockwise}\\ \text{moments about } A\end{matrix}\right\} = \left\{\begin{matrix}\text{sum of clockwise}\\ \text{moments about } A\end{matrix}\right\}$$

$$(1\ \text{m})(0.707\ F_2) = (0.8\ \text{m})(5\ \text{kgf}) + (0.3\ \text{m})(10\ \text{kgf}). \tag{9}$$

The moment on the left of the above equation is that of the vertical component of $\boldsymbol{F}_2$; the horizontal component has zero lever arm. This equation gives

$$0.707F_2 = 7\ \text{kgf},$$

or

$$F_2 = 9.90\ \text{kgf}.$$

Substitution of this result in (7) gives

$$H = 7\ \text{kgf},$$

and from (8),

$$V = 8\ \text{kgf}.$$

We then find

$$\theta = \arctan 8/7 = 48^{\circ}.8,$$

$$F_1 = 8/\sin\theta = 10.63\ \text{kgf}.$$

The student should try other axes in this example, verifying that he obtains the same results as above. Verification that the torques about another axis balance out can be used as a good check on the accuracy of the solution.

PROBLEMS

1. In Fig. 7, if the weight of the bar is negligible, $F_1 = 12$ lbf, $F_2 = 15$ lbf, and $l_1 = 5$ ft, find l_2 and the force exerted by the fulcrum. Ans: 4 ft; 27 lbf.

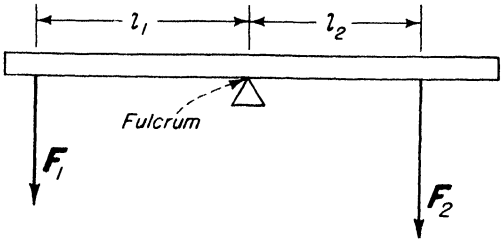


Fig. 7.

2. In Fig. 7, if the weight of the bar is negligible, $F_1 = 6$ kgf, $l_1 = 60$ cm, and $l_2 = 50$ cm, find F_2 and the force exerted by the fulcrum.

3. In Fig. 8, a weight of $F_1 = 9$ kgf hangs at a point 20 cm from the left end of a meter stick of negligible weight, and a weight of $F_2 = 16$ kgf hangs 80 cm from the left end. At what distance x must a string be tied in order that the meter stick will 'balance' when lifted by the string? Ans: 58.4 cm.

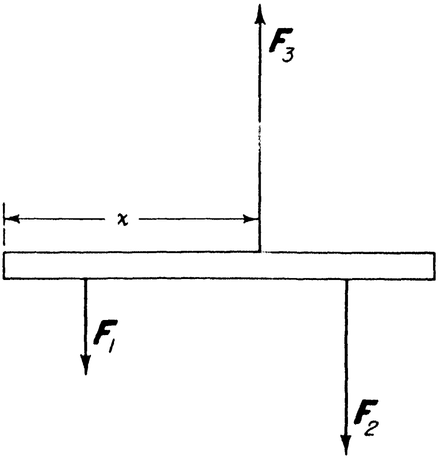


Fig. 8.

4. In Fig. 8, a weight of $F_1 = 10$ lbf hangs at a point 8 in from the left end of a yardstick of negligible weight, and a weight of $F_2 = 15$ lbf hangs 8 in from the right end. At what distance x must a string be tied in order that the yardstick will 'balance' when lifted by the string?

5. In Fig. 9, a rigid stick 160 cm long, of negligible weight, carries weights $F_2 = 5$ kgf, $F_3 = 5$ kgf, and $F_4 = 3$ kgf at points 40 cm, 105 cm, and 120 cm from the

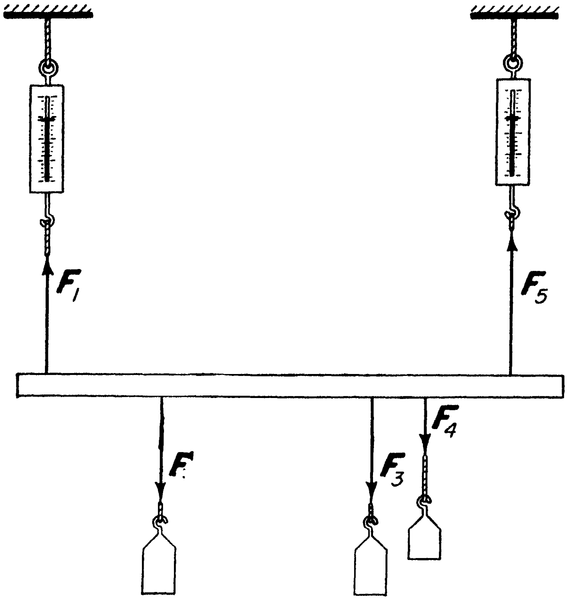


Fig. 9.

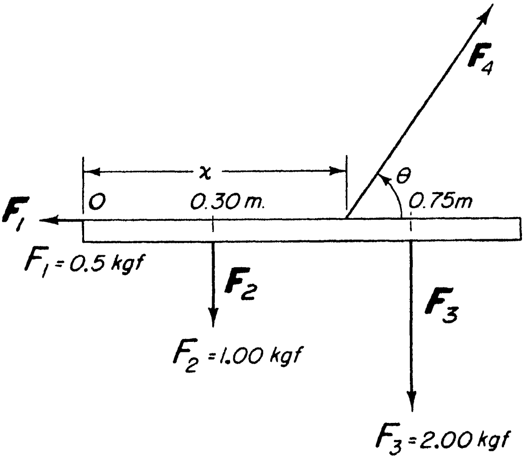


Fig. 10.

left end. It is supported by vertical strings 10 cm from the left end and 15 cm from the right end. Find F_1 and F_5. Ans: 5.93, 7.07 kgf.

6. In Fig. 9, a rigid stick 50 in long carries weights $F_2 = 9$ lbf, 13 in from the left end, $F_3 = 11$ lbf, 32 in from the left end, and an unknown weight F_4 at an unknown position. The force $F_1 = 11$ lbf is applied 3 in from the left end, and the force $F_5 = 14$ lbf is applied 5 in from the right end. Determine the magnitude of F_4 and its position of application, for equilibrium.

7. In Fig. 10, forces of 1.00 and 2.00 kgf are applied at the 30-cm and 75-cm points of a meter stick of negligible weight. A force $\boldsymbol{F}_1$ of 0.5 kgf is applied horizontally at the left end and a force $\boldsymbol{F}_4$ at angle θ at a point x meters from the left end. Find F_4, θ, and x, for equilibrium. Ans: 3.05 kgf; 80°5; 0.60 m.

8. Repeat Prob. 7 for $F_1 = 4.00$ kgf.

9. In Fig. 11, if $l_2 = 12$ in, $l_3 = 32$ in, $l_4 = 40$ in, $F_2 = 6$ lbf, $F_3 = 9$ lbf, and $\theta_1 = 60°$, find F_1, F_4, and θ_4, for equilibrium. Ans: 6.93 lbf, 9.64 lbf, 69°0.

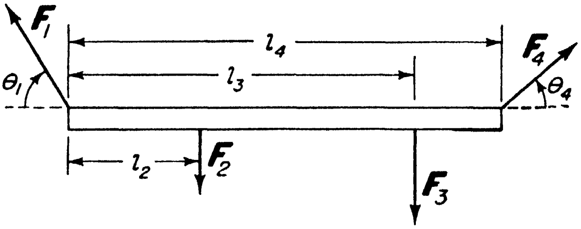


Fig. 11. Four forces act on a stick of negligible weight.

10. In Fig. 11, if $F_1 = 6$ kgf, $\theta_1 = 60°$, $l_2 = 30$ cm, $l_3 = 80$ cm, $l_4 = 100$ cm, and $F_3 =$ 5 kgf, find F_2, F_4, and θ_4, for equilibrium.

6. CENTER OF GRAVITY OF AN OBJECT

An object near the earth's surface experiences an appreciable force of attraction toward the earth because of gravitation. This force of attraction is called the *weight* $\boldsymbol{W}$ of the object. If the object is very small and can be regarded as a particle, the weight is simply a single force exerted vertically downward. If the object is extended, it can be regarded as made up of a large number of connected particles; each of the constituent particles of such a body experiences a downward gravitational force, and the entire object is subjected to a set of parallel downward forces.

It is an easily demonstrated fact that an object suspended by a single string above the surface of the earth finally assumes a definite orientation in which it is in equilibrium. When the point of attachment of the string to the body is changed and it is again suspended above the earth, the body assumes another definite equilibrium orientation. If the extended vertical line of the supporting string is marked in the body when it is in equilibrium for each of several points of attachment, all the lines so obtained intersect in a common point called the *center of gravity* of the body. This point is particularly easy to find for a thin flat piece of material such as a piece of cardboard.

The center of gravity of an object is a point fixed relative to the object, but it does not necessarily lie inside the material of the object. In the case of a carpenter's square or an automobile tire, the center of gravity lies outside the material of the body. The center of gravity of a uniform regularly shaped object such as a sphere or a meter stick lies at its geometrical center, whereas the center of gravity of a tapered pole is a point on the symmetry axis between the ends but closer to the thicker than to the thinner end.

When an object is suspended by a single string, of course the string hangs vertically. This familiar fact follows from the conditions of equilibrium if we remember that the forces of gravity on all the particles of the body act vertically downward and have no horizontal components. Hence, in equilibrium the force exerted by the string must be vertically upward and can have no horizontal component, since there would be no other horizontal component force to balance such a component.

The position $\bar{x}$ of the center of gravity of the tapered rod in Fig. 12 is that point at which the rod will be in equilibrium under the action of the single upward force F and the downward forces of gravity on the particles of the rod. We consider the rod divided into a large number of portions of weights w_1, w_2, w_3, $\cdots$ centered at distances x_1, x_2, x_3, $\cdots$ from the end of the rod, as in Fig. 12. The larger the number and the smaller the sizes of the portions into which the rod is divided, the more accurately does such a force system represent the action of gravity.

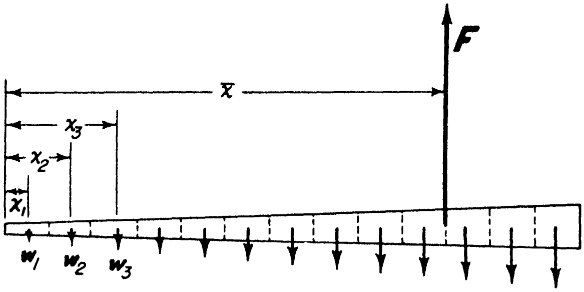


Fig. 12. A single force F in equilibrium with the system of forces w_1, w_2, w_3, $\cdots$.

Now apply the equilibrium conditions to the rod of Fig. 12. Balancing the magnitudes of the vertical forces, we find that

$$F = w_1 + w_2 + w_3 + \cdots = W,$$

where we denote the whole weight of the rod by W. The condition of rotational equilibrium about an axis through the left end of the rod gives the equation

$$F\bar{x} = w_1x_1 + w_2x_2 + w_3x_3 + \cdots,$$

or, since $F = W$, we have

$$W\bar{x} = w_1x_1 + w_2x_2 + w_3x_3 + \cdots. \qquad (10)$$

This extremely important equation shows that *the torque of the forces of gravity on the various parts of the body is the same as the torque of a single* *force equal to the whole weight concentrated at the center of gravity,* as in Fig. 13.

The statement in italics is true for torque about any axis whatsoever. It is proved in (10) only for an axis A through the left end of the rod, but we can readily show that the statement will hold for any other axis whatsoever.

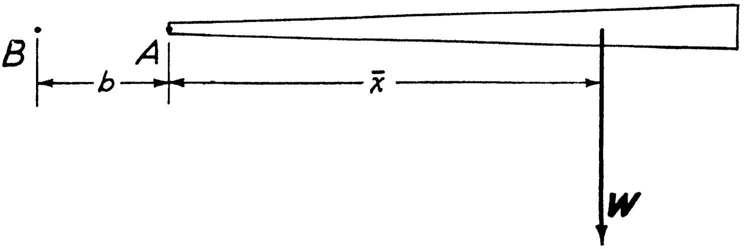


Fig. 13. The whole weight may be concentrated at the center of gravity in computing the torque of gravity.

Consider for example an axis B a distance b further to the left in Fig. 13. Add the constant Wb to both sides of (10) to obtain

$$W\bar{x}+Wb=w_1x_1+w_2x_2+\cdot\cdot\cdot+Wb.$$

On the right side substitute $w_1+w_2+\cdot\cdot\cdot$ for W:

$$W\bar{x}+Wb=w_1x_1+w_2x_2+\cdot\cdot\cdot+w_1b+w_2b+\cdot\cdot\cdot,$$

or

$$W(\bar{x}+b)=w_1(x_1+b)+w_2(x_2+b)+\cdot\cdot\cdot. \tag{11}$$

The expressions in parentheses are the lever arms of $\boldsymbol{W}$, concentrated at the center of gravity, and of $\boldsymbol{w}_1$, $\boldsymbol{w}_2$, $\cdot\cdot\cdot$, about the axis B, therefore (11) proves the statement given above in italics for any axis such as B. The statement may be similarly proved for any axis whatsoever.

These considerations lead to the most useful definition of center of gravity:

> The **center of gravity** of a body is that point at which the whole weight may be considered as concentrated for purposes of computing the torque of the gravitational forces about any axis.

According to (10), the distance $\bar{x}$ of the center of gravity from the left end of the rod of Figs. 12 and 13 is then given by

$$\bar{x}=\frac{w_1x_1+w_2x_2+\cdot\cdot\cdot}{W}. \tag{12}$$

This equation is only an approximation when the rod is divided into a finite number of pieces. The center of gravity is actually the limit of this expression as the number of pieces goes to infinity and the size of each piece goes to zero.

It is not obvious from the above definition that the center of gravity is a unique point independent of the orientation of the object. A proof can readily be given that this is so for the general three-dimensional object, but we shall be content here with proving the uniqueness of the center of gravity for a two-dimensional object such as might be cut out from a sheet of metal or wood by means of a jigsaw. The object of Fig. 14 is an example. In this figure, the x- and y-axes are fixed relative to the object and turn when the object turns.

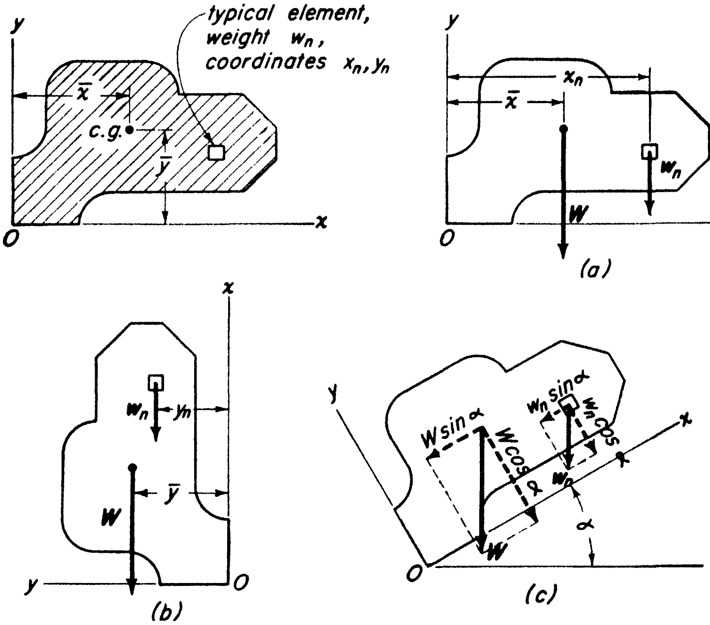


Fig. 14. If the center of gravity of a two-dimensional body is determined from two orientations (a) and (b), the same point will furnish the center of gravity in any other orientation such as (c).

We determine the x-coordinate $\bar{x}$ of the center of gravity by considering the torque of gravity in the orientation of Fig. 14(a). By definition,

$$W\bar{x} = \Sigma\, w_n x_n, \tag{13}$$

where w_n, x_n are the weight and x-coordinate of a typical small element of the body and Σ indicates a summation over all such elements. By considering the body in orientation (b), we see that

$$W\bar{y} = \Sigma\, w_n y_n. \tag{14}$$

The question is whether, in another orientation such as (c), the torque of gravity is correctly given by putting the whole weight at the point $\bar{x}$, $\bar{y}$ determined by (13) and (14). Let us compute the torque of gravity in (c) about the origin O by replacing each force $\boldsymbol{w}_n$ by components parallel to the coordinate axes. It is then seen that the torque of $\boldsymbol{w}_n$ about O is

$$x_n w_n \cos\alpha - y_n w_n \sin\alpha \quad \text{clockwise,}$$

and the whole torque of gravity is

$$\Sigma\, x_n w_n \cos\alpha - \Sigma\, y_n w_n \sin\alpha = \cos\alpha\ \Sigma\, x_n w_n - \sin\alpha\ \Sigma\, y_n w_n \quad \text{clockwise.}$$

But by (13) and (14), this can be written as

$$W\bar{x}\cos\alpha - W\bar{y}\sin\alpha \quad \text{clockwise,}$$

which is exactly the torque of the whole weight $\boldsymbol{W}$ concentrated at the center of gravity $\bar{x}$, $\bar{y}$. This is what we set out to prove.

In the calculus course, the student will learn to compute the position of the center of gravity of objects of various shapes by integration. Here we merely point out that many objects have sufficient symmetry so that there is an obvious geometrical center at which the center of gravity lies. This statement applies to spheres, cubes, rectangular parallelepipeds, uniform rods, plane rectangles, hoops, balanced wheels of all types, and so on.

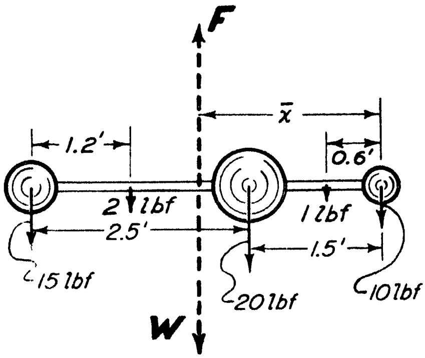


Fig. 15. Determination of the center of gravity of a composite object.

The center of gravity of a *composite* body can be readily determined if the center of gravity of each of its parts is known. Frequently an object can be artificially divided up into pieces of regular shape whose centers of gravity are known, and then treated as a composite body. We shall give two examples.

First, consider the object of Fig. 15, which consists of 3 spheres connected by rods, all arranged in a straight line. The center of gravity of each sphere and rod is at its geometrical center, so for purposes of computing torques its whole weight can be concentrated at this point as indicated in Fig. 15. We wish to determine the distance $\bar{x}$ of the center of gravity of the whole object from the center of the right-hand sphere. This is by definition the point where the whole weight $\boldsymbol{W}$ could be considered as concentrated for purposes of computing the torque of gravity, or alternatively, the point where a single force $\boldsymbol{F}$, equal and opposite to ***W***, would be in equilibrium with the forces of gravity. The total counterclockwise torque of gravity about the center of the right-hand sphere is given in lbf·ft by

$$\begin{array}{rl} 15 \cdot 4.0 & \text{(left-hand sphere)} \\ +\ 2 \cdot 2.8 & \text{(left-hand rod)} \\ +20 \cdot 1.5 & \text{(center sphere)} \\ +\ 1 \cdot 0.6 & \text{(right-hand rod)} \\ +10 \cdot 0 & \text{(right-hand sphere)} \\ & =96.2 \text{ lbf·ft.} \end{array}$$

When this torque is equated to $W\bar{x}$, where $W = 48$ lbf is the total weight, we find

$$\bar{x} = (96.2 \text{ lbf·ft})/(48 \text{ lbf}) = 2.00 \text{ ft}.$$

As another example of a composite body, consider a uniform sheet of metal or plywood of the shape shown in Fig. 16. Taking x- and y-axes with origin at the point O, we can divide up the sheet as shown by broken lines into a square of area 4 in^2 with c.g. at $x = -1$ in, $y = 1$ in; a rectangle of area 10 in^2 with c.g. at $x = 3$ in, $y = 2.5$ in; a rectangle of area 7 in^2 with c.g. at $x = 4.5$ in, $y = -1.5$ in; and a rectangle of area 24 in^2 with c.g. at $x = 1$ in, $y = -6.5$ in.

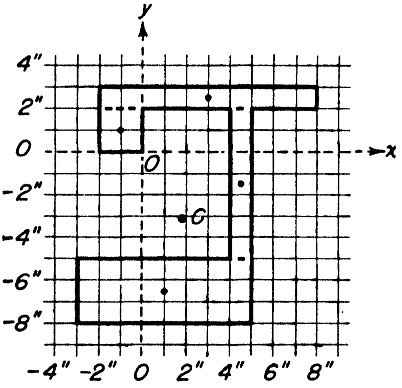


Fig. 16. Determination of the center of gravity of a piece cut from a uniform sheet of material.

We can determine the center of gravity by substituting into (13) and (14). Since the sheet is uniform, the weight is proportional to the area in square inches, so we can use as one unit of weight the weight of 1 in^2. The whole weight W is then 45 units. From (13) and (14):

$$45\bar{x} = 4(-1) + 10(3) + 7(4.5) + 24(1) = 81.5,$$
$$45\bar{y} = 4(1) + 10(2.5) + 7(-1.5) + 24(-6.5) = -137.5.$$

Hence, $\bar{x} = 1.81$ in, $\bar{y} = -3.06$ in.

These are the coordinates of the center of gravity, marked C in Fig. 16.

PROBLEMS

1. A bar of iron 1 cm square and 10 cm long weighing 79 gf and a bar of aluminum 1 cm square and 10 cm long weighing 27 gf are joined end to end. Find the distance of the center of gravity from the center of the resulting 20-cm bar.

Ans: 2.45 cm.

2. A round steel bar 1 inch in diameter and 3 ft long is put into a lathe and a 1-ft length at one end is turned down to $\frac{5}{8}$-in diameter. Find the position of the center of gravity of the resulting shape.

3. On a uniform plank 10 ft long weighing 40 lbf is placed a concentrated weight of 20 lbf, at 1 ft from one end, and another concentrated weight of 15 lbf, at 3 ft from this same end. Find the distance of the center of gravity of plank plus weights from this same end. Ans: 3.53 ft.

4. A steel rod 2 meters long weighing 3.5 kgf carries a collar weighing 1.5 kgf centered at a point 10 cm from one end and another collar weighing 0.5 kgf centered at a point 80 cm from the other end. Find the center of gravity of the rod with its two collars.

5. A piece of plywood 3 ft × 6 ft weighing 10 lbf is lying horizontally. A brick weighing 4 lbf is placed near one corner of the piece with its center 1 ft in from each edge. At what point should a vertical cord be attached to the plywood sheet so that it would 'balance' in a horizontal position when lifted by this single cord? Ans: 2.43 ft from short edge, 1.36 ft from long edge (on line connecting center of brick with center of sheet).

6. A circular piece of plywood 1 m in diameter weighing 5 kgf is lying horizontally. A brick weighing 1 kgf is placed on the piece with its center 50 cm due north of the center of the plywood circle; another brick weighing 2 kgf is placed with its center 40 cm due southeast of the center of the circle. At what point should a vertical cord be attached to the plywood sheet so that it would 'balance' in a horizontal position when lifted by this single cord?

7. Find the center of gravity of a circular sheet of metal 6 ft in diameter which has a 2-ft square cut out of it, two sides of the square lying along diameters of the circle. Ans: 0.233 ft from center of circle—in what direction?

8. Find the center of gravity of a 2-m square sheet of metal from which a circle 80 cm in diameter has been cut, the center of the circle being along a diagonal of the square 70 cm from one corner of the square.

9. A uniform bar 8 ft long is bent so that a 3-ft arm makes an angle of 45° with a 5-ft arm. If the bent bar is hooked over a horizontal wire, what angle with the vertical will the long arm assume? Ans: 16°6.

10. A uniform bar 8 ft long is bent at right angles at a point 3 ft from one end. If the bent bar is hung over a horizontal wire, what angle with the vertical will the long arm assume?

11. Find the center of gravity of the shape shown in Fig. 16 by dividing it up into pieces in a manner different from that used in the text; namely, beginning at the upper left, into a piece 2 in × 3 in lying to the left of the y-axis; a piece 1 in × 8 in lying to the right of the y-axis; a piece 1 in × 10 in lying between $x = 4$ in and $x = 5$ in, $y = 2$ in and $y = -8$ in; and a piece 3 in × 7 in at the lower left.

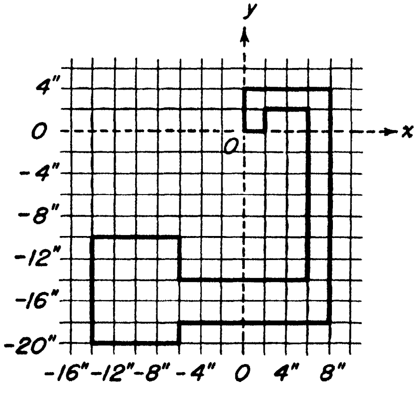


Fig. 17.

12. Find the center of gravity of the shape shown in Fig. 17, which is cut out of a uniform plate of metal.

13. If the shape of Fig. 16 is 'hooked' over the edge of a table top so that the point O rests on the table, what will be the angle between the y-axis drawn on the figure and the vertical? Ans: 30°6.

14. If the shape of Fig. 17 is 'hooked' over the edge of a table with the top part of the figure above the table and the lower part under the table, in what attitude will the shape balance?

15. Fig. 18(a) shows three thin uniform rods, all cut from the same bar and joined at one end so they are mutually perpendicular. The rod lying along the x-axis is 4 ft long; that along the y-axis is 2 ft long; that along the z-axis is 3 ft long.

(a) Find the coordinates of the center of gravity of this object.

(b) If a string is tied around the object at the point where the rods join (the origin) and the object is lifted by this string, what angle will each of the rods make with the vertical? Ans: (a) $\bar{x}=0.889$ ft, $\bar{y}=0.222$ ft, $\bar{z}=0.500$ ft; (b) 31°.7, 77°.7, 61°.4.

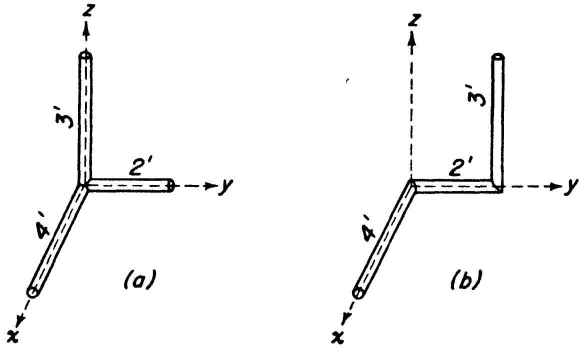


Fig. 18.

16. The three rods of Fig. 18(b) are all cut from the same bar and joined as shown. The rod along the x-axis is 4 ft long, the rod along the y-axis is 2 ft long, and the rod parallel to the z-axis is 3 ft long.

(a) Find the coordinates of the center of gravity of this object.

(b) If the object is suspended by a string tied at the point which is the origin in Fig. 18(b), what angle will each of the rods make with the vertical?

7. EQUILIBRIUM OF FORCES ACTING IN A COMMON VERTICAL PLANE

We are now prepared to include the gravitational pull on the body among the forces considered in connection with the equilibrium of a body. A downward force equal to the whole weight of the body, acting at the center of gravity, represents correctly the effect of the force of gravity in regard to both translational and rotational motion, since we have seen in the previous section that the torque of such a force about any axis equals the resultant torque of the forces of gravity about that axis. The effect of gravity is represented by a vertical force through the center of gravity. If all the other forces acting on the body lie in a common vertical plane passing through the center of gravity, we are back to the case of equilibrium of coplanar forces which was considered in Sec. 5.

The problems below will serve as examples of this case. In Fig. 19, the 800-lbf weight of the uniform bar can be considered as concentrated at its center. In Fig. 20, the center of gravity of the bar lies directly over the axis. The weight of the bar is represented by a downward force **W**. The force **F** exerted by the axis is vertical because no horizontal component is needed to balance any other horizontal forces. The magnitude F is given by $W+F_1+F_3-F_2$. In Figs. 21 and 22, the force **R** of the axis must have both horizontal and vertical components because other forces acting on the body have both types of components. The same comment applies to the force of the ground on the ladder of Fig. 23.

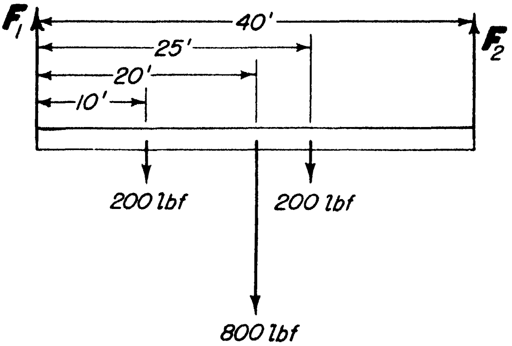


Fig. 19.

PROBLEMS

1. In Fig. 19, a uniform bar weighing 800 lbf carries two concentrated loads of 200 lbf each in the positions shown. The bar is supported at the ends. Find the magnitudes of the supporting forces F_1 and F_2. Ans: 625 lbf; 575 lbf.

2. A horizontal nonuniform bar 3 m long is supported by vertical cords at its ends which are attached to spring balances. If the balances read 4.6 kgf and 3.2 kgf, what is the weight of the bar and where is its center of gravity?

3. When just the two front wheels of an empty four-wheel truck are placed on a platform scale, the scale reads 3500 lbf. When just the two rear wheels are placed on the scale, the reading is 2500 lbf. The wheelbase is 14 ft. What is the weight of the empty truck, and how far ahead of the rear axle is its center of gravity? Ans: 6000 lbf; 8.17 ft.

4. When the truck of Prob. 3 is loaded, the weight on the front wheels is 5500 lbf and the weight on the rear wheels is 8500 lbf. What is the weight *of the load* and where is the center of gravity *of the load?*

5. An acquaintance of one of the authors hauls his small two-wheel automobile trailer to a country coal yard to buy a ton of coal. The dealer instructs him to place the car so that just the trailer wheels are on the platform scale. Then leaving the trailer attached to the car, the dealer reads the weight and loads the trailer with coal until the weight indicated by the scale is one ton-force greater. Under what circumstances is the dealer cheating himself? cheating the purchaser?

6. The person referred to in Prob. 5, who is a physicist, has noted that the center of the pile of coal in his trailer is approximately 1 ft ahead of the trailer axle, whereas the distance from the trailer axle to the ball joint which connects the trailer to the car is 5 ft. Approximately how much coal does he get when he buys a ton? (He has tried in vain to tell the dealer that his method of weighing is unsound.)

7. In Fig. 20 a beam weighing 70 kgf has its c.g. immediately above the axis. It supports a weight of $F_1 = 90$ kgf at a distance $l_1 = 80$ cm from the axis and a weight of $F_3 = 75$ kgf at a distance $l_3 = 110$ cm from the axis. What upward force F_2 at $l_2 = 50$ cm from the axis is necessary for equilibrium, and what is the force $\boldsymbol{F}$ exerted by the axis?

Ans: 21 kgf; 214 kgf.

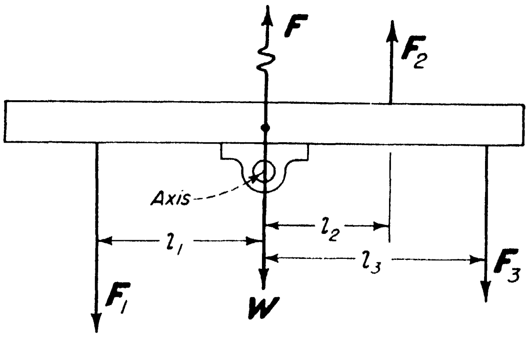


Fig. 20.

8. In Fig. 20, if the beam weighs $W = 80$ lbf and has its c.g. immediately above the axis, $F_2 = 40$ lbf at $l_2 = 6$ ft; $F_1 = 100$ lbf at $l_1 = 8$ ft, and $F_3 = 80$ lbf, find l_3 and the force exerted by the axis when the beam is in equilibrium.

9. In Fig. 21, the beam weighs 100 kgf and has its center of gravity 1.8 m from the axis. It supports a weight of 30 kgf at a distance of 4 m from the axis. If the cable makes an angle of 30° with the horizontal, find the tension in the cable and the magnitude and direction of the force $\boldsymbol{R}$ exerted by the axis on the beam.

Ans: $T = 150$ kgf; $R = 141$ kgf at $\theta = 22.9°$

10. If the beam of Fig. 21(a) weighs 150 lbf, has its center of gravity 12 ft from the axis, and supports a weight of 600 lbf at a distance of 30 ft from the axis, find the tension T in the supporting cable and the magnitude and direction of the reaction $\boldsymbol{R}$ at the support.

11. In the derrick of Fig. 22, if the weight of the boom is $W = 2500$ lbf, the load $F = 800$ lbf, the center of gravity of the boom is 15 ft from its axis of rotation, the point of attachment of the supporting cable is 20 ft from the axis and that of the load is 25 ft from the axis, find the tension T in the cable and the magnitude and direction of the thrust $\boldsymbol{R}$ of the axis.

Ans: $T = 2100$ lbf; $R = 2890$ lbf; $\theta = 51.0°$.

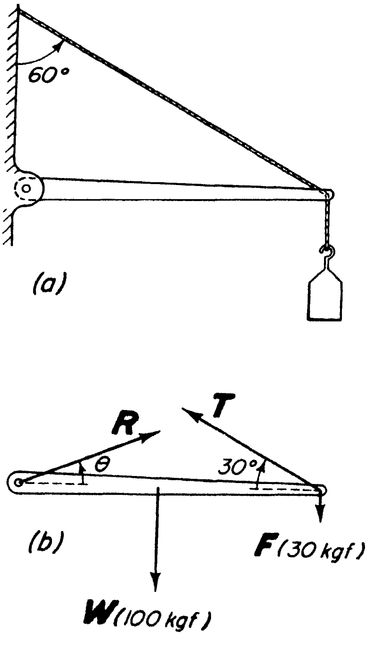


Fig. 21.

12. In Fig. 22, if the boom weighs 400 kgf, the load F is 1200 kgf, the center of gravity of the boom is 5 m from the axis of rotation, the cable is attached 10 m from the axis, and the load is attached 15 m from the axis, find the tension T in the cable and the magnitude and direction of the thrust $\boldsymbol{R}$ of the axis.

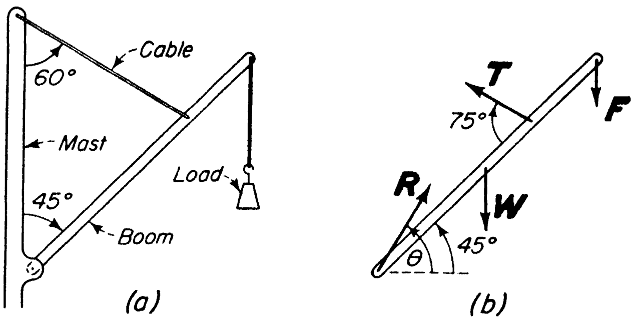


Fig. 22.

13. In Fig. 23, the ladder is 20 ft long, its weight is 100 lbf, and its center of gravity is 7.5 ft from its lower end. Its lower end rests on rough ground and its upper end leans at an angle of 32° against a smooth vertical wall. A man whose weight is 150 lbf stands 12 ft up the ladder and exerts force $\boldsymbol{F}$ on the ladder. The smooth vertical wall can exert only a push $\boldsymbol{P}$ perpendicular to its surface against the top end of the ladder. Find the magnitude P of the push of the wall and the magnitude and direction of the force $\boldsymbol{R}$ of the ground.

Ans: $P = 79.6$ lbf; $R = 262$ lbf; $\theta = 72\overset{\circ}{.}3$.

14. In the ladder problem of Fig. 23 and Prob. 13, find the horizontal component of the ground reaction $\boldsymbol{R}$ as a function of the slant height s (measured along the ladder) to which the 150-lbf man has climbed. In particular, if 90 lbf is the greatest horizontal force which the ground can exert to keep the ladder from slipping, what distance s can the man climb before the ladder starts to slip?

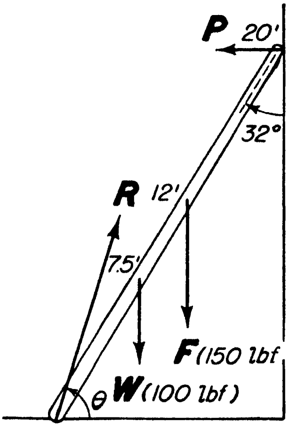


Fig. 23.

8. FRICTION BETWEEN SOLID SURFACES

Everyone is familiar with the friction that is encountered whenever the surface of one solid object moves or merely tends to move along the surface of another solid object with which it is in contact. It is friction that holds a nail in a board and keeps the individual fibers in a thread or rope from falling apart. Friction between the soles of our shoes and the ground or floor makes it possible for us to walk; one readily recalls how difficult it is to walk on an icy street or polished floor where the force of friction on the soles of our shoes is very small. Friction is essential for the propulsion and the braking of an automobile. The reduction of friction in the moving parts of machinery poses many problems in design and lubrication for the engineer.

The force of friction has, like all forces, both direction and magnitude. These two properties of a frictional force are easily observed in simple experiments. The direction of a force of friction on a moving object is always parallel to the surface along which motion occurs and opposite to the direction of motion. This statement is illustrated in Fig. 24, in which a block is shown moving along an inclined board. The force of friction exerted *by the surface on the block* is always directed parallel to the surface and opposite to the direction of motion of the block along the surface. The force of sliding friction always *opposes* the motion of one object along the surface of another object, no matter how the surface is inclined. Friction never causes a motion of this type to start, and it is effective only when some other force either produces or attempts to produce motion.

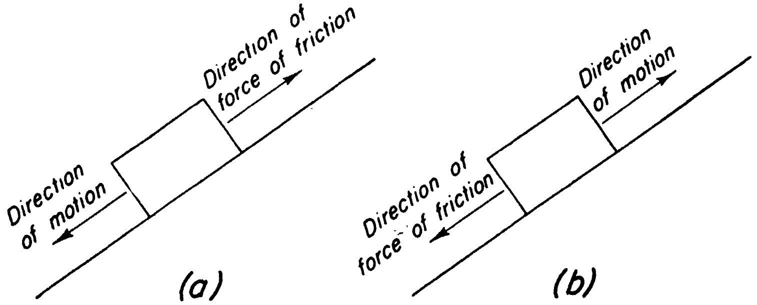

Direction of force of friction
Direction of motion
(a)
Direction of motion
Direction of force of friction
(b)

Fig. 24. The force of friction exerted *by a plane on a block.* This force is in the direction opposite to the motion of the block, as indicated. The force of friction which the *block* exerts *on* the *plane* would be opposite to the force shown here, by Newton's third law.

Two surfaces in contact exert tangential forces of friction on each other, not only when one slides past the other but also when the surfaces are stationary and one *tends* to slide past the other. If a man leans against a heavy table and the table does not move, a force of friction must be exerted by the floor on the table to prevent the motion. A horizontal force of friction must oppose the horizontal force the man is exerting if the table is to remain in equilibrium. Such a force is known as a force of *static friction* and is in such a direction as to oppose the tendency to move and to keep the body in equilibrium. If the man pushes hard enough on the table to make it move, the force of friction exerted by the floor while the table is moving is opposite in direction to the direction of motion of the table and is known as a force of *kinetic friction.*

One can readily demonstrate several important facts about the magnitude of the force of friction with the following simple equipment: a flat wooden board or sheet of some material with uniform surface, a block, a weight pan, a well-oiled pulley, and a set of weights. The apparatus is shown in Fig. 25, where a piece of light string is attached to the center of one end of the block, passes horizontally to the pulley, and supports the weight pan at its other end. The forces acting on the block are the following:

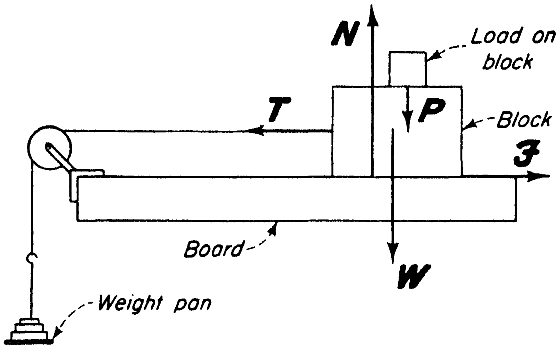


Fig. 25. Experiment to determine the force of friction.

W = weight of the block,
P = downward force exerted by the load on the block,
N = normal* force exerted by the board against the surface of the block,
T = tension in the string, which equals in magnitude the weight of the weight pan and its load,
$\mathfrak{F}$ = force of friction.

The tension T in the string, which equals in magnitude the weight of the weight pan and its load, tends to pull the block forward along the surface of the board, and the force $\mathfrak{F}$ of friction is directed backward parallel to the surface of the board. When the block is in equilibrium, it either remains stationary or moves forward with constant speed. The condition of translational equilibrium requires that

$$N = W + P \quad \text{and} \quad T = \mathfrak{F}.$$

Thus, the force of friction is equal in magnitude to the tension in the string when the block is in equilibrium.

Let us start with the weight pan detached. In this case $\mathfrak{F} = 0$ because $T = 0$. Let us now attach the weight pan and start adding weights, gradually increasing T. Until T reaches a certain magnitude, the block does not move. During this period, the force of static friction is gradually increasing, since it must always equal T. Static friction is very accommodating and always exerts just enough force to keep the block in equilibrium—up to a certain maximum. If T is increased sufficiently, the block will start to move, indicating that T has exceeded the maximum force of static friction. Thus *the force of static friction is a variable quantity*, depending on how much force is needed to keep the block from moving. But there is a definite maximum force of static friction; if the magnitude of the applied force T exceeds the maximum force of static friction, the block will move.

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* The word *normal* is used here with its geometrical meaning, to indicate a direction perpendicular to a plane.

The maximum force of static friction $\mathfrak{F}_s^{\max}$ can be found from the apparatus of Fig. 25 by gradually increasing the load on the weight pan until the block starts to move and noting at what load this occurs. It will be found, by varying the load P on the block, that for a given pair of surfaces $\mathfrak{F}_s^{\max}$ is proportional to the normal force N between the two surfaces. Hence, we write

$$\mathfrak{F}_s^{\max} = \mu_s N, \tag{15}$$

where μ_s is called the *coefficient of static friction.* This coefficient depends on the material and roughness of the surfaces (cf. Table I) but, over a wide range, *is independent of the area of contact between the surfaces.* Thus if the block of Fig. 25 were sawed in two by a vertical cut so that the area of contact with the board were reduced, the board would still exert the same maximum force of static friction if the smaller block were loaded up to give the same normal force $N = W + P$.

It is harder to get the block of Fig. 25 started than it is to keep it moving at constant velocity after it has once started. Once the weight pan has been loaded so that the block starts moving, it will be found that the block moves to the left with increasing speed, indicating that T is greater than $\mathfrak{F}_k$, the force of kinetic friction exerted by the board on the moving block. Once the block has started moving, T can be reduced somewhat and the block will still continue to move, indicating that $\mathfrak{F}_k$ *is less than* $\mathfrak{F}_s^{\max}$. One can determine $\mathfrak{F}_k$ by loading the weight pan to a value somewhat less than $\mathfrak{F}_s^{\max}$, giving the block a push with the finger to start it; if it slows down and comes to rest, it indicates that $T < \mathfrak{F}_k$; if it speeds up, it indicates that $T > \mathfrak{F}_k$. In this way one can adjust the weight till the block moves at constant speed, once it has been started.

By varying the load P on the block, it will be found that for a given pair of surfaces, $\mathfrak{F}_k$ is proportional to the normal force N between the two surfaces. Hence, we can write

$$\mathfrak{F}_k = \mu_k N, \tag{16}$$

where μ_k is called the *coefficient of kinetic friction.* This coefficient depends on the material and roughness of the surfaces (cf. Table I), but over a wide range it is *independent of the area of contact of the surfaces and of the relative velocity of the surfaces.*

TABLE I
TYPICAL VALUES OF COEFFICIENT OF STATIC FRICTION μ_s AND OF KINETIC FRICTION μ_k

Materials	μ_s	μ_k
Steel on steel	0.15	0.09
Steel on ice	0.03	0.01
Hemp rope on wood	0.5	0.4
Leather on wood	0.5	0.4
Oak on oak	0.5	0.3
Wrought iron on cast iron or bronze	0.19	0.18
Rubber tire on dry concrete road	1.0	0.7
Rubber tire on wet concrete road	0.7	0.5

For well-lubricated metal-to-metal surfaces the friction coefficients vary from about $^1/_{16}$ to $^1/_{10}$ the values given above, which are for dry machined surfaces. For automobile tires, both the coefficient of static friction (wheel rolling) and the coefficient of kinetic friction (skidding) are found to decrease with increasing car speed, and to be greater for forces applied transverse to the car than for forces applied in the direction of motion.

Note that the coefficient of static friction μ_s determines the *maximum* force of static friction but that $\mathfrak{F}_s$ can have any magnitude from zero up to this maximum. The force of static friction is $\mathfrak{F}_s^{\max}$ only when the surfaces are just on the point of starting to move relative to each other. On the other hand, μ_k determines *the* force of kinetic friction which exists *whenever* the surfaces are moving relative to each other.

It is found that relations (15) and (16) are valid regardless of the inclination of the surfaces. For example, in Fig. 26 *the plane pushes on the block* with a force which has two components: the component normal to the plane is called the *normal force* or the *normal reaction* of the plane and is denoted by $\boldsymbol{N}$; the component tangent to the plane is called the *force of friction* and is denoted by $\boldsymbol{\mathfrak{F}}$. Relation (15) gives the maximum force of static friction, and (16) gives the force of kinetic friction, with the same values of μ_s and μ_k as if the same block and plane were lying horizontally.

In Fig. 26, a block rests on an inclined plane and is acted on by forces $\boldsymbol{N}$ and $\boldsymbol{\mathfrak{F}}$ exerted by the plane and by force $\boldsymbol{W}$ of gravity. $\boldsymbol{W}$ can be resolved into components parallel and perpendicular to the plane as shown. So long as the block does not move, it is in equilibrium and

$$N = W\cos\theta, \quad \mathfrak{F}_s = W\sin\theta.$$

Thus, $\mathfrak{F}_s$ increases as the inclination θ of the plane increases. The inclination θ cannot exceed a certain value θ^{max} which makes $\mathcal{F}_s = \mathcal{F}_s^{max}$, or else the block will slide. We can determine θ^{max} by noting that the block is just on the point of slipping when $\theta = \theta^{max}$. Then

$$\mathcal{F}_s^{max} = W \sin\theta^{max} = \mu_s N;$$

also

$$W \cos\theta^{max} = N.$$

Dividing the first equation by the second gives

$$\tan\theta^{max} = \mu_s. \tag{17}$$

Thus, the plane can be raised to an angle of inclination whose tangent is μ_s before the block starts to slip. This limiting angle is called the *angle of repose* or the *angle of slip*.

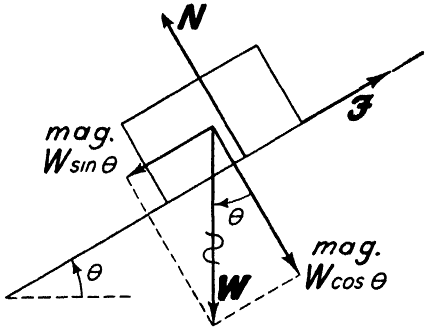


Fig. 26. The maximum value θ can have before the block starts to slip is called the *angle of repose*.

In Figs. 25, 26, and 27 a single vector N is drawn to represent the normal reaction of the plane on the block. However, this reaction is actually distributed over the area of contact in some fashion. In connection with these figures, we can only write the equations for translational equilibrium. We cannot make use of the condition for rotational equilibrium because we do not know the effective line of action of the distributed normal reaction. Rather, the normal reaction will automatically so distribute itself as to guarantee rotational equilibrium, and the equation of rotational equilibrium could be used to determine its effective line of action. In a case such as that of Fig. 29, the normal reaction is reasonably concentrated in a known position and the law of rotational equilibrium gives a useful equation. In case a block is on the point of tipping over, the normal reaction will be concentrated at the edge about which the block will tip, as one can see if one imagines the block to be already tipped by an infinitesimal angle so that it makes contact with the plane only along this edge. Problems 21–27 below are concerned with the matter of tipping over.

The 'laws' of friction that we have discussed above do not apply very well to lubricated surfaces, where there is a continuous film of liquid between the solid surfaces. Consideration of friction between lubricated surfaces is essentially a problem in hydrodynamics that we shall not treat in this text.

PROBLEMS

1. A table weighs 75 lbf. The coefficient of static friction between the table legs and the floor is 0.5. If a man pushes due north on the table with a horizontal force of 25 lbf, what is the force $\mathfrak{F}_s$ of static friction? Ans: 25 lbf south.

2. In Prob. 1, if the man pushes with 20 lbf toward the east, what is $\mathfrak{F}_s$?

3. In Prob. 1, with what horizontal force must the man push to make the table start moving toward the north? Ans: 37.5 lbf north.

4. In Fig. 25, if $W = 2$ kgf, $P = 3$ kgf, $\mu_s = 0.5$, and $\mu_k = 0.4$, what weight is required to make the block start to move? What weight will keep the block moving uniformly after it has started to move?

5. In Fig. 26, if $\mu_s = 0.5$, $\theta = 25°$, $W = 12$ lbf, and the block is at rest, find the magnitude of the force of friction. Ans: 5.07 lbf.

6. In Fig. 26, if $\mu_s = 0.6$, $\theta = 20°$, $W = 6$ kgf, and the block is at rest, find the magnitude of the force of friction.

7. In Fig. 26, if $\mu_s = 0.5$ and $\mu_k = 0.4$, what is the maximum angle θ at which the block will remain at rest on the plane? What is the minimum angle θ at which the block will continue to move down the plane if it is given a start? Ans: 26°6; 21°8.

8. Answer the questions in Prob. 7 for the case in which $\mu_s = 0.6$, $\mu_k = 0.5$.

9. In Fig. 27, if $\mu_s = 0.6$, $W = 50$ kgf, and $\theta = 30°$, find the magnitude of the pull $\boldsymbol{P}$ which is required to overcome static friction and start the block moving. Ans: 25.7 kgf.

10. In Fig. 27, show that the magnitude of the pull $\boldsymbol{P}$, acting at an angle θ above the horizontal, which is required to overcome static friction and start the block moving is given by $P = \mu_s W/(\cos\theta + \mu_s \sin\theta)$.

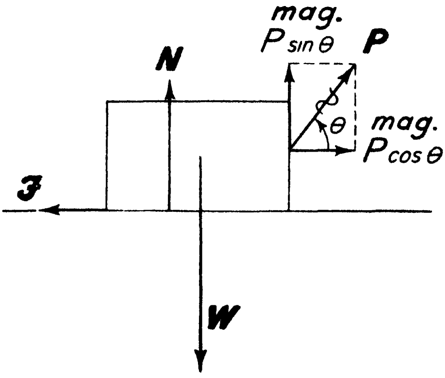


Fig. 27.

11. A block weighing 20 lbf is at rest on a plane inclined at 40° with the horizontal. If $\mu_s = 0.5$, what is the minimum push on the block, applied parallel to the plane, which will keep the block from starting to slide down the plane? What push, applied parallel to the plane, will start the block sliding up the plane? Ans: 5.20 lbf; 20.5 lbf.

12. A block weighing 12 kgf is at rest on a plane inclined at 20° to the horizontal. If $\mu_s = 0.6$, what push, applied parallel to the plane, is required to start the block moving down the plane? to start the block moving up the plane?

13. In Fig. 28(a), if $\theta = 20°$, $\mu_s = 0.6$, and the block weighs 15 kgf, what horizontal push P is required to start the block moving up the plane? Ans: 18.5 lbf.

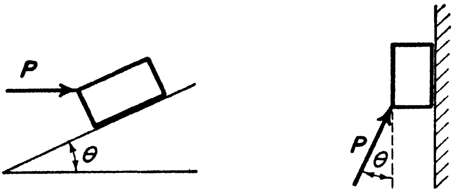
14. In Fig. 28(a), show that the horizontal push P required to start the block of weight W moving up the plane is

$$P = W\frac{\mu_s \cos\theta + \sin\theta}{\cos\theta - \mu_s \sin\theta},$$

and that the normal force exerted by the plane is

$$N = W/(\cos\theta - \mu_s \sin\theta).$$

Show that it is impossible to start a block moving up a plane by a *horizontal* push if the angle of the plane is greater than arccotμ_s.



(a) (b)

Fig. 28.

15. Show that the horizontal push P required in Fig. 28(a) to keep the block from starting to slide down the plane is given by

$$P = W\frac{\sin\theta - \mu_s \cos\theta}{\cos\theta + \mu_s \sin\theta}.$$

Notice that this is positive only when θ is greater than the angle of repose, because only then is a force required to keep the body from sliding down the plane.

16. What *horizontal* push P will hold a 20-lbf block against a vertical wall and prevent it from sliding down if the coefficient of static friction between block and wall is 0.4?

17. In Fig. 28(b), what push P acting at an angle of $\theta = 30°$ is required to prevent a 15-kgf block from sliding down a vertical wall if $\mu_s = 0.44$? What push is required to start the block moving up the wall?
Ans: 13.8 kgf; 23.2 kgf.

18. Show that it is impossible to start the block of Fig. 28(b) moving up the wall by any magnitude of push P if θ is greater than arccotμ_s.

19. Figure 29 represents a ladder 20 ft long weighing 80 lbf, with its center of gravity 7 ft from the bottom end. The base rests on a floor with $\mu_s = 0.6$. The top leans against a smooth wall (no friction). The force $\boldsymbol{F}$ is due to the weight of a 180-lbf man.

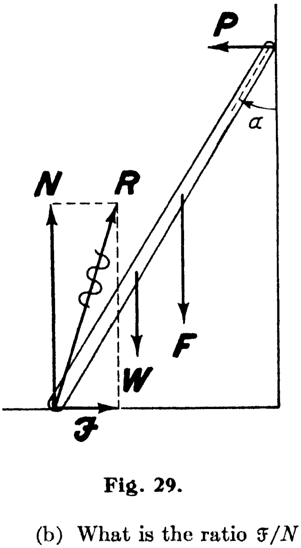


Fig. 29.

(a) What is the greatest angle α which the ladder can have and still permit the man to climb to a rung 19 ft from the base, measured along the ladder?

(b) What is the ratio $\mathfrak{F}/N$ when $\alpha \Rightarrow 20°$ and the man climbs to a rung 19 ft from the base?
Ans: (a) 38°.1; (b) 0.279.

20. If the ladder in Fig. 29 is 20 ft long and weighs 80 lbf, with its center of gravity 7 ft from the base, $\alpha = 30°$, the coefficient of static friction between ladder and floor is 0.3, and the wall is frictionless, how high up the ladder can a 180-lbf man climb before the ladder starts to slip? What would be the minimum coefficient of friction between ladder and floor which would permit the man to climb to a rung 19 ft from the base with the ladder at this angle?

NOTE: In Fig. 30, the force $\boldsymbol{F}$ is applied halfway along the table, which assures sufficient symmetry so that the equilibrium problem can be handled in two dimensions, as shown in the force diagram to the right. N_1 represents the normal force on the two left legs, N_2 the normal force on the two right legs. If μ_s is low, the table will slide when sufficient force F is applied. If μ_s is high, the table will tilt rather than slide. If one assumes that the table will slide and finds that N_1 comes out negative when the table is on the point of sliding, this result indicates that the table would already have tilted with a smaller force F just sufficient to make $N_1 = 0$ and put all the weight on the right legs. If, on the other hand, one assumes that the table will tilt, setting $N_1 = \mathcal{F}_1 = 0$, and finds that $\mathcal{F}_2$ comes out greater than $\mu_s N_2$, this result is an indication that friction is not sufficient and that the table would already have begun to slide at a lower value of F.

21. In Fig. 30, if $W = 60$ lbf, $w = 3$ ft, $h = 2.5$ ft, and $\mu_s = 0.5$, will the table slide or tilt when F is applied? What force F is required to make it move? Find N_1 and N_2. Ans: slide; 30 lbf; 5 lbf, 55 lbf.

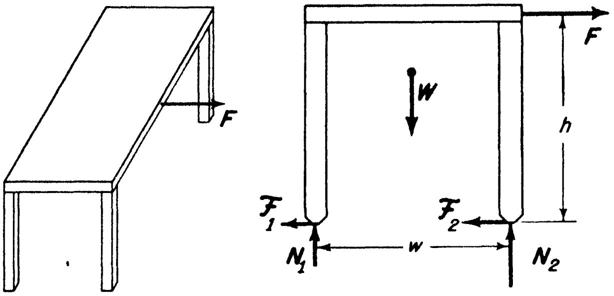


Fig. 30.

22. In Fig. 30, if $W = 60$ lbf, $w = 3$ ft, $h = 2.5$ ft, and $\mu_s = 0.7$, will the table slide or tilt when F is applied? Find the value of F required to make the table move. Find $\mathcal{F}_1$, N_1, $\mathcal{F}_2$, N_2.

23. Show that the table of Fig. 30 will slide or tilt according to whether μ_s is less than or greater than $\tfrac{1}{2}w/h$.

24. In Fig. 30, if $w = 3$ ft, $h = 2$ ft, what coefficient of friction is required to make the table tilt rather than slide? Answer the same question for $w = 2$ ft, $h = 4$ ft.

25. A packing crate is in the form of a cube 5 ft on an edge. It is loaded so its center of gravity is at its geometrical center. It rests on a concrete floor with coefficient of friction $\mu_s = 0.8$. A man pushes horizontally in the middle of one face at a height h from the floor. What is the maximum value of h in order that the cube should slide rather than overturn? Ans: 3.12 ft.

26. Take a straight-backed chair with a heavy back and try pushing horizontally on the back of the chair at various heights from the floor, both in the direction which makes the chair move or tilt backward and in the direction which makes the chair move or tilt forward. Explain your observations.

27. A straight-backed chair on a carpet has 16 in between its front and back feet. A horizontal force applied to the back of the chair at a height of 24 in or more will cause the chair to tilt backward; whereas in order to cause the chair to tilt forward, a horizontal force must be applied at least 36 in from the floor. Determine the coefficient of static friction. Ans: $\mu_s = 0.27$.

28. Part of a chain 8 ft long hangs over the edge of a horizontal table. The coefficient of static friction between chain and table is 0.45. What is the greatest length of chain which can hang over the edge without the whole chain sliding off the table?

9. DISCUSSION OF THE LAWS OF FRICTION*

Perhaps the most surprising experimental fact in regard to friction is that the force of friction between two surfaces, with a given normal force, is independent of the area of contact. This law is true provided that the area of contact is not so small that the normal force per unit area is sufficient to cause deformation of the surfaces. The law does not, for example, apply if one of the bodies is equipped with points which actually dig into the other body.

A simple treatment of an idealized model of the surfaces in contact will serve to make the independence of area seem quite reasonable. Hence it seems desirable to consider such a model.

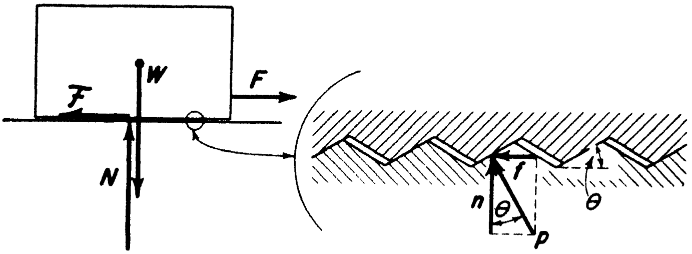


Fig. 31. Force of friction for a highly idealized model of the surfaces.

The friction between surfaces is assumed to be due to visible and microscopic irregularities in the contours of the surfaces. The force of friction arises from the fact that motion requires actual 'lifting' of one of the surfaces against the normal force. We can visualize the situation by considering two surfaces that have regular sawtooth shapes as illustrated in the enlarged section of Fig. 31.

We assume in Fig. 31 that the force F is sufficient so that it has just started the surfaces moving relative to each other. The force of static friction then has its maximum value, and the upper surface can be imagined to be lifted slightly relative to the lower so that the sawteeth make contact only on one side as in the enlarged section.

We assume that the whole friction is due to the sawtooth *shape* of the surfaces; that there is no inherent friction if the surfaces are imagined to be perfectly smooth as in the case of the surfaces in contact in the enlarged section. Then the force p between the surfaces in the enlarged section is normal to these surfaces, but this force makes an angle θ with the plane of the surfaces as defined in the ordinary picture to the left. In other words, if normal and tangent are defined by the picture to the left, p has a component $n = p\cos\theta$ in the normal direction, and a component $f = p\sin\theta$ in the tangential direction. The sum of all the forces n makes up N; the sum of all the forces f makes up $\mathfrak{F}$.

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* This section can be omitted without loss of continuity.

Since for each force, $f/n = \tan\theta$, or $f = n\tan\theta$, this same relation holds for the total forces:

$$\mathfrak{F} = N\tan\theta.$$

But μ_s is defined as $\mathfrak{F}/N$, so that for this model,

$$\mu_s = \tan\theta. \tag{18}$$

Here we have a model that gives a maximum force of static friction which depends on the character of the surfaces (the angle θ of the irregularities) but which is independent of the area of the surfaces, since nowhere in the above discussion have we had occasion to refer to the area of the surfaces. Consideration of this idealized model helps make it seem reasonable that in the real case of surfaces of more random irregularity, the maximum force of static friction should come out independent of contact area.

The above model has one more instructive feature. Clearly, if we have surfaces such as those in the enlarged section of Fig. 31 and do not pull the block, but rather tilt the plane supporting the block, the block will begin to slide when the angle of tilt exceeds θ, the angle of the sawteeth, because then the upper surfaces can slip downhill. But the angle θ of the sawteeth in (18) bears just the same relation to the coefficient of friction as the angle of repose we found in (17). Thus this model accounts in obvious fashion for the magnitude of the angle of repose.

The laws of friction that we have been considering apply to ordinary types of surfaces in which the friction arises from roughness on a scale large compared to atomic dimensions. Highly polished, optically flat surfaces of glass or steel in contact exhibit a different type of friction which arises directly from interatomic forces. Such surfaces, 'wrung' together so that the surface layers of atoms are in intimate proximity, can exhibit large forces of friction even when there is no normal force between the surfaces.

10. GENERAL CASE OF NONCOPLANAR FORCES

In Sec. 4, we considered the equilibrium of noncoplanar forces that acted at the same point, so that rotational considerations were not involved. In the general case of nonconcurrent forces acting on a solid object, we must choose three mutually perpendicular axes (x, y, z), resolve all forces into components parallel to these axes, and write six conditions for equilibrium:

(1) The resultant force in the x-direction vanishes.
(2) The resultant force in the y-direction vanishes.
(3) The resultant force in the z-direction vanishes.
(4) The resultant torque about the x-axis vanishes.
(5) The resultant torque about the y-axis vanishes.
(6) The resultant torque about the z-axis vanishes.

Problems involving noncoplanar forces in static equilibrium do not introduce any new principles. It is considered that detailed exercise in the solution of such problems is beyond the scope of this course and can be left to specialized courses in statics.

CHAPTER 4

STATICS OF FLUIDS

The term *fluid* refers to a substance that does not have a fixed *shape* but that is able to *flow* and take the shape of its container; in other words, to a *liquid* or a *gas*. Although there is no sharp line of demarcation between solids and liquids (witness such materials as gelatin, heavy grease, and cold tar), we shall restrict our attention in this chapter to substances that are obviously liquid because they flow with reasonable rapidity, and to gases. We are concerned here with fluids at rest in static equilibrium. In Chap. 12 we shall discuss fluids in motion.

It is perhaps somewhat surprising to find that the same basic laws govern the static and dynamic behavior of both liquids and gases, in spite of their very different appearance. We shall discuss the distinction between liquids and gases in some detail in Chap. 18. In discussing the mechanical behavior of fluids we need make use only of properties that liquids and gases have in common and that are associated with their ability to flow.

1. FLUID PRESSURE

It is a matter of common experience that fluids exert forces on material solids with which they are in contact; examples are the forces that tend to burst a high-pressure air tank and those that support a floating boat or balloon. The liquid in the vessel in Fig. 1 exerts a force on the bottom which is equal and opposite to the force exerted by the solid bottom on the fluid. That the latter force exists is apparent if we consider what would happen if we took away a piece of the bottom. The liquid would no longer be in static equilibrium but would flow away when the upward force of the bottom was removed.

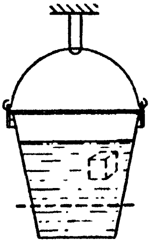

Fig. 1.

Similarly, the liquid *below* the imaginary horizontal plane shown by the dashed line in Fig. 1 must be exerting an upward force on the liquid *above* this plane, because this force is required to hold the latter part of the liquid up; remove this force and the liquid above the broken line would fall to the bottom of the pail. By Newton's third law (p. 36), the fluid above this imaginary plane must be exerting an opposite downward force on the fluid below the plane. Thus we conclude that not only does a fluid exert a force on a solid surface but also that if we 'draw' an imaginary plane in the fluid, the fluid on one side of this plane exerts a force across the plane on the fluid on the other side. The plane need not be horizontal. Consider the small imaginary cube sketched in Fig. 1. The liquid in this cube must exert horizontal forces across the vertical walls of the cube on the surrounding liquid, since such forces are needed to prevent the surrounding liquid from flowing in and occupying the space occupied by the liquid within the cube. The same argument applies to an imaginary cube drawn in a gas such as the atmosphere.

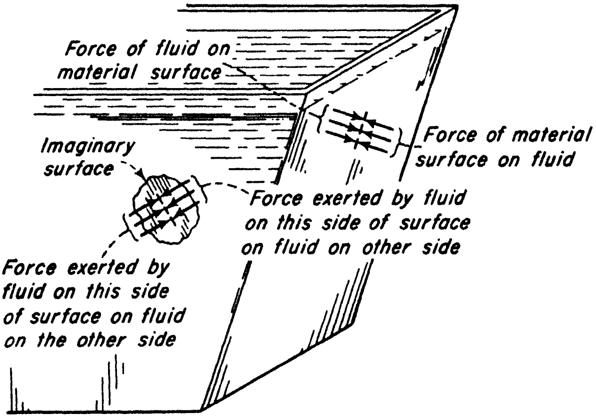
Force of fluid on material surface
Imaginary surface
Force of material surface on fluid
Force exerted by fluid on this side of surface on fluid on other side
Force exerted by fluid on this side of surface on fluid on the other side

Fig. 2.

Thus we see the general necessity for the existence of forces across any imaginary surface in the fluid as well as for forces between the fluid and solid surfaces. Experience completely justifies the assumption that in static equilibrium these forces are *normal* to the surfaces in question and are of the nature of a *push* rather than a *pull*. This statement is illustrated in Fig. 2. Such forces are described by giving the *pressure p* defined as *normal force per unit area*. The pressure is always a positive quantity.*

Our description of the forces across any imaginary surface in a substance in static equilibrium applies to a solid substance (see Chap. 9, Elasticity and Strength of Materials) as well as to a fluid, *up to the point* where we assert that the force is normal to the surface and always compressive. We readily see that in a solid there must be both tangential forces and tensile forces. For example, in Fig. 3, which shows a stick protruding from the edge of a table, there must be a vertical force across the imaginary plane at the edge of the table to balance statically the downward force of gravity on the protruding part of the stick if the stick is in equilibrium. The system of horizontal forces across this plane must have a resultant of zero but must have a moment about an axis at the edge of the table sufficient to balance the moment of the force of gravity.

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* Under certain very exceptional circumstances a liquid (never a gas) can withstand a tensile force (a negative pressure of the nature of a pull across a surface), but the rule that pressure is always positive is valid for all engineering and technical purposes.

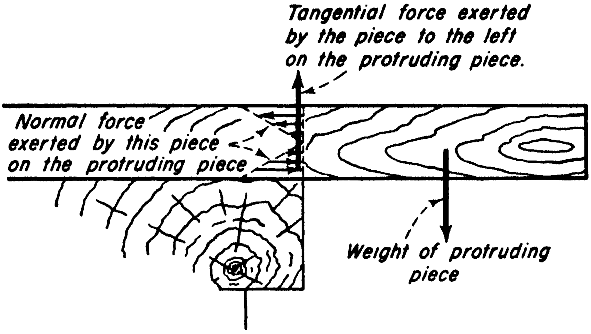
Tangential force exerted by the piece to the left on the protruding piece.
Normal force exerted by this piece on the protruding piece
Weight of protruding piece

Fig. 3. A stick protruding over the edge of a table. The forces acting on the portion of the stick that protrudes are shown.

Thus the force system across this plane in the solid has not only a tangential component but also has pulls as well as pushes (tensile as well as compressive forces). The properties peculiar to a fluid medium are obtained when we require that the force across any plane be entirely of the nature of a *pressure.* Obviously, a fluid could not have the configuration of the stick in Fig. 3.

For our purposes we adopt the following definition:

> A **fluid** is a material substance which in static equilibrium cannot exert either tangential or tensile forces across a surface, but can exert merely pressure (compressive force normal to a surface).

From this definition we shall be able to derive all the laws that govern the experimental behavior of ordinary liquids and gases in static equilibrium. Certain highly viscous substances such as tar that are borderline between liquids and amorphous solids are excluded from consideration.

We note that the pressure exerted on a real surface or across an imaginary surface in the fluid may in general vary from point to point on the surface. Hence, to define the pressure at a point P on the surface in Fig. 4, we should take a small area δA surrounding P, on which force of magnitude δF acts, and write

$$p = \lim_{\delta A \to 0} \frac{\delta F}{\delta A}. \qquad (1)$$

Suitable units for measuring pressure are lbf/in², lbf/ft², kgf/m², and so on.

From the discussion so far, one would be tempted to treat pressure as a vector quantity having the direction of the force δF in Fig. 4. We might expect the pressure to be different for every different choice of the plane through P. But it is very convenient that in the case of fluids the pressure at P is *independent* of the direction of the plane through P, so that we can treat p as a scalar function associated with the point and from it derive the vectorial force across any plane whatever through P. The theorem that the pressure is independent of the direction of the plane is proved as follows:

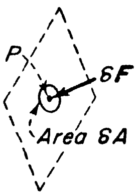

Fig. 4.

Let the slant face of the prism in Fig. 5 be any plane whatsoever through P; let this plane make dihedral angle θ with the horizontal plane. Choose a rectangular coordinate system with the z-axis vertical and the x-axis parallel to a horizontal line in the plane, and construct the small prism shown in Fig. 5.

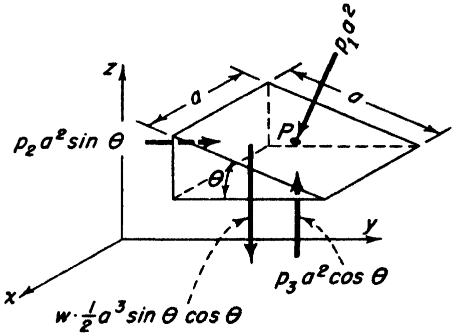

Fig. 5.

Let the average pressure on the slant face be p_1; then the force on the slant face is p_1a^2. Let the average pressure on the left face be p_2; the area of this face is $a^2 \sin\theta$, and hence the force on this face is $p_2a^2 \sin\theta$. Let the average pressure on the bottom face be p_3; the area of this face is $a^2 \cos\theta$, and hence the force on this face is $p_3a^2 \cos\theta$. Let the average specific weight of the material in the prism be w (*specific weight is weight* *per unit volume*);* the volume of the prism is $\frac{1}{2}a^3 \sin\theta \cos\theta$, and therefore the weight of the prism is $w \cdot \frac{1}{2}a^3 \sin\theta \cos\theta$.

Now consider the equilibrium of forces on the prism. In the y-direction we have

$$p_2 a^2 \sin\theta = p_1 a^2 \sin\theta,$$

or

$$p_2 = p_1. \tag{2}$$

In the z-direction we have

$$p_3 a^2 \cos\theta = p_1 a^2 \cos\theta + w \cdot \tfrac{1}{2}a^3 \sin\theta \cos\theta,$$

or

$$p_3 = p_1 + \tfrac{1}{2}wa \sin\theta. \tag{3}$$

Equations (2) and (3) are true for any size prism; in particular they must remain true in the limit as $a \to 0$. In the limit, p_1 approaches the pressure on the slant face at P; p_3 approaches the pressure on a horizontal surface *at P*; and p_2 the pressure on a vertical surface at P. Also, in (3), the last term approaches zero as $a \to 0$ and can be dropped. Hence in the limit we conclude that $p_1 = p_2 = p_3$, and hence that the pressure on any slant surface or any vertical surface through P equals the pressure on a horizontal surface through P. In other words:

At a given point in a fluid, the pressure is independent of the direction of the surface through the point.

Because of this theorem, which has no counterpart in the theory of stresses in solids, it is convenient to *treat the pressure p in a fluid as a scalar function of position.*

Although the convenient instruments for measuring pressure employ fluids and hence depend for their operation on laws of fluid statics which we have not yet derived, one could conceive of constructing a little 'gadget' that would measure fluid pressure by compression of a spring, as in Fig. 6. A very flexible rubber diaphragm is backed by a plate attached to a spring. *The inside space is evacuated.* The force of the fluid on the diaphragm is measured by the compression of the spring.

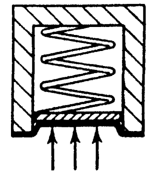

Fig. 6. 'Gadget' for pressure measurement.

Such a gadget, in the normal atmosphere, would give a pressure reading of 14.7 lbf/in², independent of whether the diaphragm were pointed up, down, sidewise, or in any other orientation, in accordance with the theorem we have just proved, which says that pressure is independent of orientation. Similarly, 35 ft down in a lake, the gadget would give a constant reading of about 30 lbf/in², independent of the way it was pointed.

---

* The relationship between *specific weight* and *density* will be pointed out in Chap. 6.

2. THE LAWS OF FLUID STATICS

In this section we shall prove a set of theorems which give the laws of fluid statics. We do not assume that the fluid is homogeneous; rather, it is important to consider the case where more than one type of fluid is present, as in the case of oil floating on water or the ever-present case of air 'floating' on water. In such cases we shall *prove* that the interface between the fluids is horizontal, with the less dense fluid above. After we are equipped with these theorems, we shall discuss their applications in the rest of the chapter.

Theorem: The pressure at every point in a horizontal layer of a fluid at rest is the same.

Consider the fluid within a thin square prism which has its axis horizontal as in Fig. 7. Let the y-axis of a coordinate system be parallel to the axis of the prism. The only forces acting in the y-direction on the fluid within this prism are the normal forces on the ends. If p_1 is the average pressure on the left end, p_2 the average pressure on the right end, equilibrium requires that

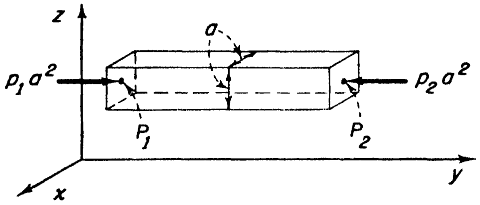

Fig. 7.

$$p_1a^2 = p_2a^2, \quad \text{or} \quad p_1 = p_2.$$

If now a approaches zero, p_1 approaches the pressure at P_1, and p_2 that at P_2; therefore the pressures at P_1 and P_2, two arbitrary points in the same horizontal plane, are equal. This identity proves the theorem only for two points P_1 and P_2 that can be joined by such a prism of fluid as in Fig. 7. It may be that because of a solid obstruction, the two points cannot be connected by a single horizontal prism but that they can be connected by a succession of horizontal prisms which remain entirely in the fluid. In this case the theorem is still clearly true. Where the two points cannot be so connected, the theorem is not in general true. For example the pressure is not the same at the same horizontal level in the two arms of a U-tube unless the U-tube is filled with homogeneous fluid. The pressure is definitely not the same at a point in a pail of water and at a point at the same horizontal level in the air outside. So the 'horizontal layer' of the theorem extends as far as one can go in a horizontal plane in every direction without crossing a solid boundary.

Theorem: The pressure at a height h_1 *in a fluid at rest is greater than the pressure at a greater height* h_2 $(h_2 > h_1)$ *by the weight of a column of fluid of unit cross section and height* $h_2 - h_1$ *lying between these two levels. For a homogeneous fluid this pressure difference is* $w(h_2 - h_1)$, *where* w *is the specific weight.*

Consider fluid levels a vertical distance $h_2 - h_1$ apart, as in Fig. 8. Connect them by a prism of area a^2. If the fluid in this prism has weight W, equilibrium of vertical forces requires that

$$F_1 - F_2 = W.$$

Dividing this equation through by the area a^2 gives

$$\frac{F_1}{a^2} - \frac{F_2}{a^2} = \frac{W}{a^2},$$

or $p_1 - p_2 =$ weight per unit cross-sectional area of prism,

which proves the theorem. One is tempted, in proving this theorem, to say, 'Take a column of fluid of unit cross-sectional area.' This instruction can be followed if we are talking about the atmosphere or a large tank of liquid, where there is no difficulty in cutting out a column of fluid having a cross-sectional area of 1 in², 1 ft², or 1 m². But if we are talking about mercury in a capillary tube, we cannot cut out a column of such area. Nevertheless, the pressure difference is the weight that a column of fluid of such an area *would have.*

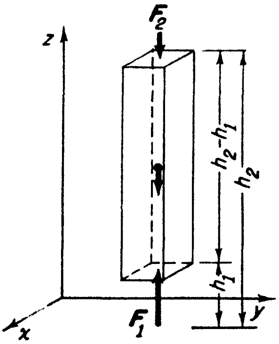

Fig. 8.

If the fluid is homogeneous, in that it has the same *specific weight* (*weight per unit volume*) w at all points, it is seen that the weight of a column of fluid of unit cross section is just w times the volume $h_2 - h_1$, as stated in the theorem.

Although this theorem refers explicitly to pressures at points vertically above one another, combined use of this theorem and the previous one which says that the pressure is constant in a horizontal layer enables us to relate the pressure at any two points in a fluid, as indicated in Fig. 9. If the U-tube of Fig. 9 contains a denser liquid to the left and a lighter liquid (one of less specific weight) in the right-hand tube down to the level marked 'interface,' the pressure at *4* will be *greater* than that at *1*, because a unit column of fluid connecting *4* and *3* will weigh less than a unit column of fluid connecting (in two steps) *1* and *3*. Algebraically, with an obvious notation:

$$p_3 - p_1 = W_{1,3},$$

$$p_3 - p_4 = W_{4,3}.$$

Subtracting: $$p_4 - p_1 = W_{1,3} - W_{4,3} > 0.$$

Theorem: If a vessel contains two fluids of different specific weights, unmixed and in equilibrium, the interface between the fluids will be horizontal.

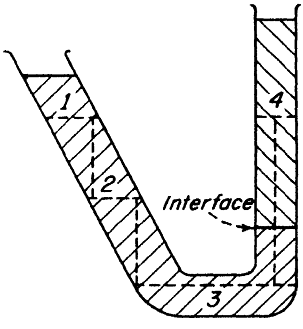

Fig. 9. Pressure is constant on layers *1*, *2*, *3*, and *4*. The pressure difference between two layers that can be connected by a vertical line is the weight of a unit column of fluid connecting these layers. The pressure at *4* is not necessarily the same as at *1*, on the same horizontal level, unless the liquid is homogeneous throughout.

Consider the two fluids, such as oil and water or air and water, in Fig. 10. The pressures at A and B, on the same horizontal level, are equal; also, the pressures at C and D are equal. Therefore the weight of a unit column of fluid along the line AC must be the same as the weight of a unit column of fluid along BD. But with fluids of different specific weights, this identity can result only if the interface is at the same height along AC as along BD. Hence the interface is horizontal. After deriving Archimedes' theorem, we shall be able to see that the denser liquid must be below, the lighter one above, for a *stable* equilibrium (see Sec. 4).

Archimedes' theorem: A fluid acts on a foreign body immersed in it with a net force that is vertically upward and equal in magnitude to the weight of the fluid displaced by the body. (This upward force is called the buoyant force.)

Let the body be immersed in a fluid which is not necessarily homogeneous but which in accordance with the previous theorem will at least be homogeneous in horizontal layers. A floating log, for example, is immersed in a fluid that is partly water, partly air. For the body of Fig. 11, we shall prove first that the net horizontal force that results from the fluid pressure is zero; then that the net vertical force equals the weight of displaced fluid.

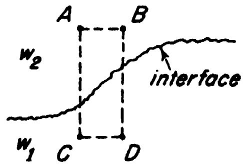

Fig. 10.

To prove that the net force in the y-direction vanishes, we divide the body into a large number of small horizontal prisms with axes parallel to the y-axis. A typical prism, of this sort, of cross-sectional area δA_3, is shown in Fig. 11. This prism intersects the left side of the body in a quadrilateral of area δS_4 and the right side in a quadrilateral of area δS_3. The fluid pressure on these two quadrilaterals will have the same value, p_3, since they are at the same horizontal level, and will exert the normal forces shown in the diagram. If ϕ_4 and ϕ_3 are the angles between the normals and the y-direction, the net y-component of fluid force on this prism will be

$$p_3\,\delta S_4\cos\phi_4 - p_3\,\delta S_3\cos\phi_3.$$

But now $\delta S_4\cos\phi_4$ is the area of the projection of δS_4 on the xz-plane, or

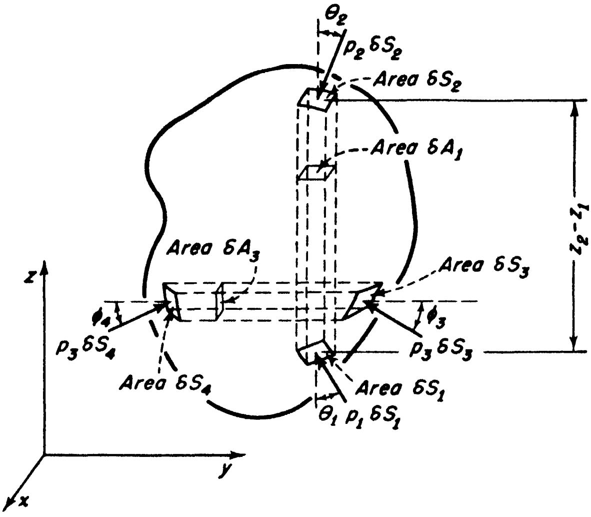

Fig. 11.

δA_3. Similarly, $\delta S_3\cos\phi_3$ is the projected area of δS_3 on the xz-plane, also δA_3. Hence the net y-component of fluid force on this prism is

$$p_3\,\delta A_3 - p_3\,\delta A_3 = 0.$$

Since the net y-component of force vanishes on each prism, it will vanish for the body as a whole.

Similarly, by dividing the body into prisms with axes parallel to x, one proves that there is no resultant fluid force in the x-direction.

To obtain the net buoyant force in the z-direction, divide the body into very small prisms parallel to the z-axis, of which a typical one is shown in Fig. 11. The net upward force exerted by the fluid on this prism will be, by arguments similar to the above,

$$p_1\,\delta S_1\cos\theta_1 - p_2\,\delta S_2\cos\theta_2 = p_1\,\delta A_1 - p_2\,\delta A_1 = (p_1-p_2)\,\delta A_1.$$

In this case, $p_1 \neq p_2$; rather, the difference p_1-p_2 is the weight of a unit column of fluid of height z_2-z_1. Hence $(p_1-p_2)\,\delta A_1$ would be the weight of a column of fluid of area δA_1 and height $z_2 - z_1$, or just the weight of the volume of fluid *displaced* by the volume of the prism we have been considering. Since the buoyant force on each such prism is the weight of the fluid it displaces, the buoyant force on the whole body is the weight of the fluid the whole body displaces. This completes the proof of this theorem, which was first proved by Archimedes (c. 287–212 B.C.).

Applications of the above theorems will be discussed in the succeeding sections.

3. THE BAROMETER, PRESSURE GAUGES

Torricelli, in 1643, first devised a method for measuring the pressure of the atmosphere by means of the mercury barometer. Knowledge of atmospheric pressure is fundamental to most other measurements of pressure because most pressure gauges use the atmosphere as a reference level and measure the difference between an actual pressure and atmospheric pressure.

> Actual pressure at a point in a fluid is called *absolute pressure.*
>
> The difference between absolute pressure at a point in a fluid and the pressure of the atmosphere at the location of the pressure gauge is called *gauge pressure.*

Gauge pressure is stated as either above or below atmospheric. A gauge that reads pressures below atmospheric is ordinarily called a *vacuum* gauge. In this book, the word *pressure* will refer to absolute pressure unless gauge pressure is specifically indicated.

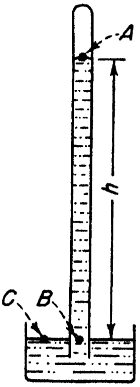

Fig. 12. The mercury barometer.

To make a simple mercury barometer, take a straight glass tube about a meter long and sealed at one end. Fill the tube *completely* with mercury, close the open end with the finger, and invert the tube in a dish of mercury as in Fig. 12. The mercury column in the tube will fall to a height h of about 76 cm above the level of the mercury in the reservoir if the experiment is done at sea level.

If the barometer of Fig. 12 is properly constructed, no air will be trapped in the space above the mercury column. Hence this space must be free from substance—a vacuum.* Since a vacuum certainly exerts no pressure, the pressure at point A is zero. The pressure at point B is the weight of a unit column of mercury of height h. The pressure at C equals that at B, since these are two points at the same horizontal level in a homogeneous fluid. But the pressure at C is the pressure of the atmosphere. Hence the pressure of the atmosphere is the weight of a unit column of mercury of height h.

---

* Actually, there will be a small amount of mercury *vapor* in this space; but since at ordinary temperatures the pressure this vapor exerts will be quite negligible, we ignore it here. See Chap. 18 for a discussion of vapor pressure.

The pressure of the atmosphere is the weight of a unit column of air extending all the way to the top of the atmosphere (99 per cent of the air lies within 30 km of the earth's surface). The whole of this air, down to sea level, has exactly the weight of a layer of mercury 76 cm thick on the average. Atmospheric pressure decreases with altitude because there is less total weight of atmosphere above a point at higher altitude than above a point at sea level. This is the principle on which the airplane altimeter works. At a given location, there are day-to-day variations in atmospheric pressure which may amount to as much as five per cent. These variations have important meteorological significance.

A pressure equivalent to the weight of exactly 76 cm of mercury (at 32° F and under standard gravity) is called *one standard atmosphere* (1 atm). Since the specific weight of mercury under these standard conditions is 13.5950 gf/cm^3, the weight of a column 1 cm^2 in cross section and 76 cm high is 76 cm$^3 \times$13.5950 gf/cm^3 = 1033.2 gf. Hence

$$1 \text{ atm} = 1033.2 \text{ gf/cm}^2 = 10{,}332 \text{ kgf/m}^2$$
$$= 14.70 \text{ lbf/in}^2 = 2116 \text{ lbf/ft}^2.$$

Pressures are frequently specified in *atmospheres* (meaning standard atmospheres). They are also conveniently specified by giving the equivalent height of mercury column, as *centimeters of mercury* or as *inches of mercury* (at 32° F under standard gravity). Thus,

$$1 \text{ atm} = 76 \text{ cm of Hg} = 29.92 \text{ in of Hg}.$$

They are sometimes specified in inches or feet of water by giving the equivalent height of water column. At 32° F, mercury is 13.60 times as heavy as water, so that

$$1 \text{ atm} = 29.92 \text{ in of Hg} = 29.92 \times 13.60 \text{ in of water}$$
$$= 406.8 \text{ in of water} = 33.90 \text{ ft of water}.$$

The pressure of the atmosphere may be measured (less accurately) by means of an *aneroid barometer*. This consists in principle of a sealed evacuated metal box, flat and circular in shape, with a corrugated top as in Fig. 13. The top is bent in by the pressure of the atmosphere, the amount of bending being a measure of the pressure. Motion of the top is amplified by a system of levers and gears and is transmitted to a pointer which reads centimeters or inches of mercury on a scale that has been calibrated against a mercury barometer.

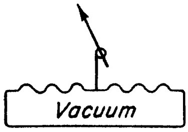

Fig. 13. Schematic cross-section of aneroid-barometer element.

The normal specific weight of air is 1.292 kgf/m^3, or 0.0807 lbf/ft^3, at 32° F, 1 atm pressure, and standard gravity. The manner in which the specific weight varies with pressure and temperature will be discussed in detail in Chap. 17; its variation with variation in the force of gravity will be discussed in Chap. 6. This specific weight is so small that variations in atmospheric pressure with a change in height of a few feet are ordinarily negligible, so that if one reads barometric pressure in a room, the value obtained will ordinarily apply to apparatus at any height in the room. The decrease in pressure in going up 10 ft, for air of the normal specific weight given above, is the weight of a column 10 ft high, which is 0.807 lbf/ft^2. Since 1 atm = 2116 lbf/ft^2, this is only 0.0004 atm decrease in pressure, a value negligible for most purposes. On the other hand, altimeters can be made sufficiently sensitive to respond to this change in pressure and give a reading accurate to about 10 ft change in altitude.

TABLE I

TYPICAL SPECIFIC WEIGHTS OF LIQUIDS AND SOLIDS AT 68° F AND STANDARD GRAVITY

Liquids	kgf/m^3	lbf/ft^3	Solid metals	kgf/m^3	lbf/ft^3
Water (32–50° F)	1,000	62.43	Aluminum	2,700	169
Sea water	1,030	64.4	Cast iron	7,200	449
Benzene	879	54.9	Copper	8,890	555
Carbon tetrachloride	1,594	99.5	Gold	19,300	1205
Ethyl alcohol	789	49.3	Lead	11,340	708
Gasoline	680	42.5	Magnesium	1,740	109
Kerosene	800	49.9	Nickel	8,850	553
Lubricating oil	900	56.2	Silver	10,500	656
Methyl alcohol	792	49.4	Steel	7,800	487
100% sulfuric acid	1,831	114.3	Tungsten	19,000	1190
Turpentine	873	54.5	Zinc	7,140	446
Mercury (32° F)	13,595	848.7	Brass or bronze	8,700	543

Nonmetallic solids	kgf/m^3	lbf/ft^3	Woods	kgf/m^3	lbf/ft^3
Ice (32° F)	922	57.5	Balsa	130	8
Concrete	2300	144	Pine	480	30
Earth, packed	1500	94	Maple	640	40
Glass	2600	160	Oak	720	45
Granite	2700	169	Mahogany	560	35

The specific weights of various liquids and solids (at 68° F and standard gravity) are given in Table I. The variation with temperature is small, as is the variation in the force of gravity, so the values of Table I can be used for ordinary purposes over a fairly wide range of temperature and at any point on the earth's surface.

We shall now describe two types of pressure gauges in common use, both of which measure gauge pressure. One is the *manometer*, illustrated in Fig. 14. The manometer is essentially a U-tube open to the atmosphere on one side and to the fluid whose pressure is being measured on the other, and filled with a suitable liquid. From the difference in heights of column in the two sides, and the laws of fluid statics, the difference between the fluid pressure and the atmospheric pressure is readily determined.

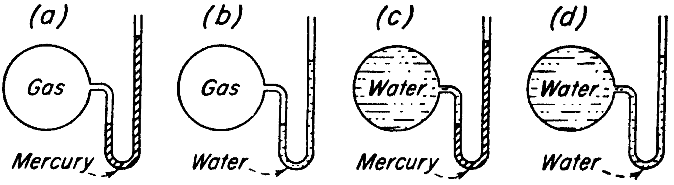

Fig. 14. The manometer.

Another type of pressure gauge, used in many technical applications, is the *Bourdon gauge*. This gauge contains a sealed spiral of flat metal tubing as in Fig. 15. The inside of the tubing is filled with the fluid whose pressure is being measured (or with air in communication with the fluid and hence at the same pressure); the outside of the tubing is exposed to the atmosphere. The elastic properties of such a tube are such that when the pressure inside is increased, the tube tends to 'unwind' and straighten out. This unwinding effect is communicated to a pointer. It turns out that the configuration of the tube and hence the position of the pointer depends only on the difference between the pressure inside the tube and the atmospheric pressure outside. Such gauges can be constructed and calibrated to cover various pressure ranges and to read pressures either above or below atmospheric, or both.

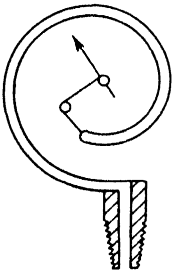

Fig. 15. The sensitive element of a Bourdon-type pressure gauge (schematic).

The hand gauge ordinarily used to measure the pressure in automobile tires is of still another type, in which the high-pressure air pushes back a piston against the force of a spring. Since the atmosphere acts on the spring side of the piston, the compression of the spring is a measure of gauge pressure, that is, the difference between absolute pressure in the tire and atmospheric pressure.

PROBLEMS

NOTE: Use specific weights from Table I.

1. If the liquid in a barometer is water, at what height in feet would the water stand when a mercury barometer read 27 in? (Neglect pressure of vapor above the water column.) **Ans: 30.6 ft.**

2. If the liquid in a barometer is carbon tetrachloride, at what height in meters would the CCl_4 stand when a mercury barometer read 76 cm? (Neglect pressure of CCl_4 vapor above the liquid.)

3. In Fig. 14(a), if the mercury stands 40 cm higher in the right-hand tube than in the left-hand and if the barometer reads 72 cm, what is

(a) the gauge pressure of the gas in cm of Hg? in gf/cm^2?
(b) the absolute pressure of the gas in cm of Hg? in atmospheres?

Ans: (a) 40 cm, 544 gf/cm^2; (b) 112 cm, 1.47 atm.

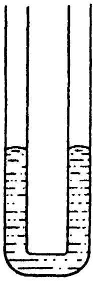

Fig. 16. A U-tube of uniform bore.

4. In Fig. 14(c), if the mercury stands 70 cm above the bottom of the water tank in the right-hand tube and 20 cm above the bottom of the tank in the left-hand tube and if the barometer reads 73 cm, find the gauge pressure and the absolute pressure at the bottom of the water tank (a) in cm of Hg, (b) in atm, (c) in lbf/in^2.

5. A manometer such as in Fig 14(b) attached to a household gas line records, typically, 10 inches difference in water level. What is the gauge pressure in lbf/in^2? What is the absolute pressure in lbf/in^2 and in atm if the barometer reads 27 inches of mercury?

Ans: 0.361 lbf/in^2; 13.6 lbf/in^2; 0.927 atm.

6. If the gauge pressure of the water at the bottom of the tank in Fig. 14(d) is 10 lbf/in^2 and if the barometer reads 0.95 atm, how high about the bottom of the tank does the water stand in the manometer tube?

7. If the U-tube of Fig. 16 contains mercury, how many centimeters of water column must be poured into the right-hand arm to depress the mercury level 1 cm in this arm and raise it 1 cm in the other?

Ans: 27.0 cm.

8. If the U-tube of Fig. 16 contains water and if a 20-cm column of kerosene is poured over the water in the right arm, by how many cm is the water level depressed in this arm and raised in the other?

9. In Fig. 17, the right column contains CCl_4, the center column water, the left column kerosene. If the water stands at a height of 20 in, at what heights do the other two columns stand?

Ans: 12.5 in, 25.0 in.

10. In Fig. 17, the center column contains water and stands at a height of 30 cm. The other two columns contain unknown liquids and stand at heights of 41 cm and 23 cm. Determine the specific weights of the unknown liquids.

11. In Fig. 9, the left tube and the bottom part of the right tube contain water; the rest of the right tube contains kerosene. The top of the kerosene in the right tube is 2 ft above the top of the water in the left tube. If the pressure at level *1* in the left tube is 1.5 lbf/in^2 gauge, what is the pressure at level *4* in the right tube?

Ans: 1.89 lbf/in^2 gauge.

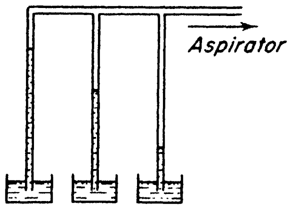

Fig. 17. Balanced columns of fluid.

12. In Fig. 9, the left tube and the bottom part of the right tube contain water, and the rest of the right tube contains kerosene. The interface is 15 ft below the top of the water in the left tube. How far below the top of the kerosene is the pressure 5 lbf/in^2 gauge? What is the gauge pressure in the water at the same vertical height?

13. Show that the force F required to pull apart the evacuated hemispheres (Magdeburg hemispheres) of Fig. 18 is $\pi R^2 p$, where p is the pressure of the atmosphere and R the outside radius of the hemispheres. (Assume that the vacuum seal occurs at the outside radius.)

14. If the hemispheres were 24 inches in diameter and imperfectly evacuated to 0.1 atm inside in the demonstration by Otto von Guericke in Magdeburg in 1654, what force would the horses have had to exert to pull them apart if the atmospheric pressure was 1 atm?

15. If the earth's atmosphere were homogeneous and of normal specific weight throughout, what would be the thickness of the atmosphere in km when the barometer read 1 atm? Ans: 8.00 km.

16. If the glass tube of the barometer in Fig. 12 is supported by a string attached to the upper end, so that the lower end just dips below the level in the reservoir, determine the tension in the string. Analyze the forces carefully and determine whether this tension is (a) just the weight of the glass tube, (b) the weight of the tube plus the weight of the mercury it contains, or (c) some other weight.

17. Find the magnitude of the resultant force tending to open the cap on the end of a pipe of 2-ft diameter, if this end is 75 ft below the surface of the water in the reservoir to which the pipe leads and if the barometer reads 14.7 lbf/in². Ans: 147,000 lbf.

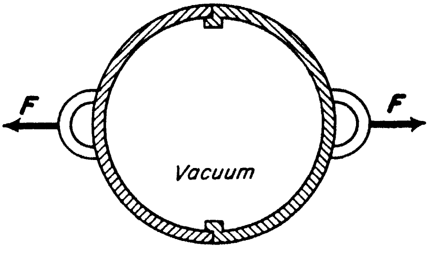

Fig. 18. Magdeburg hemispheres.

18. Why should gauge pressure rather than absolute pressure be specified in giving the bursting strength of a tank?

19. Why do bottles containing liquids tend to leak if taken up in an airplane?

20. A company manufacturing double-pane windows with a layer of air hermetically sealed between two panes of glass found the windows all broken in shipments sent by rail from the Eastern Seaboard to the West Coast. Can you explain this damage?

4. BUOYANT FORCES

The existence of buoyant forces on bodies immersed in or floating on liquids is familiar, as well as the existence of buoyant forces on bodies such as balloons and lighter-than-air craft immersed in air. The buoyant force of the atmospheric air on solids and liquids immersed in it is for most purposes negligible compared to the weight of solid or liquid, but it must be taken into account in *very accurate* determinations of the weight of a substance.

From Archimedes' theorem, it is seen that a body completely immersed in a homogeneous fluid will sink if its own average specific weight is greater than the specific weight of the fluid, will rise if its specific weight is less than that of the fluid, and will be in equilibrium only if its average specific weight exactly equals that of the fluid. A body will 'float' at the interface of two fluids if the specific weight of one fluid is greater and that of the other fluid less, than the average specific weight of the body.

The 'body' mentioned in the preceding paragraph may be a quantity of fluid as well as a quantity of solid. A quantity of denser fluid immersed in a lighter fluid with which it does not mix will sink; similarly, a quantity of lighter fluid immersed in a denser fluid will rise. A layer of denser fluid floating on lighter fluid would constitute an unstable situation which is seldom observed except in narrow tubes, since a slight disturbance would cause the denser fluid to sink down through the lighter. In stable static equilibrium the denser fluid is below, the lighter above.

PROBLEMS

1. When an iceberg floats on sea water, what percentage of the iceberg is under water? Ans: 89.5%.

2. When a block of oak of specific weight 45.0 lbf/ft^3 floats in methyl alcohol, what percentage of the block is submerged?

3. When a block of material of specific weight 950 kgf/m^3 floats at the interface between water and kerosene, what fraction of the material is below the interface? Ans: $\frac{3}{4}$.

4. When a piece of steel floats at the interface between mercury and water, what fraction of the steel is above the interface?

5. In each of the three vessels of Fig. 19, the water has the same depth and the bottom the same area. Hence the pressure and the force on the bottom is the same in all cases. If we imagine the bottom of each vessel to be a frictionless watertight piston, we would have to exert the same force F to hold up each piston. This is sometimes considered as a paradox, because while in (a) this force is just equal to the weight of water, in (c) the force is less than the weight of the water and is sufficient to hold up only the vertical cylinder of water indicated by broken lines, while in (b), more remarkably, the force is greater than the weight of water present and is sufficient to hold up all the water which would fill the vertical column indicated by broken lines.

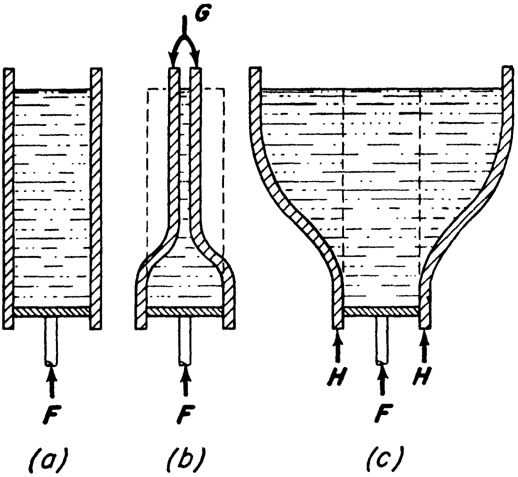

Fig. 19. Pascal's vases.

Resolve the paradox by analyzing the forces acting on the material of the sides of the vessels. Show that, over and above the weight of this part of the vessel, in (c) a lifting force H is required that is exactly equal to the weight of the excess fluid not supported by F, whereas in (b) a downward force G is required that is equal to the weight of the fluid apparently supported by F but not present. Considering the forces on the fluid, we have, in (c), $\boldsymbol{F}+\boldsymbol{H}$ upward, balancing the weight; in (b), $\boldsymbol{F}-\boldsymbol{G}$ upward, balancing the weight.

6. How much does a ship weigh if it displaces water weighing 10,000 tons-force when fully loaded?

7. How many cubic feet of water does a 300-lbf floating log displace? Ans: 4.81.

8. The total weight of a balloon (except for the lifting gas) is 3000 kgf. What volume of helium of specific weight 0.178 kgf/m^3 is required to lift the balloon in air of specific weight 1.292 kgf/m^3?

9. The bag of a balloon is a sphere 50 m in diameter filled with hydrogen of specific weight 0.090 kgf/m^3. What total weight of fabric, car, and contents can be lifted in air of specific weight 1.292 kgf/m^3? Ans: 78,700 kgf.

10. A solid heavier than water is weighed in air and then suspended by a string and weighed in water. The difference between these weights is called the 'loss of weight in water.' This is a convenient method of determining the specific weight of the solid. Show that

$$\text{specific weight of solid} = \frac{\text{weight of solid in air}}{\text{loss of weight in water}} \times \text{specific weight of water.}$$

11. The specific weight of a liquid may be determined by weighing a solid in air, then in water, then in the liquid. Show that

$$\text{specific weight of liquid} = \frac{\text{loss of weight of solid in liquid}}{\text{loss of weight of solid in water}} \times \text{specific weight of water.}$$

12. Devise a scheme for determining the specific weight of a solid lighter than water by tying it to a solid heavier than water, of known specific weight, and weighing the combination under water.

13. A *hydrometer* (Fig. 20) is a convenient instrument for measuring the specific weight of a liquid by a determination of the depth at which the hydrometer floats.

If it is desired to construct a hydrometer in which the calibration marks for 1.0 gf/cm^3 and 2.0 gf/cm^3 are 15 cm apart on a stem of 1 cm^2 cross section, what must be the volume of the bulb below the 2.0 mark and what must be the total weight of the hydrometer? How far from the 1.0 mark will the 1.5 mark fall? Ans: 15 cm^3; 30 gf; 10 cm.

14. It is desired to construct a hydrometer for fluids lighter than water in which the calibration marks for 0.7 gf/cm^3 and 1.0 gf/cm^3 are 15 cm apart on a stem of 1 cm^2 cross section. What must be the volume of the bulb below the 1.0 mark and what must be the total weight of the hydrometer? How far from the 1.0 mark will the 0.9 and 0.8 marks fall?

15. An iron casting weighs 60 lbf in air and 39 lbf in water. What is the volume of the cavities in the casting? Ans: 0.202 ft^3.

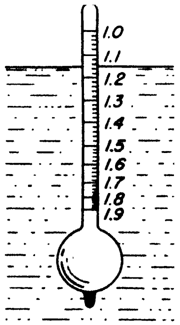

Fig. 20. A hydrometer giving specific weights in gf/cm^3.

16. A bar of gold is balanced by brass weights totalling 10,364.8 gf weight, on a precision balance in normal air. Determine the weight of the gold bar to an accuracy of 0.1 gf.

5. PASCAL'S PRINCIPLE

Pascal's principle states:

If pressure is applied anywhere on a confined liquid, this pressure is transmitted undiminished to all portions of the confining vessel.

This principle follows from the theorems of Sec. 2, since the pressures at different points in a confined liquid differ by amounts that are entirely determined by the differences in vertical heights and the specific weight of the liquid. Any increase in pressure at one point must result in the same increase at all points because the pressure *differences* do not change. This argument supposes the liquid to be negligibly compressible, a supposition that is valid for all ordinary liquids over pressure ranges ordinarily encountered.

This is the principle underlying the hydraulic press, the hydraulic jack, the hydraulic elevator, and other hydraulic devices in which one essentially obtains hydraulic amplification of force. Consider the fluid-filled vessel of Fig. 21 equipped with two pistons, a small one of area s and a larger one of area S. If a force f is applied to the small piston and if the fluid is in equilibrium, the gauge pressure at the piston will be $p=f/s$. This will require force $F=pS=f(S/s)$ on the large piston to maintain equilibrium. If the force on the large piston is slightly less than this value, the large piston will move up and the small piston will move down. Thus a small force will be capable of lifting a very large weight. It will not lift it very far with one 'stroke' of the small piston; but by equipping the tubing with valves so that the large piston cannot move back down when the small piston is retracted and so that new fluid enters the vessel from a reservoir on the return stroke, the large piston can be raised as much as desired by repeated strokes of the small piston.

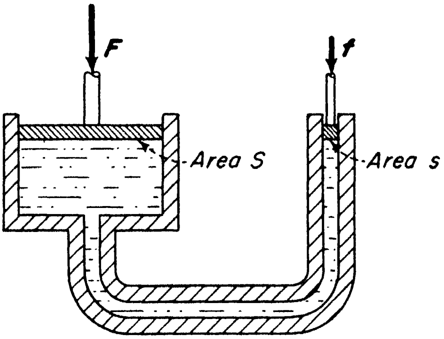

Fig. 21.

PROBLEMS

1. Draw a schematic design of a valve system for a hydraulic jack.

2. If it is desired to raise 3000 lbf by a hydraulic jack employing a small piston of 2-inch stroke on which a force of 150 lbf is applied, what area ratio S/s is needed? How many strokes are required to raise the 3000-lbf weight a distance of 4 inches?

CHAPTER 5

KINEMATICS

Motion of a body can be defined as a continuous change in the position of the body. In this chapter we shall discuss methods by which the motion of bodies can be described quantitatively. The quantitative description of motion is a branch of mechanics called *kinematics*. The scope of kinematics is limited to a mere *description of motion* and does not include a treatment of the forces and torques which cause the motion. As pointed out in an earlier chapter, any change in the position of a body can be treated as a combination of a translation and a rotation. Here we shall devote our attention chiefly to translational motion; a detailed treatment of rotational motion will be given in a later chapter.

In describing the translational motion of a body, the essential problem is one of giving the position of some reference point or particle on the body as a function of time. In order to treat this problem, it will be necessary to introduce and define carefully certain quantities such as *speed*, *velocity*, and *acceleration*. In our treatment of kinematics, we shall consider the motion of a particle in a straight line, the motion of a particle in a plane, and the motion of a particle in three-dimensional space. The motion of a particle experiencing constant acceleration will be treated in detail.

1. SPEED

Perhaps the most familiar quantity used in the description of translational motion is *speed*. Everyone is familiar with the automobile speedometer. A glance at the dial of this instrument tells the driver 'how fast the vehicle is going' at any given instant; the speedometer reading gives the instantaneous speed of the automobile in mi/hr or in some other convenient unit. In order to understand clearly the meaning of the term *instantaneous speed*, it will be desirable for us to define first the term *average speed*.

Let us consider the motion of an automobile along a path on a map like the one shown in Fig. 1. The curve shown in this figure might represent a portion of a level roadway. The average speed of a car in moving from point A to point B is defined as the ratio of the total path length or distance traversed *along the road* to the total time which has elapsed while the motion is taking place; that is,

$$\text{average speed} = \frac{\text{path length traversed}}{\text{elapsed time}}.$$

The path length and the elapsed time are both scalar quantities, and hence the ratio of these quantities is also a scalar; that is, *the average speed of the car is a scalar quantity.* If the total path length from A to B is L and the total elapsed time is t, the average speed between A and B will be L/t.

It should be noted that this expression gives the average speed maintained during the entire trip from A to B but gives no information concerning the average speed during any shorter part of the journey. For example, a delay for refueling might result in a considerably smaller value for the average speed of the entire trip than the value of the average speed for shorter distances. If we happen to be interested in determining the average speed during the portion of the trip between points A' and B', we would compute $\Delta l/\Delta t$ where Δt is the time elapsed during the motion of the car from A' to B'.

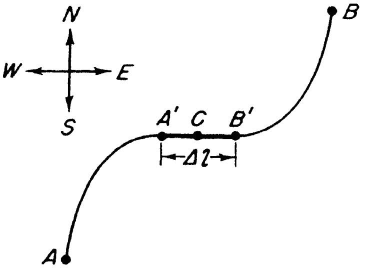

Fig. 1.

By considering successively smaller path lengths Δl, always including the point C, we obtain values for average speed which approach the *instantaneous speed of the car at point C, defined as the limit of the average speed over a path length which approaches zero but always includes the point C.* It might be noted that although the path length Δl becomes extremely small, the time interval Δt by which Δl is to be divided also becomes extremely small and the ratio in general approaches a finite limit. The instantaneous speed v is defined in calculus notation by the expression

$$v = \lim_{\Delta t \to 0} \frac{\Delta l}{\Delta t} = \frac{dl}{dt}.$$

This definition of instantaneous speed implies that the traversed path length l is a function of *time,* which can be regarded as the independent variable. The length of path measured from A is indeed a function of the time elapsed after the car has left point A.

Speed is a scalar quantity. The units in which it is commonly expressed include ft/sec, m/sec, mi/hr, and km/hr. If we have complete information concerning the path length traversed by a body as a function of time, we may find the instantaneous speed of a body at any given instant or at any given point on the path traversed. It is possible to construct devices like the conventional speedometer which give direct readings for 'instantaneous' speed. Owing to finite response times, most conventional speedometers do not give exact values for instantaneous speed when the speed is changing rapidly; however, accurate speed determinations can be obtained with properly calibrated speedometers when the speed is constant or is changing slowly.

It can be shown that the average speed, as we have defined it, is the *time average* of the instantaneous speed.*

The definitions given above presuppose a definition of *time interval.* The fundamental unit of time interval is the *second,* defined by

$$1\text{ second} = \tfrac{1}{86400}\text{ mean solar day},$$

where

> one mean solar day is the average length of the day, from one passage of the sun across the meridian to the next, averaged over a long period of years.

Time intervals are measured by clocks, with which everyone is familiar. One very reproducible type of clock is furnished by the apparent revolution of the fixed stars around the earth. The period of revolution of a star is called a *siderial day.* Since the period of rotation of the earth is constant, siderial days do not vary in length. Solar days do vary in length during the year because the speed of the earth is not constant in its motion around the sun. The word *day,* used without qualification, denotes the mean solar day of 24 hr, or 1440 min, or 86,400 sec.

PROBLEMS

1. A speedometer indicates that a car is moving at a speed of 60 mi/hr. Assuming that the speedometer reading is correct, find the speed of the car in ft/sec.
Ans: 88 ft/sec.

2. Express the speed of the car in Prob. 1 in m/sec.

3. A motorist makes a certain trip in 2.5 hr. If the recorded mileage for this trip is 60.0 mi, what is the average speed of the car? From these data, what information can be obtained concerning instantaneous speed? Ans: 24 mi/hr; instantaneous speed must have been at least 24 mi/hr during some portion of the trip.

4. A railroad train makes a trip between two stations in 4 hr 20 min. If the

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* The proof of this statement is simple but may involve more calculus than the reader has yet studied. The time average of the instantaneous speed, between times t_0 and t_1, is

$$\frac{1}{t_1-t_0}\int_{t_0}^{t_1} v\,dt.$$

Since $v=dl/dt$, this becomes

$$\frac{1}{t_1-t_0}\int_{t_0}^{t_1}\frac{dl}{dt}\,dt=\frac{1}{t_1-t_0}\int_{l_0}^{l_1}dl=\frac{l_1-l_0}{t_1-t_0},$$

if l_1-l_0 is the path length traversed in time t_1-t_0. But the last expression is just our definition of average speed.

length of the track between the two stations is 210 mi, what is the average speed of the train?

5. A runner completes the 100-yd dash in 10 sec. Find his average speed in ft/sec and mi/hr. Ans: 30 ft/sec; 20.4 mi/hr.

6. If the runner mentioned in Prob. 5 could maintain this average speed, what would be his time for the 220-yd dash? for a run of 1 mi?

7. A motorist makes a round trip from city *A* to city *B*, which are 120 mi apart by road. If the time required for the trip from *A* to *B* is 3 hr and the time required for the return trip is 4 hr, find (a) the average speed of the car in going from *A* to *B*, (b) the average speed of the car on the return trip, and (c) the average speed of the car for the entire trip. Ans: (a) 40 mi/hr; (b) 30 mi/hr; (c) 34.3 mi/hr.

8. An automobile makes a round trip between two points 200 mi apart by highway. If the outward trip requires 4 hr and the return trip 8 hr, find the average speeds for the outward trip, the return trip, and the complete round trip.

9. Over a 10-sec period, the distance l in feet traveled by a certain automobile is accurately described by the equation

$$l = 20t + 0.5t^2$$

where t denotes elapsed time in seconds. Find the instantaneous speed of the car when $t = 2$ sec. Ans: 22 ft/sec.

10. Find the instantaneous speed of the automobile in Prob. 9 at the end of the fourth, sixth, eighth, and tenth seconds.

11. An automobile travels at 30 mi/hr for 1 hr and then at 10 mi/hr for 1 hr. What is its average speed? Ans: 20 mi/hr.

12. An automobile travels at 30 mi/hr for 15 mi and then at 10 mi/hr for 15 mi. What is its average speed?

13. A *knot* is a unit of speed equal to one nautical mile (6080 ft) per hour. Show that

$$1 \text{ knot} = 1.152 \text{ mi/hr} = 1.689 \text{ ft/sec}$$

14. Express the speed of a 20-knot ship in mi/hr and in ft/sec.

2. VELOCITY

Although a knowledge of the speed of an object is an important bit of information in many situations, its value in kinematics is limited. The general problem of kinematics is one of finding the *position* of a body as a function of time. If we know the speed of an object, we know *how fast* the object is moving but have no information concerning the *direction* in which it is moving. Therefore, a knowledge of the speed of an object is of no use whatever in finding the position of the object at a given time unless other information is also available. For example, let us consider a car moving at a constant speed of 60 mi/hr. We wish to find the location of the car at the end of one hour. Knowing the speed of the car, we can say immediately that the total *path length* traversed by the car is 60 mi but are unable to say anything about the net displacement of the car from the starting point. The *magnitude* of the displacement would be 60 mi if the path traversed were a straight line, zero if the car traversed a closed path of the proper length, or some intermediate value for other paths.

From the above example, it would appear desirable to define some physical quantity which would give not only the *speed* of a body but also the *direction* in which the body is moving. The quantity which has these properties is the vector quantity called *velocity*. We shall first show how velocity is defined and then consider in some detail how this quantity may be used in describing the motion of a particle.

The *average velocity* of a moving body is defined as the ratio of its *displacement* to the length of the time interval in which the displacement occurred:

$$\text{average velocity} = \frac{\text{displacement}}{\text{elapsed time}}.$$

Since the displacement is a vector quantity and time is a scalar quantity, the ratio is a vector quantity that has the same direction as the displacement.

In order to understand the definition of average velocity, consider Fig. 2(a), which shows again the path of Fig. 1. In Fig. 2(a) the actual path of the car is given by the dotted curve. The total displacement s of the car in making the complete trip from A to B is given by the arrow. Denoting the total elapsed time for the trip by t, we may write the average velocity $\bar{v}$ as

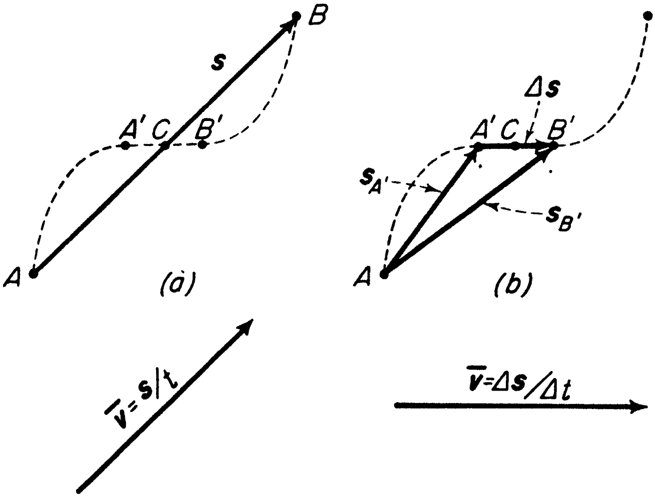

Fig. 2.

$$\bar{v} = s/t. \qquad (A \rightarrow B)$$

The average velocity $\bar{v}$ can be represented by the arrow shown below Fig. 2(a); the length of the arrow gives a measure of the magnitude of the average velocity and the direction of the arrow is the same as that of the arrow representing $\boldsymbol{s}$.

Let us now write down the expression for the average velocity for the car in moving from A' to B'. As measured from the starting point, the total displacement of the car at A' is represented in Fig. 2(b) by the arrow labeled $\boldsymbol{s}_{A'}$, and the total displacement of the car at B' is given by the arrow labeled $\boldsymbol{s}_{B'}$. The change in displacement of the car in going from A' to B' is given by the short arrow labeled $\Delta\boldsymbol{s}$. This vector $\Delta\boldsymbol{s}$ must be added to the total displacement $\boldsymbol{s}_{A'}$ to obtain the total displacement $\boldsymbol{s}_{B'}$. Since $\boldsymbol{s}_{B'}=\boldsymbol{s}_{A'}+\Delta\boldsymbol{s}$, $\Delta\boldsymbol{s}$ is the vector difference

$$\Delta\boldsymbol{s}=\boldsymbol{s}_{B'}-\boldsymbol{s}_{A'}.$$

The average velocity of the car in moving from A' to B' is given by the relation

$$\bar{\boldsymbol{v}}=\Delta\boldsymbol{s}/\Delta t, \qquad (A'\rightarrow B')$$

where Δt is the time required for the car to move from A' to B'. This average velocity can be represented by the arrow shown below Fig. 2(b). The direction of this arrow is parallel to the arrow representing $\Delta\boldsymbol{s}$; the length of the arrow gives a measure of the magnitude of the average velocity.

Following a procedure similar to that used in defining instantaneous speed, we may define the *instantaneous velocity* $\boldsymbol{v}$ of the car at some point such as C by the expression

$$\boldsymbol{v}=\lim_{\Delta t\rightarrow 0}\frac{\Delta\boldsymbol{s}}{\Delta t}=\frac{d\boldsymbol{s}}{dt},$$

where $\Delta\boldsymbol{s}$ is the change in displacement of the car in passing along a short path length Δl that includes point C.

It can be shown that the average velocity, as we have defined it, is just the *time average* of this instantaneous velocity.*

The units in terms of which the magnitude of the velocity is measured are the same as those in terms of which speed is measured, but the statement of the velocity must include a direction. For example, the velocity of a particle might be stated as 30 mi/hr eastward or 12 m/sec upward. Hereafter, unless otherwise indicated, we shall use the term *velocity* to indicate *instantaneous velocity*.

Now let us see how we can compute the velocity of a particle if we are given its x-, y-, and z-coordinates as functions of time. Suppose that

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* The proof is like that for the case of speed, except that vector relations are involved. The time average of the instantaneous velocity is

$$\frac{1}{t_1-t_0}\int_{t_0}^{t_1}\boldsymbol{v}\,dt=\frac{1}{t_1-t_0}\int_{t_0}^{t_1}\frac{d\boldsymbol{s}}{dt}\,dt=\frac{1}{t_1-t_0}\int d\boldsymbol{s}=\frac{\boldsymbol{s}_1-\boldsymbol{s}_0}{t_1-t_0},$$

which is the average velocity as we have defined it.

the particle traverses the space path indicated in Fig. 3 and we are given functions $x(t)$, $y(t)$, and $z(t)$ which give its coordinates at any time t. We desire to determine the velocity of the particle at a certain point P on its path. Starting from P, the particle will undergo a displacement $\Delta \boldsymbol{s}$ in a time Δt, as indicated on Fig. 3. The vector $\Delta \boldsymbol{s}$ has rectangular components Δx, Δy, and Δz as indicated. The velocity at P is defined as

$$\boldsymbol{v} = \lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{s}}{\Delta t},$$

a vector which has components

$$v_x = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t};$$

$$v_y = \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t};$$

$$v_z = \lim_{\Delta t \to 0} \frac{\Delta z}{\Delta t}.$$

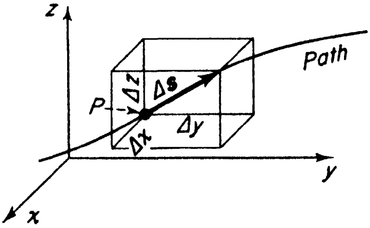

Fig. 3.

But, by definition, these three limits are just the derivatives of the three functions $x(t)$, $y(t)$, and $z(t)$, and so we obtain the components of the velocity by just differentiating these three functions:

$$v_x = dx/dt; \quad v_y = dy/dt; \quad v_z = dz/dt. \tag{1}$$

By considering what happens in Fig. 3 as the magnitude of $\Delta \boldsymbol{s}$ becomes smaller and smaller, we see that

> the *magnitude* of the instantaneous velocity of the body at point P is equal to the instantaneous speed of the body at P, and the *direction* of the instantaneous-velocity vector at P is the direction of the tangent to the path at that point.

The simplest case to which to apply these ideas is the motion of a particle along a straight line. Let us take the straight-line path as the x-axis of our coordinate system. Then there is only one non-vanishing velocity component,

$$v_x = dx/dt.$$

This can be evaluated at any point or at any time, provided the relation between x and t is known. For example, suppose that the position of the particle as a function of time is given by the relation

$$x = a + bt + ct^2, \tag{2}$$

where a, b, and c are constants. Then the x-component of the velocity of the particle is given as a function of time by

$$v_x = dx/dt = b + 2ct. \tag{3}$$

If we wish to find the x-component of the velocity when the particle has some particular coordinate $x = X$, we substitute this value for x in (2) and solve for t; substitution of this value of t in (3) will give the desired velocity.

The motion of a body with *constant velocity* is easily described. Since neither the magnitude nor the direction of the velocity changes in time, the motion is necessarily one with constant speed along a straight line. If we let this line be the x-axis and denote the constant x-component of velocity by V, the particle will move the distance Vt in the time t. Hence if the particle is at $x = x_0$ at $t = 0$, the particle will be at

$$x = x_0 + Vt$$

at time t. We see by differentiation that this expression correctly gives $v_x = dx/dt = V$.

PROBLEMS

1. In Fig. 2, point B is located exactly 100 mi northeast of point A, but the total path length between the two points is 135 mi. If the time required for the completion of the trip from A to B is 3 hr, find the average velocity and average speed of the car for the trip. Ans: 33.3 mi/hr NE; 45 mi/hr.

2. If the time required for the trip described in Prob. 1 had been 4 hr, what would have been the average velocity and average speed for the trip?

3. In Fig. 2 point B' is located 22.5 mi east of point A', and the road is straight. Thirty minutes is required for the car to move from A' to B'. Find the average velocity and average speed of the car in the portion of the trip between A' and B'. If the speed of the car is constant for this part of the trip, what is the instantaneous velocity of the car at point C? Ans: 45 mi/hr E; 45 mi/hr; 45 mi/hr E.

4. If the time required for the part of the trip between A' and B' in Prob. 3 had been 24 min, what would have been the average velocity and average speed of the car for this portion of the trip?

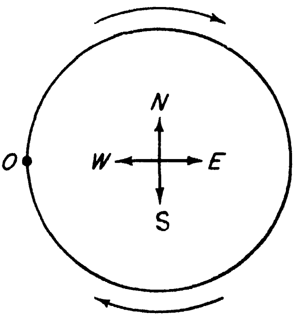

Fig. 4.

5. A car is traversing the circular racetrack shown in Fig. 4 at a constant speed of 60 mi/hr. If the circumference of the racetrack is 1 mi and if the car started at point O, find the instantaneous velocity at the end of 15 sec, 30 sec, 45 sec, and 60 sec. What is the average velocity during the first minute?
Ans: 60 mi/hr E; 60 mi/hr S; 60 mi/hr W; 60 mi/hr N; zero.

6. For the car described in Prob. 5, find the average velocity during the first 15 sec, the first 30 sec, and the first 45 sec. What is the average velocity during the first 2½ min? During the time interval $t = 30$ sec to $t = 60$ sec?

7. Returning to Prob. 1, find the total displacement of the car from point A: (a) if the average velocity of 33.3 mi/hr northeastward has been maintained for 6 hr; (b) if the average speed of 45 mi/hr has been maintained for 6 hr.
Ans: (a) 200 mi NE; (b) indeterminate.

8. An airplane left the Chicago airport at 2:30 P.M. and was sighted at 5:30 P.M. at a point 600 mi due south of the airport. Find the average velocity and the average speed of the plane during the elapsed time interval.

9. Returning to Prob. 1, find (a) the northward component of the average velocity of the car in going from A to B; (b) the westward component.
Ans: (a) 23.6 mi/hr; (b) −23.6 mi/hr.

10. Returning to Prob. 2, find the northward and westward components of the average velocity.

11. The motion of a steel ball rolling down an inclined plane is noted at 1-sec intervals, and the data obtained are recorded in Table I. From the data given, calculate the magnitude of the average velocity during (a) the first second, (b) the fifth second, and (c) the first 5 seconds.
Ans: (a) 2 cm/sec; (b) 18 cm/sec; (c) 10 cm/sec.

12. From the data given in Table I, calculate the magnitude of the average velocity of the steel ball during (a) the sixth second, (b) the ninth second, and (c) the last 5 seconds of motion.

13. To the data shown in Table I, fit a simple equation giving the displacement s down the inclined plane as a function of time. Using this equation, find the instantaneous velocity of the ball at the end of the fifth second.
Ans: 20 cm/sec, down the plane.

14. As in Prob. 13, find the instantaneous velocity of the steel ball at $t=3$ sec, $t=6$ sec, and $t=9$ sec.

15. The x-coordinate in cm of a particle at time t in seconds is given by the expression

$$x=4+2t+3t^2.$$

Find (a) the x-component of the average velocity of the particle during the first 5 seconds and (b) the x-component of the instantaneous velocity of the particle at the end of the fifth second. Ans: (a) 17 cm/sec; (b) 32 cm/sec.

TABLE I

Time t *(sec)*	*Position* s *(cm)*
0	0
1	2
2	8
3	18
4	32
5	50
6	72
7	98
8	128
9	162
10	200

16. For the particle mentioned in Prob. 15, find the x-component of the average velocity during the first 3-sec interval and during the second 3-sec interval, and find the x-component of the velocity when $t=6$ sec.

3. ACCELERATION

In most cases, the velocity of a moving body changes as the motion proceeds, and the body is said to move with *accelerated motion* or to have an *acceleration.* The *average acceleration* of a body is the ratio of the change in the body's velocity to the time interval in which the change occurs:

$$\text{average acceleration}=\frac{\text{change in velocity}}{\text{elapsed time}}.$$

Since the change in velocity is a vector quantity and time is a scalar, the ratio is a vector quantity. If $\boldsymbol{v}_0$ is the velocity of a body at time t_0 and $\boldsymbol{v}_1$ is the velocity of the body at time t_1, the average acceleration $\bar{\boldsymbol{a}}$ is given by

$$\bar{\boldsymbol{a}}=\frac{\boldsymbol{v}_1-\boldsymbol{v}_0}{t_1-t_0}=\frac{\Delta\boldsymbol{v}}{\Delta t}.$$

It should be remembered that this equation gives a relation between *vector* quantities. The relationship involved can be understood by considering the diagram shown in Fig. 5. In this diagram, $\boldsymbol{v}_0$ represents the velocity of a body at time t_0 and $\boldsymbol{v}_1$ represents the velocity at later time t_1. The vector $\Delta\boldsymbol{v}$ is the change in velocity $\boldsymbol{v}_1-\boldsymbol{v}_0$ that has occurred in the time interval $\Delta t=t_1-t_0$. The average acceleration vector $\bar{\boldsymbol{a}}=\Delta\boldsymbol{v}/\Delta t$ has the same direction as $\Delta\boldsymbol{v}$. Since $\Delta\boldsymbol{v}$ is measured in ft/sec or m/sec in a given direction, the average acceleration $\bar{\boldsymbol{a}}$ is measured in (ft/sec) *per sec* or (m/sec) *per sec* in this direction, that is, in ft/sec^2 or m/sec^2.

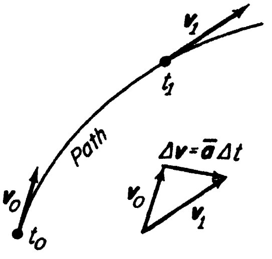

Fig. 5.

The instantaneous acceleration of a body is defined in a manner analogous to that in which instantaneous velocity was defined. The instantaneous acceleration $\boldsymbol{a}$ of a particle at a given point in its path is defined by

$$\boldsymbol{a}=\lim_{\Delta t\to 0}\frac{\Delta\boldsymbol{v}}{\Delta t}=\frac{d\boldsymbol{v}}{dt},$$

where the elapsed time interval Δt includes the time at which the particle is at the point in question.

Just as in the case of velocity, it can be shown that the average acceleration, as defined above, is just the *time average* of the instanteneous acceleration.

Also, since the vector $\Delta\boldsymbol{v}$ has components Δv_x, Δv_y, and Δv_z, we conclude that in the motion of a particle along a space curve as in Fig. 3, the components of the acceleration vector are given by

$$\left.\begin{aligned} a_x&=\lim_{\Delta t\to 0}\frac{\Delta v_x}{\Delta t}=\frac{dv_x}{dt}=\frac{d^2x}{dt^2};\\ a_y&=\lim_{\Delta t\to 0}\frac{\Delta v_y}{\Delta t}=\frac{dv_y}{dt}=\frac{d^2y}{dt^2};\\ a_z&=\lim_{\Delta t\to 0}\frac{\Delta v_z}{\Delta t}=\frac{dv_z}{dt}=\frac{d^2z}{dt^2}.\end{aligned}\right\}\quad(4)$$

In the case of motion along a straight line, the acceleration is directed along the path and has magnitude equal to the rate of change of speed, but in the case of motion in a curved path *neither of these statements is true*, as we shall demonstrate later in this chapter.

Let us first consider the simplest case of motion along a straight line, which we shall take as the x-axis of our coordinate system as in Fig. 6. In this diagram, the particle is at point A at time t_0 and has velocity component v_{x0}. At a later time t_1, the particle is at B and has velocity component v_{x1}. The x-component of the average acceleration during this interval is given by the relation

$$\bar{a}_x=\frac{v_{x1}-v_{x0}}{t_1-t_0}=\frac{\Delta v_x}{\Delta t}.$$

This is positive or negative according to whether v_{x1} is greater than or less than v_{x0}. If the velocity is toward the right (v_x positive) but the speed is decreasing (dv_x/dt negative), a_x is negative and is said to represent a *deceleration*. If the velocity is given by formula (3), the acceleration is given by

$$a_x = dv_x/dt = 2c;$$

in other words, for the motion of (2), the acceleration is constant, with magnitude $2c$.

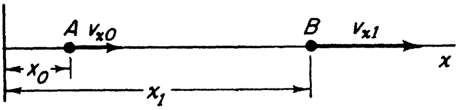

Fig. 6.

In order to understand how the acceleration of a body can be determined from experimental observations of the position of a body as a function of time, it may be desirable to consider a numerical example. Let us consider data which can be obtained by observation of a ball starting from rest and rolling down an inclined plane. Denoting the distance traveled down the inclined plane by x, we might observe the position of the ball at one-second intervals. Data which might be obtained in such an experiment are given in the first two columns of Table II; these data give the values of x for various values of t. In order to find the values of the average velocity during successive 1-sec intervals, it is necessary only to find the distance Δx traversed during the time interval $\Delta t = 1$ sec. For example, for the first 1-sec interval, $\Delta x = 2$ ft and hence $\bar{v}$ for this second is 2 ft/sec; for the second 1-sec interval,

TABLE II

Time (sec)	*Position x (ft)*	*Average velocity $\Delta x/\Delta t$ (ft/sec)*	*Change in average velocity per sec (ft/sec^2)*
0	0		
		2	
1	2		4
		6	
2	8		4
		10	
3	18		4
		14	
4	32		4
		18	
5	50		4
		22	
6	72		4
		26	
7	98		

$\Delta x = 6$ ft and $\bar{v}$ for this interval is 6 ft/sec; for the third 1-sec interval, $\Delta x = 10$ ft and $\bar{v}$ for this interval is 10 ft/sec; and so on. Now, in general, the *average* velocity during each observation *interval* is the only quantity that can be determined directly from the data without interpolations between the observations of position. However, we can get a rough idea of the average acceleration for various 1-sec intervals by noting *the change in average velocity* occurring each second. This is not, in general, the same as the change in instantaneous velocity during a 1-sec interval, which we have defined as average acceleration, but in this particular example it does give us an idea of the magnitude of the average acceleration, since the average velocity for successive 1-sec intervals is changing at a constant rate of 4 ft/sec^2. This statement can readily be verified from Table II, since the average velocity during the second 1-sec interval is 4 ft/sec greater than the average velocity during the first 1-sec interval, and so on.

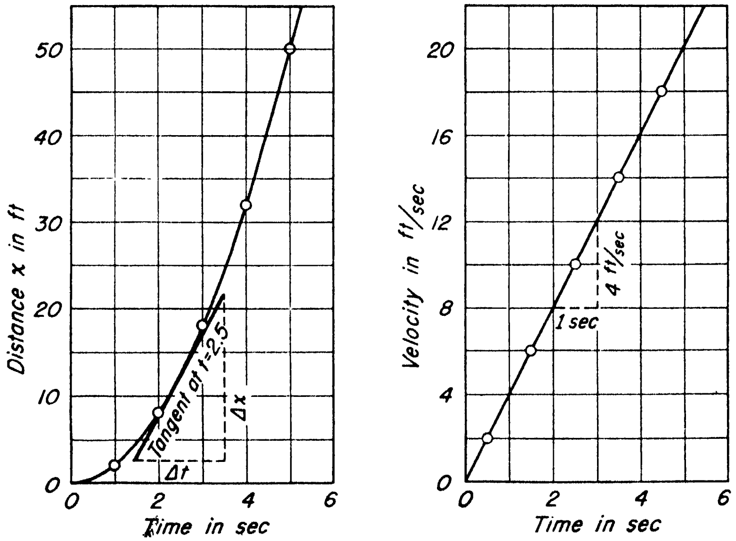

Fig. 7.

Experimental data like those in Table II may be handled graphically. We plot x as a function of t in Fig. 7(a) and draw a smooth curve through the points. We can then, by drawing tangents to the curve at various points and measuring the slopes $\Delta x/\Delta t$ of the tangents, plot a velocity curve as in Fig. 7(b). The slope of the velocity curve at any value of t gives the acceleration at that time. In this example the slope has the constant value (4 ft/sec)/(1 sec) = 4 ft/sec^2.

Such graphical methods are inherently inaccurate. A better method is to fit the data with a simple analytical formula, if one can be found, and then to differentiate. In this case it is readily seen that the data of Table II fit the formula

$$x = 2t^2 \text{ ft}.$$

Hence, $$v = dx/dt = 4t \text{ ft/sec},$$

and $$a = d^2x/dt^2 = 4 \text{ ft/sec}^2.$$

The problem of rectilinear motion with constant acceleration is an important one which we shall consider in detail in the next section.

PROBLEMS

1. In 4 sec a car starting from rest attains a velocity of 50 ft/sec southward. What is the average acceleration of the car? Ans: 12.5 ft/sec² southward.

2. A car starting from rest attains a velocity of 15 m/sec southward in ½ min. What is the average acceleration of the car?

3. An automobile is moving eastward along a level road at a speed of 60 mi/hr when the brakes are applied. If the car is brought to rest in 20 sec, what is the average acceleration of the car during this interval? Ans: 4.4 ft/sec² westward.

4. A car moving northward along a level road at a speed of 45 mi/hr is brought to rest in 30 sec. Find the average acceleration of the car during this interval.

5. A man is driving a car eastward along a highway at a speed of 15 mi/hr. If he steps on the accelerator and one minute later the speedometer reads 60 mi/hr, what is the average acceleration of the car? Ans: 1.1 ft/sec² eastward.

6. A car is rolling down a long slope. If the initial speed of the car is 2 m/sec and its speed at the end of 8 sec is 14 m/sec, what is the average acceleration of the car?

7. A car is equipped with a special speedometer that registers speed in ft/sec. While the car is moving eastward along a level street, an occupant of the front seat observes the speed and at various times records the data shown in the table. From these data calculate the average acceleration (a) during the first 6 seconds of motion, (b) during the interval $t = 4$ sec to $t = 6$ sec, and (c) during the interval $t = 6$ sec to $t = 8$ sec.

Ans: (a) 3 ft/sec² eastward; (b) 4 ft/sec² eastward; (c) −1 ft/sec² eastward or 1 ft/sec² westward.

Time (sec)	Speed (ft/sec)
0	0
2	4
4	10
6	18
8	16
10	8
12	4
14	0

8. From the data given in Prob. 7, find the average acceleration (a) during the first 4 seconds, (b) during the interval $t = 2$ sec to $t = 8$ sec, and (c) during the entire period of motion.

9. A ball rolls from rest down an inclined plane and has a constant acceleration of 6 ft/sec² down the slope. What is the velocity of the ball at the end of each of the first four seconds? Ans: 6, 12, 18, 24 ft/sec.

10. A ball rolling from rest down an inclined plane has a constant acceleration of 3 m/sec² down the plane. Find the velocity of the ball at the end of the third, fifth, and seventh seconds.

11. A ball thrown vertically upward with an initial velocity of 80 ft/sec has a constant *downward* acceleration of 32 ft/sec². Find its velocity after 1, 2, and 3 seconds. At what time is its velocity zero?

Ans: 48, 16, −16 ft/sec upward; 2.5 sec after it is thrown.

12. A ball thrown vertically upward with an initial velocity of 25 m/sec has a constant downward acceleration of 9.8 m/sec^2. Find its velocity after 1, 3, and 5 seconds. At what time is its velocity zero?

13. A train increases its speed at the rate of 15 mi/hr per minute. Express its acceleration in ft/sec^2. Ans: 0.367 ft/sec^2.

14. If a V-2 rocket increases its upward velocity from zero to 1.2 km/sec in one minute, what is its average acceleration in km/min^2 and in m/sec^2?

15. The speedometers on ships and many planes read in knots (cf. Prob. 13, p. 94). Show that 1 knot/sec = 1.69 ft/sec^2.

4. LINEAR MOTION WITH CONSTANT ACCELERATION

The simplest type of accelerated motion is that of a body moving along a straight line with *constant* acceleration, as in the case of the numerical example discussed in the preceding section. We shall take our x-axis along this line with a specified positive sense, letting v represent the x-component of velocity and a the x-component of acceleration. v will be positive when the velocity vector points along the axis in the positive sense, otherwise negative; a will be positive when the acceleration vector points along the axis in the positive sense, otherwise negative. For convenience in discussing this simple problem, we shall speak of v and a as *the* velocity and *the* acceleration, keeping in mind, however, that they are really the x-components of these quantities.

The velocity of the body executing linear motion with constant acceleration increases or decreases by the same amount in each unit of time. The change Δv in velocity in any time interval Δt is given by

$$\Delta v = a\,\Delta t, \tag{5}$$

where a is the constant value of the acceleration.

Let the initial x-coordinate of the body (at time $t=0$) be x_0, and the initial velocity of the body be v_0. At any later time t the velocity will have changed by $\Delta v = at$, as we see from (5), so that v will be

$$v = v_0 + \Delta v = v_0 + at. \tag{6}$$

Similarly, the position x of the particle at any time t is given by

$$x = x_0 + \Delta x \tag{7}$$

where Δx, the change in position of the particle, is given by

$$\Delta x = \bar{v}t \tag{8}$$

where v is the average velocity during the entire time of motion t. Now, since the velocity of the particle is changing at a constant rate, the average velocity during any time interval equals one-half the sum of the velocity at the beginning of the interval and the velocity at the end of the interval. Thus, for the situation we are considering here, we may write

$$\bar{v} = \tfrac{1}{2}(v_0 + v). \tag{9}$$

It should be noted that equation (9) can be used for determining the average velocity of a body only if it is experiencing constant acceleration along a straight line. Substituting the value of v, from (6), in (9), we obtain

$$\bar{v} = \tfrac{1}{2}[v_0 + (v_0 + at)], \quad \text{or} \quad \bar{v} = v_0 + \tfrac{1}{2}at.$$

Substitution of this value in (8) gives, for the displacement during time interval t,

$$\Delta x = [v_0 + \tfrac{1}{2}at]\, t,$$

or

$$\Delta x = v_0 t + \tfrac{1}{2}at^2. \tag{10}$$

Hence, (7) gives the position x of the particle at any time t as

$$x = x_0 + v_0 t + \tfrac{1}{2}at^2. \tag{11}$$

Equation (11) enables us to locate the particle at any time t after the motion at constant acceleration has commenced, provided x_0, v_0, and a are known.

It is sometimes useful to employ a relation that gives directly the velocity v at any position x. This relation can be obtained by solving equation (6) for t and substituting this expression for t in equation (11). The resulting expression is

$$v^2 = v_0^2 + 2a(x - x_0), \tag{12}$$

where v is the velocity when the particle is at point x.

PROBLEMS

Note: In the following problems, rectilinear motion with constant acceleration is assumed.

1. A body starting from rest moves with a constant acceleration of 2 ft/sec² in a certain direction. What is the body's velocity after 8 seconds? What is its average velocity during this time? How far does the body move during this time?
Ans: 16 ft/sec; 8 ft/sec; 64 ft.

2. A sled starting from rest has a constant acceleration of 1.5 m/sec² in a certain direction. What is the change in velocity during the first six seconds of motion? What is the average velocity of the sled during this time? How far does the sled move?

3. Find the distance traversed by the body in Prob. 1 in each of the first 8 seconds. Ans: 1, 3, 5, 7, 9, 11, 13, 15 ft.

4. Find the distance traversed by the sled in Prob. 2 in each of the first 6 seconds.

5. A car starts from rest and travels eastward. If the speed of the car at the end of one minute is 60 mi/hr, how far does the car travel during this time if the acceleration is constant? What is the magnitude of the acceleration?
Ans: 2640 ft; 1.47 ft/sec².

6. A car moving west at a speed of 45 mi/hr is brought to rest in 30 sec. If the acceleration is uniform, how far does the car travel? What is the magnitude of the acceleration?

7. A motorcycle moving with an initial velocity of 20 m/sec experiences a deceleration of 3 m/sec^2. How far does the motorcycle move during the first 4 seconds after its velocity begins to decrease? Ans: 56 m.

8. A car has an initial velocity of 15 m/sec and experiences a deceleration of 2 m/sec^2. How far has the car moved by the time its velocity is reduced to 5 m/sec?

9. A car has a velocity of 60 mi/hr. If the driver applies the brakes so as to produce constant deceleration and brings the car to rest in a distance of 200 ft, what is the magnitude of the deceleration? Ans: 19.4 ft/sec^2.

10. A golf ball rolls across a green and experiences constant deceleration. If the initial velocity of the ball was 3 m/sec and if the ball comes to rest after traveling 4 m, what was the magnitude of the deceleration?

11. A certain airplane taking off from a landing strip experiences a constant acceleration during its take-off run. If the plane leaves the ground with a horizontal velocity of 120 mi/hr after a run of 2000 ft, what was the magnitude of its acceleration? Ans: 7.74 ft/sec^2.

12. An airplane moving at a speed of 90 mi/hr lands at the end of a runway and is brought to rest after a run of 1200 ft. If the deceleration was constant, what was its magnitude?

13. Derive equation (12) by the method sketched in the text.

14. A body starts from rest and moves with constant acceleration a. Show that the average velocity during the nth second of motion is equal to $(n - \frac{1}{2})a$.

15. Show that in Table II, p. 101, we are justified in identifying the last column heading "Change in average velocity per sec" with a constant acceleration. (Note that the average velocities listed are for successive 1-sec intervals.)

5. FREELY FALLING BODIES

The most common example of motion with very nearly constant acceleration is that of a falling body near the earth's surface. During ancient times and during the Middle Ages it was thought that heavy bodies fall more rapidly than light bodies, and, indeed, it is a matter of ordinary casual observation that a lead shot, a feather, a snowflake, and a water droplet fall through the air at different velocities. It remained for Galileo (1564–1642), who might be termed the first modern experimental physicist, to give a correct analysis of the problem. Galileo introduced the clearly defined concept of acceleration and, as a result of indirect experimental measurements, concluded that in vacuum all *freely falling* bodies would experience *the same constant acceleration.* This conclusion, which can easily be verified in a laboratory by modern experimental methods, was based on a series of experiments that were hampered by the fact that the only devices available for the measurement of time were crude water clocks, which had not been improved appreciably since their invention by the Egyptians and Babylonians thousands of years earlier. However, even under this handicap, Galileo arrived at a correct experimental relation describing the motion of a freely falling body.

> A freely falling body is one that experiences no force except that of the earth's gravity.

In principle, then, a body falls freely only in vacuum because otherwise air resistance acts on it. In practice, the motion of a dense body in air, if the speed is not too great, approximates closely that of a freely falling body; in this case, air resistance is small compared to the weight of the body.

The acceleration of a freely falling body, called the *acceleration of gravity,* is usually denoted by the symbol $\boldsymbol{g}$. At or near the earth's surface the magnitude of $\boldsymbol{g}$ is approximately 32.2 ft/sec^2, or 9.81 m/sec^2, and $\boldsymbol{g}$ is, of course, directed downward.* Since the acceleration of a freely falling body is constant, the equations of motion developed in Sec. 4 are immediately applicable.

We shall consider the motion as occurring along the y-axis of a rectangular coordinate system with *the positive y-axis pointing upward.* We shall denote the *magnitude* of the acceleration by g ($g \approx 32.2$ ft/sec^2 = 9.81 m/sec^2), so that the y-component of acceleration is $-g$. We shall choose the origin of our coordinate system at the initial position of the body, so that $y=0$ at $t=0$. The initial velocity will be denoted by v_0; this is positive if the initial velocity is upward, negative if it is downward. With this notation, equations (6), (11), and (12) of the preceding section give for the velocity v and position y at time t the relations

$$v=v_0-gt, \tag{13}$$

$$y=v_0t-\tfrac{1}{2}gt^2, \tag{14}$$

$$v^2=v_0^2-2gy. \tag{15}$$

In order to aid in visualizing the motion of a freely falling body released from rest, let us consider Fig. 8, which shows the positions and velocities of the body at various times after its release. In this case, since $v_0=0$, equations (13) and (14) assume the simple forms

$$v=-gt; \quad y=-\tfrac{1}{2}gt^2.$$

In the figure the positions of the body at successive 1-sec intervals after release are given by the dots on the y-axis and the velocities at corresponding times are given by the arrows.

As a second illustration, let us consider the problem of a ball thrown upward. Suppose that a man on the top of a building 300 ft high leans over the edge of the building and throws a ball vertically upward. If

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*** We shall point out in the next chapter that the acceleration of a body takes place in the direction of the resultant applied force acting on a body. As indicated in Chap. 2, the plumb line that defines the direction *down* indicates the direction of the gravitational force on the bob. Hence the acceleration produced by this gravitational force is always downward.**

the magnitude of the initial upward velocity is 64.4 ft/sec, let us find the position and velocity of the ball at successive 1-sec intervals after it leaves the man's hand and find the time required for the ball to reach the pavement on the ground at the side of the building. In this case, our equations for y and v become

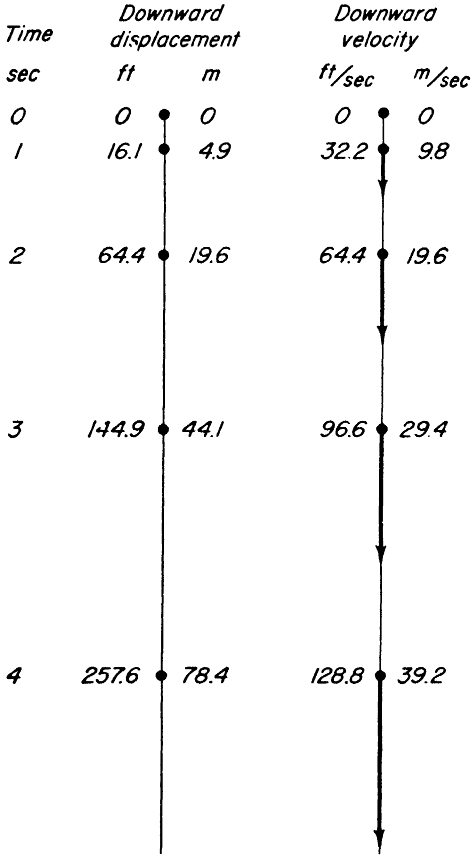

Fig. 8. Downward displacement and velocity of a freely falling body at the ends of successive one-second intervals after release from rest.

$$y = 64.4t - \tfrac{1}{2}\cdot 32.2t^2 = 64.4t - 16.1t^2 \text{ ft}, \tag{16}$$

and

$$v = 64.4 - 32.2t \text{ ft/sec}. \tag{17}$$

From these equations, the following table of positions and velocities can be obtained:

Time	*Position*	*Velocity*
0 sec	0 ft	+64.4 ft/sec
1	+48.3	+32.2
2	+64.4	0
3	+48.3	−32.2
4	0	−64.4
5	−80.5	−96.6
6	−193.2	−128.8
7	−338.1	−161.0

The values of position y and velocity v at various times are shown graphically in Fig. 9.

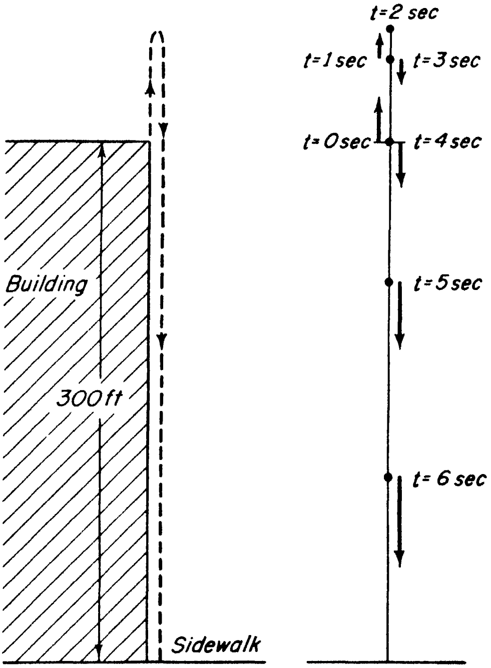

Fig. 9. The position and velocity of a ball at the end of successive one-second intervals after the ball is thrown upward from the top of a building at 64.4 ft/sec.

Two points might be noted in connection with the results shown in the table. One of these is that the ball thrown upward rose for two seconds and then required two seconds to return to its original position. This result is quite general; the time of rise and time of descent to the original level are equal whenever an object is thrown upward, provided the effect of air resistance can be ignored. The second point to be noted is that the downward velocity of the ball at the time it returns to its initial position is equal in magnitude to the initial upward velocity of the ball; this result is also quite general.

Now let us return to the problem of finding the time required for the ball to reach the ground at the side of the building. From the results in the table, we note that the time is between 6 and 7 seconds. In order to find the exact time, we substitute the value $y=-300$ ft in (16) and solve for t. For this value of y, (16) gives

$$-300=64.4t-16.1t^2,$$

or

$$16.1t^2-64.4t-300=0.$$

By solving this equation for t, we obtain the time required for the ball to reach the ground. This equation has two solutions: $t=6.76$ sec and $t=-2.76$ sec. Only the positive solution has any physical significance in the problem we are considering, so we may take 6.76 sec as the time required for the ball to reach the ground.

It should be re-emphasized that the discussion we have given holds strictly only for the idealization of a *freely falling* body that experiences no air resistance. The theory presented does form a good first-order approximation for dense compact bodies falling through the air. For example, the theory would be closely applicable to a lead shot or a golf ball dropped from moderate heights but would not be applicable to a feather or to an open parachute. The greater the velocity of a falling body, the poorer is the approximation given by the simple theory. For example, if a tennis ball were dropped from a great height, the approxi-

TABLE III

VALUES OF THE ACCELERATION OF GRAVITY*

At sea level

latitude	ft/sec²	m/sec²
0°	32.0878	9.78039
10°	32.0929	9.78195
20°	32.1076	9.78641
30°	32.1302	9.79329
40°	32.1578	9.80171
50°	32.1873	9.81071
60°	32.2151	9.81918
70°	32.2377	9.82608
80°	32.2525	9.83059
90°	32.2577	9.83217

At 40° latitude

altitude (ft)	ft/sec²	altitude (m)	m/sec²
0	32.1578	0	9.80171
500	32.1563	500	9.80017
1,000	32.1547	1,000	9.79864
2,000	32.1516	2,000	9.79554
4,000	32.1454	4,000	9.78937
8,000	32.1331	8,000	9.77702
16,000	32.1084	16,000	9.75233
32,000	32.0608	32,000	9.70296

* Values in this table are taken from *Smithsonian Physical Tables* (1934) and were calculated from the empirical formula of the United States Coast and Geodetic Survey.

mation would be good during the first few seconds of fall but would thereafter become worse; the tennis ball would eventually attain a constant downward velocity known as the *terminal velocity.* There are methods by which problems of this type can be handled, but a treatment of these is beyond the scope of this book.

In solving most engineering problems, we may use for the magnitude g the approximate value $g = 32.2\ \text{ft/sec}^2$ or $9.81\ \text{m/sec}^2$. It should be noted, however, that accurate measurements show a slight dependence of g on latitude and altitude. In Table III are given the accurate values of g for sea-level stations at various latitudes and for stations at various altitudes at 40° latitude. Even these values are only mean values, and there are local variations of smaller magnitude that depend on the character of the underlying rocks. Measurement of such local variations is a useful tool in geophysical prospecting for oil.

PROBLEMS

NOTE: In working these problems, take $g = 32.0\ \text{ft/sec}^2$ or $9.8\ \text{m/sec}^2$, and assume that air resistance is negligible.

1. A golf ball is dropped from the top of a tall building. Find the magnitude of the instantaneous downward velocity of the ball at the end of successive half-second intervals after its release, for the first three seconds.

Ans: 16.0 ft/sec, 32.0 ft/sec, 48.0 ft/sec, · · ·.

2. Find the magnitude of the downward displacements of the ball described in Prob. 1 at the end of successive half-second intervals for the first three seconds.

3. A baseball is batted vertically upward. If the initial upward velocity is 96 ft/sec, find the upward displacement and the magnitude and direction of the velocity at one-second intervals until the ball returns to its initial position.

Ans: $y = 0$, 80, 128, 144, 128, 80, 0 ft; $v = 96$, 64, 32, 0, −32, −64, −96 ft/sec.

4. An object is hurled vertically upward with an initial velocity of 49 m/sec. Find its velocity and upward displacement at the end of successive one-second intervals until the object returns to its initial position.

5. A man at the top of a tall building throws a ball vertically downward with an initial velocity of 40 ft/sec. Find the downward displacement of the ball at the end of two seconds. Ans: 144 ft.

6. If the man in Prob. 5 had given the ball an initial downward velocity of 10 m/sec, what would have been the downward displacement of the ball after 2 sec?

7. With what velocity must a ball be thrown vertically upward in order to rise to a height of 100 ft? How long will it remain in the air? What will be the downward acceleration of the ball at the highest point reached?

Ans: 80 ft/sec; 5.0 sec; 32 ft/sec^2.

8. With what velocity must a ball be thrown vertically upward in order to reach a height of 20 m? How long will it remain in the air? What will be its downward acceleration at the highest point reached?

9. If a bullet is fired vertically upward with an initial velocity of 1200 ft/sec, at what height will its velocity be reduced to 600 ft/sec? Ans: 16,900 ft.

10. What will be the velocity of the bullet of Prob. 9 at a height of 1 mile?

6. THE MOTION OF PROJECTILES

In cases where the effects of air resistance can be neglected, the paths or *trajectories* of projectiles can be determined by the methods we have outlined thus far, by considering separately the horizontal and the vertical components of the motion. The vertical acceleration experienced by a projectile as a result of the action of gravitational forces is not affected by horizontal motion of the projectile. For example, if as shown in Fig. 10 a shell is shot horizontally from a gun mounted at a height h above a horizontal plane and a second shell at height h is released at the instant the first projectile leaves the muzzle of the gun and allowed to fall freely, the two projectiles reach the ground at the same time. This result can be easily verified in the laboratory by suitable simple experimental arrangements.

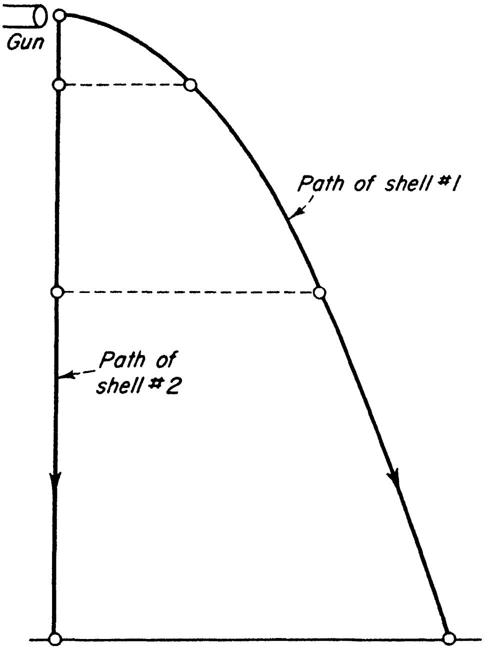

Fig. 10. A projectile fired horizontally reaches the ground at the same time as a freely falling projectile released at the instant of firing.

The *downward* component of the motion of a shell fired horizontally is similar in all respects to the motion of the shell which was allowed to fall freely. Therefore, so far as *vertical* motion is concerned, both shells can be treated as freely falling bodies. The equations previously developed to describe the motion of falling bodies are therefore applicable to the *vertical component* of the motion of the projectile fired from the gun.

This fact can be used to determine the *range* or horizontal distance traversed by the projectile fired horizontally from the gun, provided we know the muzzle velocity of the projectile. In determining the range let us place the origin of our coordinate system at the muzzle of the gun shown in Fig. 11. The projectile will reach the ground when its y-coordinate is equal to $-h$. Substituting this value in the equation

$$y=-\tfrac{1}{2}gt^2,$$

we obtain the equation

$$h=\tfrac{1}{2}gt^2.$$

From this equation, we may determine the time t required for the projectile to reach the ground. This time t is

$$t=\sqrt{2h/g}.$$

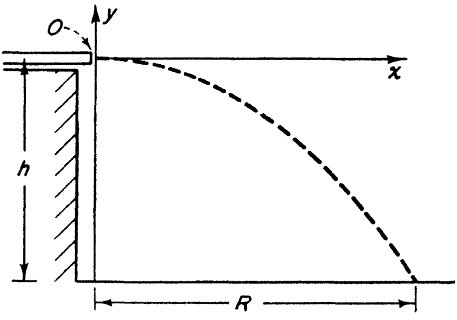

Fig. 11.

Analysis of the horizontal motion of a projectile is quite simple, since it is found that the horizontal component of the velocity is *constant.* In the case we are considering, v_x, the constant horizontal component of the velocity, is equal to the muzzle velocity v_0. Therefore the horizontal component x of the total displacement is given by the equation

$$x=v_0\cdot t.$$

Therefore the *range* R, which in this case is the maximum value of x attained during flight, is given by substituting the value $t=\sqrt{2h/g}$ of the time of flight in this equation; that is,

$$R=v_0\sqrt{2h/g}.$$

The velocity $\mathbf{v}$ of the shell at any time t during the flight can be determined from the magnitudes of its horizontal and vertical components, which are given by the relations

$$v_x=v_0;\quad v_y=-gt.$$

The magnitude of the resultant velocity is

$$v=\sqrt{v_x^2+v_y^2}=\sqrt{v_0^2+g^2t^2},$$

and the direction given by the angle θ measured from the horizontal is

$$\theta = \arctan(v_y/v_x) = \arctan(-gt/v_0),$$

as shown in Fig. 12. The negative value of the tangent indicates that θ is below the horizontal.

Let us now consider the more general problem of a projectile fired with muzzle velocity of magnitude v_0 at angle θ_0 above the horizontal. This situation is depicted in Fig. 13, in which a gun with muzzle located at ground level fires a shell that subsequently follows the path indicated by the dotted curve, which we desire to describe. The initial velocity has components

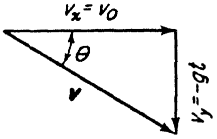

Fig. 12.

$$v_{x0} = v_0 \cos\theta_0; \quad v_{y0} = v_0 \sin\theta_0.$$

After the shell leaves the gun, the horizontal component v_x of the velocity remains constant and is at all times given by

$$v_x = v_{x0} = v_0 \cos\theta_0.$$

The horizontal motion occurring at constant velocity is described by the relation

$$x = v_{x0}t, \tag{18}$$

where x gives the horizontal position at time t in the coordinate system shown in Fig. 13.

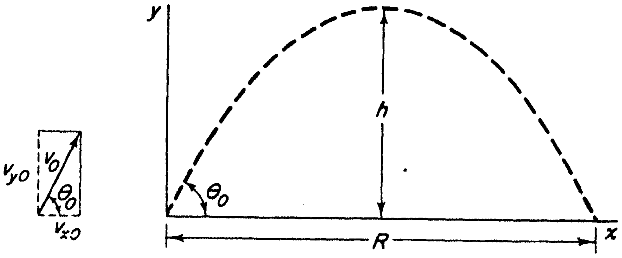

Fig. 13. Path of a projectile fired with muzzle velocity of magnitude v_0 at elevation angle θ_0.

Since the shell experiences a downward acceleration of magnitude g, the vertical component of the velocity changes in the same manner as that described for a ball thrown vertically upward. The vertical velocity component v_y at any time t is given by

$$v_y = v_{y0} - gt, \tag{19}$$

and the y-position of the shell in the coordinate system shown in Fig. 13 is given by

$$y = v_{y0}t - \tfrac{1}{2}gt^2. \tag{20}$$

The equation

$$v_y^2 = v_{y0}^2 - 2gy, \tag{21}$$

which can be derived from (19) and (20) by elimination of t, is also useful.

Equations (18) and (20) are the parametric equations for a *parabola,* which is the type of trajectory followed by the shell. These equations together with the associated velocity equations describe the motion completely. By means of these equations such quantities as the total time of flight, the range R, the maximum height h attained, and the velocity of the projectile at any time can be calculated. Determination of these quantities will be left as problems.

PROBLEMS

NOTE: In these problems, take $g = 32.0$ ft/sec² or 9.8 m/sec² and assume that air resistance is negligible.

1. A shell is fired horizontally from a gun located 64 ft above a horizontal plane. The muzzle velocity of this shell is 1000 ft/sec. How long does the shell remain in the air? What is its range? Ans: 2 sec; 2000 ft.

2. If the range attainable with the horizontally fired gun in Prob. 1 is to be 3000 ft, what should be the location of the gun above the horizontal plane?

3. A gun pointed 30° above the horizontal fires a shell with a muzzle velocity of 640 ft/sec. If the muzzle of the gun is at ground level, find the position of the projectile at the ends of successive one-second intervals after the gun is fired, until the shell returns to earth. Plot a curve showing the path of the shell.
Ans: $x = 554$ ft, 1108 ft, · · ·; $y = 304$ ft, 576 ft, · · ·.

4. Find the positions of the shell in Prob. 3 at the ends of successive two-second intervals if the gun is pointed 60° above the horizontal, and plot a curve showing the path of the shell.

5. Find the vertical and horizontal components of the velocity of the shell in Prob. 3 at the ends of successive one-second intervals.
Ans: $v_x = 554$ ft/sec (constant); $v_y = 288$ ft/sec, 256 ft/sec, · · ·.

6. Find the vertical and horizontal components of the velocity of the shell in Prob. 4 at the ends of successive two-second intervals.

7. From the parametric equations (18) and (20) for the path of a projectile, derive a single equation in x and y for the path. Ans: $y = (v_{y0}/v_{x0})x - \tfrac{1}{2}(g/v_{x0}^2)x^2$.

8. Find the maximum height attained by a projectile if the vertical component of the initial velocity is v_{y0}.

9. Derive an expression giving the range R of a shell fired from ground level on a level plain with muzzle velocity v_0 at an angle θ_0 above the horizontal.
Ans: $R = (v_0^2 \sin 2\theta_0)/g$.

10. By differentiation of the equation derived in Prob. 9, find the elevation angle θ_0 for which the range is a maximum.

11. A baseball player throws a baseball with a velocity of magnitude 96 ft/sec in a direction 30° above the horizontal. If the baseball leaves the thrower's hand at a height of 6 ft above the field, find the maximum height attained and the horizontal distance traversed by the ball before it strikes the ground. Ans: 42 ft; 259 ft.

12. If the ball in Prob. 11 had been thrown at an angle of 45° above the horizontal, what would have been the maximum height attained and the horizontal distance traversed?

13. During the First World War, the Germans bombarded Paris with a specially constructed long-range gun (Big Bertha). The length of the gun barrel was approximately 120 ft and the muzzle velocity was approximately 4800 ft/sec. Employing the simple theory we have developed and using an elevation angle $\theta_0 = 45°$, calculate the range attained in the absence of air resistance. (The range actu lly attained was about 132,000 yd with $\theta_0 = 55°$.) Ans: 240,000 yd.

14. Calculate the acceleration (assumed constant) of the projectile in passing through the gun barrel and also the maximum height attained by the projectile in Prob. 13.

15. The shell fired from a trench mortar has a muzzle velocity of 300 ft/sec. Calculate two angles of elevation at which the mortar can be fired in order to hit a target 1500 ft away and at the same level as the mortar. Calculate the height and time of flight attained in 'direct' and 'indirect' fire at the target.

Ans: $\theta_0 = 16\overset{\circ}{.}1$ and $73\overset{\circ}{.}9$; $h = 108$ ft and 1298 ft; $t = 5.20$ sec and 18.0 sec.

16. Calculate the maximum range of the trench mortar in Prob. 15 for a target at the same level as the mortar.

17. In Fig. 14, a gun is 'bore-sighted' on a target. At the instant the projectile leaves the muzzle of the gun, the target starts to fall as a freely falling body. Show that for any angle of elevation of the gun, any muzzle velocity, and any distance to the target, the projectile will always hit the target if the effect of air resistance is negligible.

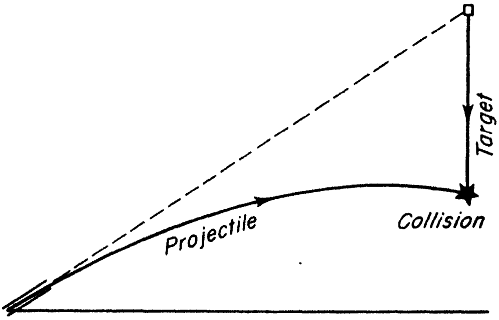

Fig. 14.

7. MOTION IN A CIRCLE AT CONSTANT SPEED

In the examples of accelerated motion considered thus far, the acceleration has resulted in a change in the *magnitude* of the velocity of a body. This is not always the case. Since velocity is a vector quantity, it is possible for acceleration to result in a change in velocity by causing a change in its *direction* without an accompanying change in its *magnitude*. This is the case when a particle moves at constant speed in a circular path.

Consider a body moving at constant speed v in a circular path of radius R as shown in Fig. 15(a). Let us consider the motion of the particle from point a to point b in time Δt. The particle's initial and final displacements $\boldsymbol{R}_a$ and $\boldsymbol{R}_b$, relative to the center O, are shown in Fig. 15(b). During the time Δt required for the particle to move from a to b, the body experiences a change in displacement $\Delta\boldsymbol{s}$, equal in magnitude to the chord of the circle joining points a and b. In moving from point a to b, the particle does not, of course, pass along the *chord* but along the *arc* of length $\Delta l = v\,\Delta t$.

Now let us consider the velocity-vector diagram shown in Fig. 15(c). Here $\boldsymbol{v}_a$ and $\boldsymbol{v}_b$ represent instantaneous velocities of the particle at points a and b, respectively, and $\Delta\boldsymbol{v}$ represents the change in the velocity of the particle in the time interval Δt. The magnitudes of $\boldsymbol{v}_a$ and $\boldsymbol{v}_b$ are both equal to v.

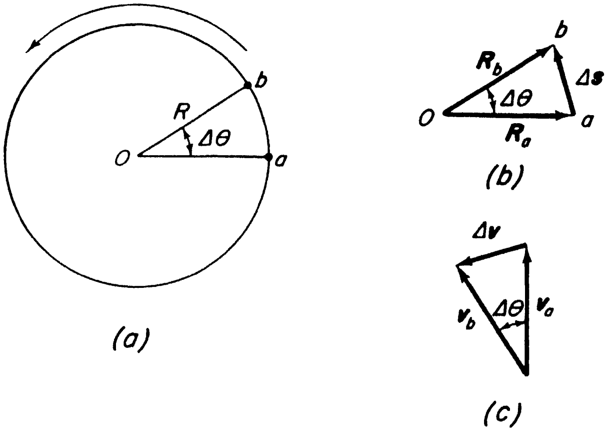

Fig. 15. (a) Particle moving at constant speed in a circular path of radius R.

(b) Displacement diagram: $\boldsymbol{R}_a$ and $\boldsymbol{R}_b$ give displacements of particle from O when particle is at points a and b respectively.

(c) Velocity diagram: $\boldsymbol{v}_a$ and $\boldsymbol{v}_b$ give velocities of particle at points a and b respectively.

In circular motion the instantaneous velocity is always perpendicular to the radius vector giving the displacement of the particle from the center. Hence, in the vector diagrams in (b) and (c), $\boldsymbol{v}_a$ is perpendicular to $\boldsymbol{R}_a$ and $\boldsymbol{v}_b$ is perpendicular to $\boldsymbol{R}_b$, hence the angle between $\boldsymbol{v}_a$ and $\boldsymbol{v}_b$ is equal to the angle $\Delta\theta$ between $\boldsymbol{R}_a$ and $\boldsymbol{R}_b$. Further, since $v_a = v_b = v$ and $R_a = R_b = R$, the velocity triangle and the displacement triangle are similar isosceles triangles. Thus, $\Delta\boldsymbol{v}$ is perpendicular to the chord $\Delta\boldsymbol{s}$ and therefore is directed toward the center of the circle. Since $\bar{\boldsymbol{a}} = \Delta\boldsymbol{v}/\Delta t$, where $\bar{\boldsymbol{a}}$ is the average acceleration during time Δt, the average acceleration during any time interval Δt is also directed toward the center of the circle.

Since the instantaneous-acceleration vector is the limit of the average-acceleration vector as $\Delta t \rightarrow 0$, *the instantaneous-acceleration vector is directed toward the center of the circle at every point in the motion.*

To obtain the magnitude of the instantaneous acceleration, we note that *if we let Δv represent the magnitude of $\Delta \mathbf{v}$ in Fig.* 15,* we obtain, from similar triangles,

$$\frac{\Delta v}{v}=\frac{\Delta s}{R}, \quad \text{or} \quad \Delta v=\frac{v}{R}\,\Delta s.$$

If we divide both sides of this equation by Δt, we obtain

$$\frac{\Delta v}{\Delta t}=\frac{v}{R}\,\frac{\Delta s}{\Delta t}.$$

We now go to the limit as $\Delta t \rightarrow 0$. Since v and R are constants, this gives

$$\lim_{\Delta t\rightarrow 0}\frac{\Delta v}{\Delta t}=\frac{v}{R}\lim_{\Delta t\rightarrow 0}\frac{\Delta s}{\Delta t}.$$

But the limit on the left is just the magnitude a of the instantaneous acceleration, whereas the limit on the right is just the speed v. Hence,

$$a=\frac{v^2}{R}. \tag{22}$$

A particle moving in a circle of radius R at constant speed v has acceleration $\mathbf{a}$ *which at each instant is directed toward the center of the circle and has magnitude v^2/R.*

This is called the *centripetal* ('center-seeking') acceleration. It is this acceleration that accomplishes the continuous turning of the velocity vector in direction, without change of speed.

In the more general case of motion in a circle with variable speed v, we shall show in Chap. 8 that the acceleration vector has two components: a centripetal component of magnitude v^2/R, which accomplishes the turning of the velocity vector; and a component tangent to the circle of magnitude equal to the rate of change of speed, which accomplishes the change in magnitude of the velocity vector.

PROBLEMS

1. A ball at the end of a string is whirled at constant speed in a horizontal plane. If the radius of the circle is 4 ft and the speed of the ball is 10 ft/sec, calculate the magnitude of the centripetal acceleration. Ans: 25 ft/sec^2.

2. If the radius of the circle in Prob. 1 were increased to 12 ft without changing the speed of the ball, what would be the magnitude of the centripetal acceleration?

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* We emphasize that we are letting Δv represent the magnitude of the change in velocity—not the change in speed, as this symbol might also be construed to imply. The change in speed and the magnitude of the change in velocity are quite different quantities, as we see from our present example, where the change in speed is identically zero, but the magnitude of the change in velocity is not at all zero.

3. A car moves at a speed of 60 mi/hr around a curve of 500-ft radius. What is the acceleration of the car? Ans: 15.5 ft/sec^2 toward the center of curvature.

4. If the speed of the car in Prob. 3 were reduced to 45 mi/hr, what would be the acceleration of the car?

5. The armature of an electric motor has a diameter of 2 ft and turns at 3000 rev/min. Calculate the magnitudes of the velocity and the acceleration of a point on the periphery of the armature. Ans: 314 ft/sec; 98,700 ft/sec^2.

6. The flywheel on a stationary steam engine is 4 ft in diameter and makes 600 rev/min. Calculate the magnitudes of the instantaneous velocity and acceleration of a point on the rim of the flywheel.

CHAPTER 6

DYNAMICS

In Chap. 3 we stated Newton's first and third laws of motion and described a method by which forces could be measured by means of a properly constructed and calibrated spring balance. These two laws of motion, together with this operational definition of force, were adequate for the treatment of problems encountered in statics; however, it will be recalled that statics dealt only with bodies at rest or moving with constant velocity. As we noted in the preceding chapter, the motion of a body generally involves *acceleration.* In order to discuss the influence of forces in producing acceleration, it is necessary to define more clearly what we mean in general by the term *force;* this definition is given by Newton's first and second laws of motion. The three Newtonian laws of motion provide the basis of classical *dynamics,* which is the study of motion in terms of the forces that produce it. The system of dynamics based on Newton's three laws of motion proves to be adequate for the treatment of all problems ordinarily encountered in engineering.

Since Newton's laws of motion play such an important role in the development of scientific theory and engineering practice, we shall discuss in detail the reasoning involved in their formulation. Briefly, we can say that *the first step was experimental,* involving observation of the motion of interacting bodies; the directly observable kinematic factors were *positions* and *times,* from which velocities and accelerations could be calculated. From accelerations occurring during interactions between pairs of bodies, Newton showed that it is possible to attribute to every body a property called its *mass.* Newton found it desirable to describe the observed interaction in terms of the three quantities *mass, length,* and *time,* which we call *fundamental quantities. The second step involved the formulation of a set of relations* giving a consistent account of the experimental observations. These relations involve certain assumptions and the introduction of a fourth quantity called *force,* which is *defined* in terms of the three fundamental quantities. The three necessary relations constitute Newton's laws of motion.

It should be noted that inasmuch as Newton's second law *defines* the relationship between force and the three fundamental quantities, it cannot be *proved* either experimentally or in terms of more fundamental relationships. The first law and (given the definition of force embodied in the second law) the third law are consequences of experimental observations. Newton formulated his laws in order to give a single coherent account of experiments actually performed. In Newton's own words, "These laws must be considered as resting on convictions drawn from observation and experiment, not on intuitive perception." The real justification or 'proof' of Newton's laws is the success with which they and the relationships based upon them can be applied to the practical problems encountered in physics and engineering. Their publication occurred in 1687, and it was not until the beginning of the twentieth century that types of problems occurred which could not be handled by the classical dynamics based upon Newton's laws. One of these types involves motion at high velocities, comparable to the velocity of light (186,000 mi/sec), and must be treated by *relativity theory;* the other type involves atomic and subatomic particles, to which *quantum mechanics* must be applied. It should be pointed out that relativity theory and quantum mechanics both reduce to classical Newtonian mechanics when applied to bodies such as automobiles, bullets, baseballs, and locomotives, moving at the velocities involved in ordinary engineering applications. We can safely assume that Newton's laws are applicable to the problems we shall discuss in this and most of the following chapters.

1. NEWTON'S FIRST LAW OF MOTION

We are led to conclude as a result of everyday experience that a body at rest will remain at rest unless it interacts in some fashion with another body. If an interaction with another body does occur, the body may be set in motion, but when *left to itself* it cannot set itself in motion.

By a series of experiments with small spherical bodies rolling on a horizontal table top and with other material bodies, Galileo and Newton were led to make the following postulate in regard to the motion of a particle:

> NEWTON'S FIRST LAW: *A particle left to itself will maintain its velocity unchanged.*

This postulate, known as *Newton's first law,* includes the case of a stationary particle, whose velocity remains zero as long as the particle is 'left to itself'; here the postulate seems obvious. In the case of moving particles the postulate is an idealization from experimental observations. For example, a marble rolling across a rug quickly comes to rest as a result of interaction with the rug; the same marble rolling across a bare floor will move for a much longer time before coming to rest because the interaction between the marble and floor is smaller than the interaction between the marble and the rug; in neither case is the marble really 'left to itself.' In no way is it possible to perform an experiment in which a body is left completely to itself, and hence, in the case of a moving object, it is clear that the above postulate is a pure idealization which cannot be realized in an experiment; the same is true in the case of a stationary particle.

Let us now introduce the concept of *force* in the manner it was introduced in the statement of the first law given in Chap. 3.

> *Every body will remain in a state of rest or of uniform motion unless acted on by external forces.*

Comparison of this statement with the one given above reveals two implications: (1) *external forces* act on a body as a result of *interactions* with other bodies, and (2) *external forces* acting on a body tend to *produce acceleration.* It should be noted that the first law leaves the quantitative relationship between the external force and the acceleration undefined. This relationship is given by the second law.

PROBLEMS

1. A marble lies at rest on the horizontal top of a large table. Is the marble *left to itself* in Galileo's sense of the term? If not, what external forces act on the marble?

2. A marble is set in motion along the horizontal floor of a hallway. After moving 30 ft, the marble comes to rest. Do external forces act on the marble? If so, list them and discuss their origins.

2. MASS

The property of a body by which it maintains a constant velocity when 'left to itself' is called *inertia.* Newton, in formulating his laws, found it desirable to introduce *mass* as a measure of inertia.

Whenever a given particle experiences an *acceleration*, the acceleration must be attributed to the influence of other particles; this follows from the first law. The other particles are said to *exert forces* on the particle under observation; as a result of these *external forces,* the particle is accelerated. In studying acceleration, Newton decided that the simplest procedure was to study the acceleration produced by interactions between *pairs* of particles. Using a pair of small spheres moving on a table top to approximate particles, Newton and others noted that *whenever there was an acceleration of one particle, there was also an acceleration of the other particle.* The accelerations are found to be oppositely directed, as shown in Fig. 1.

As to the magnitudes of the two accelerations, $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$. Newton concluded *as a result of his observations* that their ratio was always a constant for a given pair of particles, provided there were no interactions with other bodies. This conclusion is found to be valid regardless of the type of interaction between the bodies studied; whether the interaction is of elastic, electrostatic, or gravitational character, the ratio of the accelerations experienced by the two interacting bodies is always the same, provided interactions with other bodies can be ignored. As we have noted before, complete isolation of a single body is impossible of experimental realization; similarly, *two* interacting bodies cannot be *completely* isolated from all other bodies, and hence Newton's conclusion involves an idealization of the experiments actually performed.

All evidence points to the fact that Newton's conclusion is correct. This conclusion may be stated in the form

$$a_2/a_1 = \text{constant} = R_{12},$$

where the ratio R_{12} has a definite constant value for any two particles, 1 and 2, as shown in Fig. 1. Newton concluded further that the accelerations experienced during an interaction were always oppositely directed. Thus, a vector equation may be written

$$\boldsymbol{a}_2 = -R_{12}\,\boldsymbol{a}_1.$$

where the constant R_{12} is a positive number. In interpreting R_{12}, Newton proposed to assign a number m_1 to one particle and to define a corresponding number m_2 for the other particle by the equation

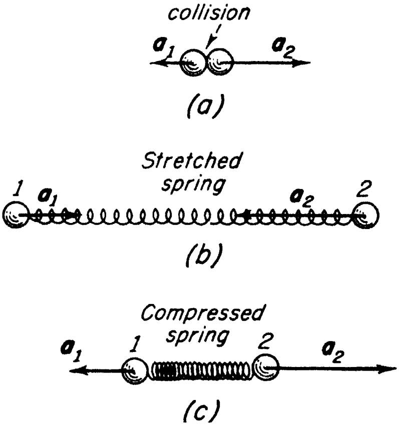

Fig. 1. Whenever two particles interact, their accelerations are oppositely directed and have a constant magnitude ratio. Since the two accelerations shown above have different magnitudes, the balls must be assumed to be of different materials, such as lead and wood.

$$R_{12} = m_1/m_2.$$

Therefore,

$$\frac{a_2}{a_1} = \frac{m_1}{m_2},$$

or

$$m_2 = (a_1/a_2)m_1, \tag{1a}$$

where the acceleration ratio (a_1/a_2) is determinable from experimental observation. In this manner, if a number m_1, called the *mass*, is arb
trarily assigned to particle 1, the value m_2 of mass to be assign
particle 2 is determined. Similarly, by an experiment in which pa
is made to interact in any manner with a third particle 3, a mas
mined for 3 by the relation

$$m_3 = (a_1/a_3)m_1.$$

In this manner a value of mass may be assigned to any pa
of the mass assigned to the standard particle 1.

We now need to introduce another generalization f that was made by Newton. If particles 2 and 3 are assigned masses m_2 and m_3 by experiments in which first one and then the other interacts with particle 1, and then particles 2 and 3 are made to interact with each other in any manner, the same values of masses may be used to determine the ratios of the accelerations in this interaction. The relation is

$$\frac{a_3}{a_2}=\frac{m_2}{m_3} \quad \text{or} \quad m_2a_2=m_3a_3.$$

This experimental fact shows that the mass values determined in terms of a standard particle to which an arbitrary mass is assigned are universally

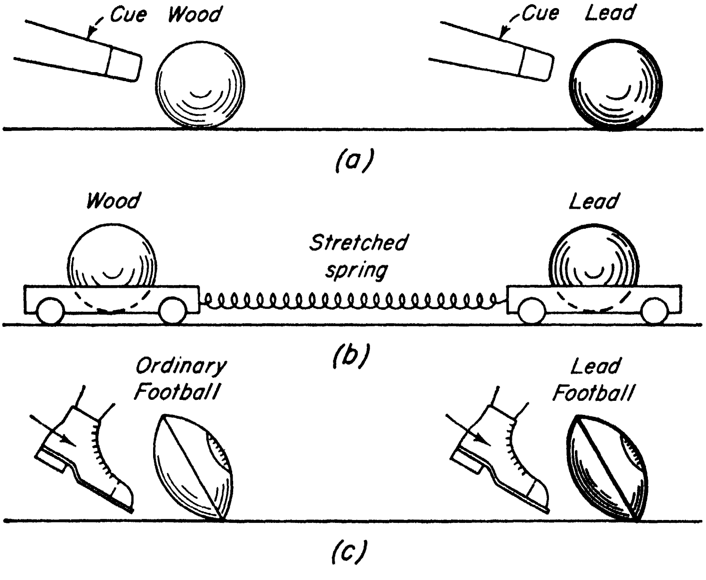

Fig. 2. Comparison of masses.

applicable in determining the ratios of the accelerations in the interaction between any two particles.

The mass is said to be a measure of the inertia of a particle since in the ation $m_1a_1=m_2a_2$ the particle of greater mass has the lesser acceleration interaction. The simple demonstrations indicated in Fig. 2(a) e this equation. We might place on a billard table two balls of diameter but composed of different materials, say wood and mpts to set the balls in motion with the cue make it evident that the mass (inertia) of the lead ball is greater than that of all, since the lead ball has a greater tendency to remain at e difficult to accelerate than the wooden ball. In a sense, s a 'standard body' to interact with the two balls. We might compare the masses of the two balls by a more direct interaction between them with the arrangement shown in Fig. 2(b). We place the two balls in small cars connected by a light spring. If the cars are separated by stretching the spring and then released, the acceleration of the wooden ball will be greater than that of the lead ball; again we conclude that the mass of the lead ball is greater. A third demonstration we might perform is indicated in Fig. 2(c). Here we have an ordinary football and another football made of lead; a kicker would have no difficulty in telling which ball has the greater mass.

We shall describe later more convenient and more accurate methods of comparing masses. However, we might point out now that equation (1) enables us to determine the mass m of any body in terms of the mass m_1 of some standard body. It is, of course, important that some one standard mass should be selected as a unit in terms of which all other masses can be measured. The *standard unit mass is called the kilogram* (kg), and is the mass of a standard cylinder of platinum-iridium kept at the International Bureau of Weights and Measures at Sèvres, France. This unit mass and the *gram* (g) mass, which is 0.001 kg, are the units used in scientific work. The mass unit commonly used in English-speak-

TABLE I

Typical Densities of Liquids and Solids at 68° F

Liquids	kg/m^3	lb/ft^3	Solid metals	kg/m^3	lb/ft^3
Water (32–50° F).....	1,000	62.43	Aluminum....	2,700	169
Sea water............	1,030	64.4	Cast iron............	7,200	449
Benzene.............	879	54.9	Copper..............	8,890	555
Carbon tetrachloride..	1,594	99.5	Gold................	19,300	1205
Ethyl alcohol.........	789	49.3	Lead................	11,340	708
Gasoline.............	680	42.5	Magnesium..........	1,740	109
Kerosene............	800	49.9	Nickel..............	8,850	553
Lubricating oil........	900	56.2	Silver...............	10,500	656
Methyl alcohol.......	792	49.4	Steel................	7,800	487
100% sulfuric acid....	1,831	114.3	Tungsten...........	19,000	1190
Turpentine....	873	54.5	Zinc................	7,140	446
Mercury (32° F)......	13,595	848.7	Brass or bronze......	8,700	543

Nonmetallic solids	kg/m^3	lb/ft^3	Woods	kg/m^3	lb/ft^3
Ice (32° F)..........	922	57.5	Balsa...............	130	8
Concrete.........	2300	144	Pine...............	480	30
Earth, packed........	1500	94	Maple..............	640	40
Glass................	2600	160	Oak................	720	45
Granite..............	2700	169	Mahogany..........	560	35

ing countries is the *pound* (lb), which in the United States is defined as 453.5924277 kg; the *ounce* (oz) = 1/16 lb and the *ton* = 2000 lb are also frequently used in commercial and engineering transactions. It should be carefully noted that the *kilogram* (kg) and the *pound* (lb) are *mass units;* they should not be confused with the *kilogram-force* (kgf) and the *pound-force* (lbf) defined in Chap. 2, which are *force units.* The relations between these units will be discussed later.

An important quantity associated with matter is its *density:*

Density is defined as mass per unit volume.

Data on the densities of various materials are given in Table I. The relationship between the density of a material and the specific weight of a material (see Chap. 4) will become apparent later in the present chapter.

PROBLEMS

1. A 1-lb body collides with a second body of unknown mass. From photographs of the collision taken with a high-speed motion-picture camera, the magnitudes of the accelerations of the bodies during collision can be determined. At a certain instant the magnitude of the acceleration of the 1-lb body is 12 ft/sec^2 and that of the second body is 2 ft/sec^2. What is the mass of the second body? Ans: 6 lb.

2. If a 1-kg body experiences an acceleration of magnitude 2 m/sec^2 when colliding with a second body and if the magnitude of the acceleration experienced by the second body is 5 m/sec^2, what is the mass of the second body? What can be said concerning the *directions* of the accelerations experienced by the two bodies?

3. A 4-lb body and a 3-lb body are connected by a stretched light spring as shown in Fig. 1(b). If the two bodies are released simultaneously and the initial acceleration of the 4-lb body is 5 ft/sec^2 eastward, what is the initial acceleration of the 3-lb body? Ans: 6.67 ft/sec^2 westward.

4. If the 3-lb body in Prob. 3 were replaced by an 8-lb body, what would be its initial acceleration if the acceleration of the 4-lb body were 5 ft/sec^2 eastward?

5. During an interaction between a 12-kg body and a 4-kg body, the 12-kg body experiences an eastward acceleration of 2.5 m/sec^2. What is the acceleration of the 4-kg body? Ans: 7.5 m/sec^2 westward.

6. A 10-kg cannonball is dropped from the top of a large building and experiences a downward acceleration of 9.81 m/sec^2. What interaction produces this acceleration? Does the other body involved in the interaction experience an acceleration? Is the acceleration of the other body observable? Discuss.

3. NEWTON'S SECOND LAW

When two particles interact in such a way that they are accelerated. they are said to exert *forces* on each other. The product $\boldsymbol{ma}$ is taken as the measure of the force, except for an arbitrary constant factor that defines the size of the particular force unit used. *This is the fundamental definition of force.* It is convenient because the above discussion shows that, when two particles interact, $m_1\boldsymbol{a}_1 = -m_2\boldsymbol{a}_2$, so the two particles exert equal and opposite forces on each other.

Fundamentally, all forces are exerted by individual particles on each other. In practice, a given particle may experience forces from a large number of other particles simultaneously, and we need to know how these forces compound. Here we have recourse to a further generalization that Newton made from experimental observation. Suppose that three particles, 1, 2, and 3, are simultaneously interacting. Suppose that if particle 3 were not present, particle 1 would experience acceleration $\boldsymbol{a}_{12}$ owing to particle 2 alone, and that if particle 2 were not present, 1 would have acceleration $\boldsymbol{a}_{13}$ owing to 3. Then it is concluded from experience that with all three particles present, particle 1 will have acceleration

$$\boldsymbol{a}_1=\boldsymbol{a}_{12}+\boldsymbol{a}_{13},$$

the vector sum. If we multiply this equation through by m_1, we obtain the relation

$$m_1\boldsymbol{a}_1=m_1\boldsymbol{a}_{12}+m_1\boldsymbol{a}_{13}.$$

Except for a constant common factor involved in the choice of the force unit, $m_1\boldsymbol{a}_{12}$ is $\boldsymbol{F}_{12}$, the force exerted by 2 on 1; $m_1\boldsymbol{a}_{13}$ is $\boldsymbol{F}_{13}$, the force exerted by 3 on 1; and $m_1\boldsymbol{a}_1$ is *defined* as the *resultant* force $\boldsymbol{F}_1$ acting on particle 1. Hence with this definition, forces compound vectorially and the preceding equation gives

$$\boldsymbol{F}_1=\boldsymbol{F}_{12}+\boldsymbol{F}_{13}.$$

Thus the resultant force acting on a particle is defined as the vector sum of the various forces exerted on that particle by other particles.

The cases we have considered in the chapters on statics are those where the resultant force is zero. If the resultant force $\boldsymbol{F}$ is not zero the particle of mass m experiences an acceleration $\boldsymbol{a}$ given by

$$k\boldsymbol{F}=m\boldsymbol{a}, \tag{2}$$

where we have explicitly written the constant k to permit various choices of size of the force unit. From equation (2) we can obtain the proportionality relation

$$\boldsymbol{a}\propto\frac{\boldsymbol{F}}{m} \tag{3}$$

which, stated in words, gives

> NEWTON'S SECOND LAW: *The acceleration of a particle is directly proportional to the resultant external force acting on the particle, is inversely proportional to the mass of the particle, and has the same direction as the resultant force.*

Figure 3 illustrates the relationship between the force and the acceleration. It is seen that Newton's first law is really a special case of the second law in which the force, and hence the acceleration, vanishes.

In equation (2), we are free to choose any value of the constant k we please. Different choices of k correspond to different choices of the size of the unit in which F is measured. The simplest choice, $k=1$, gives the simple relation

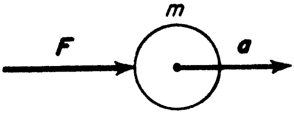

Fig. 3. The resultant external force $\boldsymbol{F}$ acting on a body has the same direction as the resulting acceleration $\boldsymbol{a}$. The magnitude of the acceleration is proportional to the magnitude of the external force.

$$\boldsymbol{F}=m\boldsymbol{a}, \tag{4}$$

according to which unit force would be defined as the force that will give unit acceleration to a unit mass.

In the MKS system of units, the unit of length is the meter, the unit of mass is the kilogram, the unit of time is the second, and the unit of force defined by equation (4) is the *newton* (nt).

> A newton is that force which imparts to a one-kilogram mass an acceleration of one meter per second per second.

The relative sizes of the *newton*, the *pound-force* (lbf), and the kilogram force (kgf) were given in Chap. 2. The newton is called an *absolute unit* of force because its value is defined in terms of its effect on an object quite independently of the position of the object in the universe.

We note that the name *newton* is an abbreviation for *kg·m/sec²:*

$$1 \text{ nt} = 1 \text{ kg·m/sec}^2. \tag{5}$$

In solving dynamical problems in metric absolute units, forces will come out initially in kg·m/sec^2, and this fact can be used as a check on the correctness of the work. *Force* is a derived quantity of physical dimensions *mass* × *length/time*^{2}.

Just as the newton is defined as the absolute unit of force in the MKS system, so also we might define an absolute unit of force for the British FPS (foot, pound, second) system of units by means of equation (4).* The resulting force unit (called the *poundal*) is inconveniently small and is not widely used; gravitational units are generally used in engineering applications and will be used in this text when British units are used. The gravitational force units will be defined in Sec. 5 of this chapter.

The *weight* of a body is by definition the force exerted on the body in its gravitational interaction with the earth. Since this force is sufficient to give the body an acceleration $\boldsymbol{g}$, the weight is

$$\boldsymbol{W}=m\boldsymbol{g},$$

in nt if m is the mass of the body in kg and g is in m/sec^2.

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* A more complete discussion of the various possible systems of units is given in the Appendix.

PROBLEMS

1. A 5-kg body experiences an upward acceleration of 2 m/sec^2. What resultant external force acts on the body? Ans: 10 nt upward.

2. A 10-kg body experiences an acceleration of 20 cm/sec^2 eastward. What resultant external force acts on the body?

3. A resultant downward force of 4 nt acts on a body of mass 3 kg. What acceleration is experienced by the body? Ans: 1.33 m/sec^2 downward.

4. If a resultant external force of 8 nt acts on a body of mass 2 kg, what is the magnitude of the acceleration experienced by the body? What can be said concerning the direction of the acceleration?

5. A resultant horizontal force of 10 nt acts on a certain body, which experiences an acceleration of 2 m/sec^2 eastward. What is the direction of the resultant force? What is the mass of the body on which the force is exerted? Ans: eastward; 5 kg.

6. A resultant force of magnitude 2 nt exerted on a certain body produces an acceleration whose magnitude is 10 m/sec^2. What is the mass of the body?

7. What is the magnitude of the gravitational force acting on a 9-kg body at a place where the gravitational acceleration is 9.81 m/sec^2? Ans: 88.3 nt.

8. What is the magnitude of the gravitational force acting on a 2-kg body at a place where the acceleration of gravity is 9.79 m/sec^2?

9. Referring to Table III of Chap. 5, p. 110, find the weight of a 1-kg body at sea level at the following latitudes: 0°, 40°, 60°, and 90°.
Ans: 9.78039 nt; 9.80171 nt; 9.81918 nt; and 9.83217 nt.

10. Using Table III of Chap. 5, p. 110, find the weight of a 2-kg mass at 40° N. latitude at the following heights above sea level: 0 m, 500 m, 1000 m, and 2000 m.

4. NEWTON'S THIRD LAW

Now that we have defined force by Newton's second law, let us return to the relationship

$$m_1\boldsymbol{a}_1 = -m_2\boldsymbol{a}_2 \tag{6}$$

that Newton used in comparing masses. By the second law,

$$\boldsymbol{F}_1 = m_1\boldsymbol{a}_1 \tag{7}$$

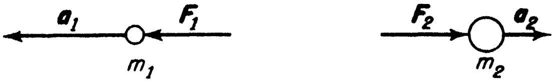
is the *resultant force acting on the particle of mass* m_1. Since equation (6) assumes that only two particles are involved, we may say that $\boldsymbol{F}_1$ *is the force exerted on particle* 1 *by particle* 2. Similarly,

$$\boldsymbol{F}_2 = m_2\boldsymbol{a}_2 \tag{8}$$

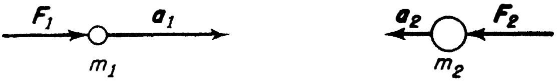
is the force exerted on particle 2 *by particle* 1. Substitution of equations (7) and (8) in equation (6) leads to the result

$$\boldsymbol{F}_1 = -\boldsymbol{F}_2, \tag{9}$$

which states that the force exerted on particle 1 by particle 2 is equal in magnitude and opposite in direction to the force exerted on particle 2 by particle 1. Figure 4(a) gives a diagram showing the results of a repulsive interaction between 'particle' 1 and 'particle' 2; Fig. 4(b) shows the results of an attractive interaction. Calling the force exerted *on* particle 1 *by* particle 2 the *action* and the force exerted *on* particle 2 *by* particle 1 the *reaction*, we may state equation (9) as follows:



(a) Repulsion



(b) Attraction

Fig. 4. When two particles interact, the magnitude of the force exerted on the first particle by the second particle is equal to the magnitude of the force exerted on the second particle by the first. The forces are oppositely directed.

> NEWTON'S THIRD LAW: *To every action there is an equal and opposite reaction.*

It can be readily shown that if this law is true for each pair of 'particles,' it is true for the more general situations depicted in Fig. 1 of Chap. 3.

Let us now give a résumé of the processes used in formulating Newton's laws. There were four conclusions based on idealizations of experiments:

(a) The velocity of a particle is constant as long as there are no interactions with other bodies.

(b) When two particles interact, they both experience accelerations. The accelerations are oppositely directed, and the ratio of the accelerations is a constant for a given pair of particles, independent of the method by which they interact.

(c) Characteristic masses can be assigned to each particle, by comparison with a standard particle, such that when *any* two particles interact the ratio of their accelerations is the inverse of the ratio of their preassigned masses.

(d) When a particle interacts with several other particles simultaneously, the acceleration it experiences is the vector sum of the accelerations it would experience by interacting with each of the other particles individually.

Conclusion (a) is the *first law.* Conclusions (b) and (c) permit the definition of *mass*, and hence of *force* as *ma*. Conclusion (d) shows that forces obey the law of vector addition and hence permits formulation of the *second law* in terms of resultant force. Conclusion (b) and the definition of force lead directly to the *third law*.

It might be mentioned that many of the ideas involved in Newton's laws had been evolved by others during and before Newton's time. For example, the first law had been stated by Galileo. However, it remained for Newton to give a *complete and consistent formulation* of three laws on which a widely applicable system of dynamics could be based.

PROBLEMS

1. In an interaction between a 2-kg body and a 3-kg body, the 2-kg body experiences an acceleration of magnitude 3 m/sec^2. What is the magnitude of the resultant external force acting on the 2-kg body? What is the magnitude of the resultant force acting on the 3-kg body? What is the magnitude of the acceleration experienced by the 3-kg body? Ans: 6 nt; 6 nt; 2 m/sec^2.

2. In an interaction between a 3-kg body and a 6-kg body, the 3-kg body experiences an acceleration of magnitude 4 m/sec^2. What is the magnitude of the force acting on the 6-kg body? What is the magnitude of its acceleration?

3. In moving a 30-kg trunk across a horizontal floor, a man exerts a horizontal force. If the trunk experiences an acceleration of magnitude 0.5 m/sec^2 and if the coefficient of kinetic friction between the trunk and floor is 0.2, what force must the man exert? What force does the trunk exert on the man? Does the man experience a resultant acceleration in a direction opposite to that of the acceleration experienced by the trunk? Ans: 73.9 nt; 73.9 nt; no, why not?

4. A man exerts a horizontal force of 80 nt on a 30-kg box in pushing it across a level floor. If the box experiences an acceleration of 0.5 m/sec^2, find the coefficient of kinetic friction between the box and the floor. What force does the box exert on the man? What horizontal force does the box exert on the floor?

5. A 12-kg body falls freely and experiences a downward acceleration of 9.79 m/sec^2. What force does the earth exert on the body? What force does the body exert on the earth? Ans: 117.5 nt downward; 117.5 nt upward.

6. A 4-kg ball rolls across a level floor. If the initial velocity of the ball is 2 m/sec and if the ball comes to rest in 10 sec, what is the average resultant horizontal force exerted on the ball? What is the average horizontal force exerted on the floor by the ball?

5. WEIGHT. GRAVITATIONAL UNITS OF FORCE

The weight of a body is the gravitational force exerted on the body by the earth. Since the acceleration g of a falling body is produced by the weight of the body, Newton's second law shows that the weight W of a body of mass m is

$$W = mg, \tag{10}$$

where W is expressed in absolute force units, newtons with m in kg and g in m/sec^2, or poundals with m in lb and g in ft/sec^2. For example, at a place where the acceleration of gravity is 9.80146 m/sec^2, the weight W of a 1-kg mass is 9.80146 nt.

Equation (10) shows that the weight of a body at a given position on the earth's surface is directly proportional to the mass of the body. This fact suggests immediately the possibility of comparing the *masses* of bodies by comparing their *weights*. The simplest scheme for doing this involves the use of a beam balance as shown in Fig. 5. A uniform beam is mounted with its center of gravity over the knife-edge support; a standard mass m_s and an unknown mass m_x are suspended from the beam and their positions are varied until the beam is balanced. When the beam is balanced, we may make use of the condition of rotational equilibrium by taking moments about the knife-edge:

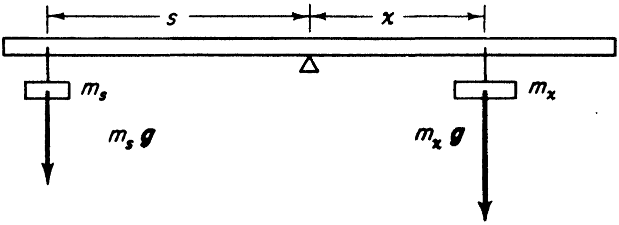


Fig. 5. Comparison of masses by means of a balance.

$$\text{clockwise torque} = \text{counterclockwise torque},$$

$$W_x\, x = W_s\, s,$$

where W_x and W_s are the weights of the bodies in absolute force units and s and x are the distances from the points of suspension of the bodies to the knife-edge support.

Since $W_x = m_x g$ and $W_s = m_s g$, this leads to

$$m_x g x = m_s g s.$$

Since g is the same for both bodies in a given locality, we find that

$$m_x = (s/x) m_s.$$

The experimental determination of the unknown mass m_x in terms of the standard mass m_s therefore involves merely a measurement of the ratio of two distances s and x. However, it should be noted that we are really comparing the interactions of each of the two bodies with a third body—the earth itself.

Let us now turn to a consideration of the gravitational force units, the kilogram-force (kgf) and the pound-force (lbf), which were introduced in Chap. 2.

> The *kilogram-force* is defined as the weight of a one-kilogram mass at a specified location near the earth's surface. The

> specified location is arbitrarily chosen as a place where the acceleration of gravity has the 'standard' value $g_0 = 9.80665$ m/sec^2.

The weight of a 1-kg mass at this location is given by (10) as 9.80665 newtons. Hence

$$1 \text{ kgf} = 9.80665 \text{ nt.}$$

Although the kilogram-force (kgf) is a convenient unit to use in the problems of statics and in laboratory measurements, since balances usually read in gf or kgf, we shall not use this metric gravitational unit directly in dynamical equations. As pointed out in the Appendix, we could use this force unit in dynamics by using an appropriate value for k in equation (2) and in all later equations derived from (2). However, it will lead to less confusion if we use equation (4) and always express forces in newtons (nt) in our work on dynamics when metric units are employed. When a force is given in kgf, the magnitude of the force in nt should first be obtained and this value used for F in the dynamical equations.

Now let us consider the British gravitational unit of force:

> The *pound-force* is the weight of a one-pound mass at a place where the acceleration of gravity has the 'standard' value g_0. In British units, $g_0 = 32.17398$ ft/sec^2.

This gravitational force unit, the pound-force, is of a magnitude that is convenient for most engineering applications, whereas, as pointed out in Sec. 3, the absolute force unit, defined in a manner analogous to that in which the newton is defined, is inconveniently small. Therefore, engineers use the pound-force (lbf). However, if the lbf is employed as the force unit and the lb is used as the mass unit, equation (4) cannot be used for Newton's second law, and equation (2) with an appropriate value for k must be used; this is done in much engineering work. We shall adopt a different procedure which seems to have the widest usage in mechanical and aeronautical engineering. This procedure permits us to use the convenient expression (4) for Newton's second law, with the proportionality constant k equal to unity. We may use this expression with F in lbf provided we select a mass unit of proper size in place of the pound; this mass unit is called the *slug:*

> One *slug* is the mass of a body that experiences an acceleration of 1 ft/sec^2 when a force of 1 lbf acts upon it.

This procedure, which involves the definition of a new mass unit, the *slug,* in terms of a force unit, the *lbf,* may at first seem rather involved, particularly in view of the fact that the *lbf* was itself defined in terms of the gravitational force exerted upon another mass, the *lb.* However, it is actually worthwhile in view of the fact that it permits us to use a force unit of *convenient* size in the *convenient* equation

$$F=ma.$$

Whenever British units are used in this equation, the force should be expressed in *lbf*, the mass in *slugs*, and the acceleration in ft/sec^2.

Since 1 lbf will by definition give a 1-lb mass an acceleration of 32.174 ft/sec^2 and will give a 1-slug mass an acceleration of only 1 ft/sec^2, it follows from Newton's second law in the form $a \propto F/m$ that the slug is 32.174 times as massive as the pound. In other words,

$$1 \text{ slug} = 32.174 \text{ lb},$$

or, to three significant figures, 1 slug = 32.2 lb. In problem work, the mass of a body may be originally stated in *lb* but should be converted to *slugs* before its value is used in the dynamical equation $F=ma$ or in another relation derived from this equation.

The pound-force is a unit equal to the slug·ft/sec^2:

$$1 \text{ lbf} = 1 \text{ slug·ft/sec}^2. \qquad (11)$$

The various commonly used systems of units are discussed in detail in the Appendix, but it may be well to describe once again the two systems we shall use. Both of these systems are based on the convenient expression

$$F=ma$$

for Newton's second law; in this equation, the proportionality factor k in (2) is unity. When International metric units are employed, we shall use the MKS *absolute system;* in this system force is measured in *newtons* (nt), mass in *kilograms* (kg), and acceleration in *meters per second per second* (m/sec^2). This system is called an *absolute* system, since the units are defined in a way that does not refer to the magnitude of terrestrial gravitation. When English units are employed, we shall use the *British engineering system* in which the force unit is the *pound-force* (lbf), the mass unit is the *slug*, and acceleration is given in *feet per second per second* (ft/sec^2). This system is called a *gravitational* system, since the force unit is defined in terms of gravitational effects at a specified position.

We note from the table on p. 110 that on the earth's surface g does not depart by more than a fraction of a per cent from its standard value, and hence that for most engineering purposes the weight of a body in lbf may be taken as equal to its mass in lb.

PROBLEMS

NOTE: Use data on g given in Table III of Chap. 5, p. 110.

1. What is the exact weight, in nt and in kgf, of a 1.00000-kg body at Denver, where the acceleration of gravity is 9.79609 m/sec^2? Ans: 9.79609 nt; 0.99892 kgf.

2. What is the exact weight, in nt and in kgf, of a 2.50000-kg body at New Orleans, where the gravitational acceleration is 9.79324 m/sec^2?

3. The mass of a certain body is exactly 1.00000 slug. What is the exact weight of this body at a place where the acceleration of a freely falling body is 32.1578 ft/sec^2? What would be the weight of this body at a sea-level station at the equator? at a sea-level station at the North Pole? Ans: 32.1578 lbf; 32.0878 lbf; 32.2577 lbf.

4. The mass of a metal cylinder is exactly 3.00000 slugs. What would be the exact weight of this body at sea-level and 40° latitude? at 32,000 ft altitude and 40° latitude?

5. What is the exact weight of a 1-lb body at sea-level stations at the following latitudes: 0°, 40°, and 80°? Ans: 0.99732 lbf; 0.99950 lbf; 1.00244 lbf.

6. The mass of a certain body is exactly 3 lb. What is the exact weight of this body at a location where the acceleration of a freely falling body is 32.2132 ft/sec^2?

7. A trunk weighing 50.0 lbf is pushed across a level floor. If the coefficient of kinetic friction between the trunk and the floor is 0.3, what horizontal force must be applied in order to give the trunk an acceleration of 2 ft/sec^2? What is the mass of the trunk in slugs? What is the magnitude of the frictional force opposing motion? Ans: 18.11 lbf; 1.55 slugs; 15.0 lbf.

8. A trunk weighing 120 lbf is pushed across a level floor. If the coefficient of kinetic friction between the trunk and the floor is 0.24, what horizontal force must be applied in order to give the trunk an acceleration of 3 ft/sec^2? What is the mass of the trunk in slugs? What is the magnitude of the frictional force opposing motion?

9. A large crate weighing 322 lbf is pushed across a horizontal loading platform. If a horizontal force of 60.2 lbf must be applied in order to give the box an acceleration of 2 ft/sec^2, what is the coefficient of friction between the box and the platform? Ans: 0.125.

10. When a horizontal force of 10 lbf is exerted on a 60-lb sled, the sled experiences an acceleration of 5 ft/sec^2. What is the coefficient of friction between the sled and the snow?

11. An elevator cage weighing 2000 lbf experiences an upward acceleration of 4.02 ft/sec^2. If friction is negligible, what upward force is exerted on the elevator cage by the supporting cable? Ans: 2250 lbf.

12. If the elevator cage in Prob. 11 experiences a downward acceleration of 8 ft/sec^2, what upward force is exerted by the supporting cable?

13. If the coefficient of static friction between the tires and the road is μ_s, show that the minimum stopping distance for an automobile of mass m travelling at speed v is $v^2/2\mu_s g$.

14. If $\mu_s = 1.0$, find the minimum stopping distance of a 3000-lb car travelling at 15, 30, 60 mi/hr. Of a 1000-kg car travelling at 60 km/hr.

6. UNIVERSAL GRAVITATION

In this and earlier chapters we have had occasion to speak of the gravitational force or gravitational pull exerted by the earth on objects near its surface. The existence of this force becomes apparent to everyone early in life. This force exerted by the earth is merely an example of a property possessed by all material bodies, called *universal gravitation*. Newton first formulated the following law, which gives a quantitative description of gravitational action:

Every particle of matter in the universe attracts every other particle. The magnitude of the force of attraction between two particles is proportional to the product of the masses of the particles and inversely propor-

tional to the square of the distance between them; the gravitational forces between two particles act along the line between the two particles.

The equation expressing this relationship for a given pair of particles of masses m_1 and m_2 is

$$F = G\frac{m_1 m_2}{r^2},$$

where r is the distance between the two particles and G is a proportionality constant called the *gravitation constant* (not to be confused with g, the acceleration of a freely falling body).

It should be noted that Newton's law of universal gravitation is *not* a defining equation like Newton's second law of mechanics and cannot be derived from defining equations. It represents an *observed relationship,* and the gravitation constant G can be determined experimentally. Newton was led to the formulation of this law, not from considerations of falling apples, but from considerations of the motions of the planets around the sun and the motion of the moon around the earth; his first publications on the subject dealt with the motion of the moon around the earth.

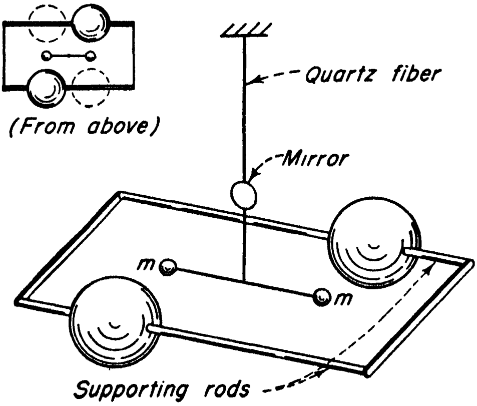


Fig. 6. The experiment of Cavendish for determining the gravitation constant G.

The classical experiment of Cavendish in 1798 gave the first accurate laboratory measurement of the gravitation constant G. The essential features of the Cavendish apparatus are shown in Fig. 6. Two small balls, each of mass m, are attached to the ends of a light rod. The resulting 'dumbbell' is suspended in a horizontal position by a fine wire or quartz fiber just strong enough to support it. Heavy iron or lead balls of mass m' can be moved close to the ends of the dumbbell by sliding them along supporting rods. When the large balls are in the position shown, the gravitational forces they exert on the small balls produce a torque tending to twist the fiber. When they are moved into the position indicated by broken lines, the fiber tends to twist the other way. The angle θ through which the fiber is twisted when the balls are moved from one position to the other is measured by observing the deflection of a beam of light reflected from the small mirror attached to the dumbbell. From a knowledge of the geometry of the apparatus and the elastic properties of the supporting fiber, the value of G can be determined from the observed angle θ. At this writing the best value for G is 6.670×10^{-11} nt·m²/kg², based on the measurements of P. R. Heyl of the National Bureau of Standards.

From a consideration of Newton's law of gravitation, it is easy to see one reason for the variations of g with latitude and altitude as given in Table III of Chap. 5. If the earth were a perfect sphere of uniform density, the magnitude of the gravitational force exerted on a particle of mass m would decrease with the distance of the particle from the center of the earth. This variation with *altitude* is observed. The observed variation of g with *latitude* is readily understood when we recall that the earth is not a perfect sphere but is an oblate spheroid, and the distance from sea level to the center of the earth becomes less as one proceeds from the equator toward the poles. Therefore, g would be expected to show a slight increase with increasing latitude, as it does.*

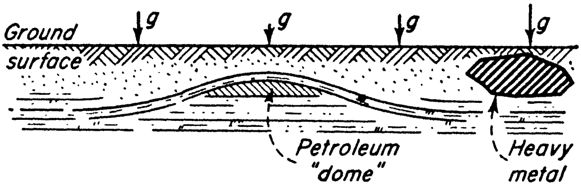


Fig. 7. Geophysical prospecting: g is slightly increased in places where heavy rock structures are close to the surface; slightly decreased in the vicinity of salt domes, where oil is likely to collect because of the tilted rock strata.

The earth is not homogeneous. Local variations in the densities of the rock strata underlying the surface cause small local variations in the acceleration of gravity. These variations of g with subsurface structure are utilized in geophysical methods of prospecting. As indicated in Fig. 7, where deposits of heavy-metal ores underly the surface, the acceleration of gravity will be slightly increased. On the other hand, oil pools are frequently associated with 'salt domes'—underground intrusions of salt, which is lighter than normal rock and hence causes a local decrease in gravity. The oil collects in the regions of the porous oil-bearing strata that have been elevated by the intrusion of the dome.

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* The rotation of the earth also has a small effect on the value of the acceleration of gravity; we shall omit discussion of this subject.

In prospecting, the g-values for a field are mapped by means of a 'gravity meter,' which in essentials consists of an extremely sensitive spring balance to which is attached a fixed mass. The gravity meter weighs the mass. Although it must be extremely sensitive, capable of detecting variations in g of only a few parts in ten million, a useful gravity meter must also be sufficiently rugged to permit transportation in trucks over rough country.

PROBLEMS

NOTE: It can be shown that the resultant gravitational attraction between two homogeneous spheres is the same as if the whole mass of each sphere were concentrated in a small particle at its center. You will need to use this theorem in the following problems.

1. Two lead balls, each of mass 4 kg, are placed with their centers 20 cm apart. What is the magnitude of the gravitational force exerted by the first ball on the second? Ans: 2.67×10^{-8} nt.

2. Two lead spheres of radius 10 cm are placed with their centers 21 cm apart. What is the magnitude of the gravitational attraction between the spheres?

3. The downward acceleration of freely falling bodies near the earth's surface is approximately 9.8 m/sec^2. Taking the radius of the earth as approximately 6.7×10^6 m, compute the mass of the earth. Assume that for purposes of computing gravitational forces, the earth behaves like a single massive particle placed at the center of the earth; this is very closely true. Ans: 6.6×10^{24} kg.

4. At what altitude above the earth's surface would the acceleration of gravity be approximately 8 ft/sec^2? Take the radius of the earth as 4000 mi.

5. Show how it follows from the law of universal gravitation that all freely falling bodies have the same acceleration at a given point near the earth's surface.

7. APPLICATIONS OF NEWTON'S LAWS OF MECHANICS

We shall now discuss the application of Newton's laws to various problems in dynamics. We shall employ the second law in the form $F=ma$, which is valid if we employ the metric units nt, kg, m/sec^2, or the British units lbf, slug, ft/sec^2. All derived relations are valid in either of these systems of units.

As our first example, let us describe the operation of the device known as Atwood's machine, which is frequently used to illustrate the second law. The arrangement is shown in Fig. 8. In part (a) of this figure are shown two bodies of equal mass m attached to the ends of a string that passes over a light frictionless pulley. Since the two masses are equal, the forces applied to the ends of the string by the weights mg are equal and both act downward, so that there is no tendency for the string passing over the pulley to be set in motion. Therefore, there is no acceleration. Since there is no acceleration, each body is in equilibrium and we can find the tension T in the string by writing the first condition of equilibrium for either body:

$$T = mg.$$

In Fig. 8(b) a small body called the 'rider,' of mass m_1, has been attached to the body of mass m on the right side. The forces on the two bodies are no longer equal and acceleration will result. The masses on the right will experience an acceleration a downward and the mass on the left will experience an acceleration of equal magnitude a but directed upward. Let us now write the second law for the bodies and use the consistent sign convention indicated in Fig. 8(c); the sign convention is chosen so as to give a the same algebraic sign in the two expressions for the second law, so that we can solve these equations simultaneously. Since the masses of the string and of the pulley are assumed negligible, and the pulley frictionless, the tension in the string will be the same throughout its length, say T'.

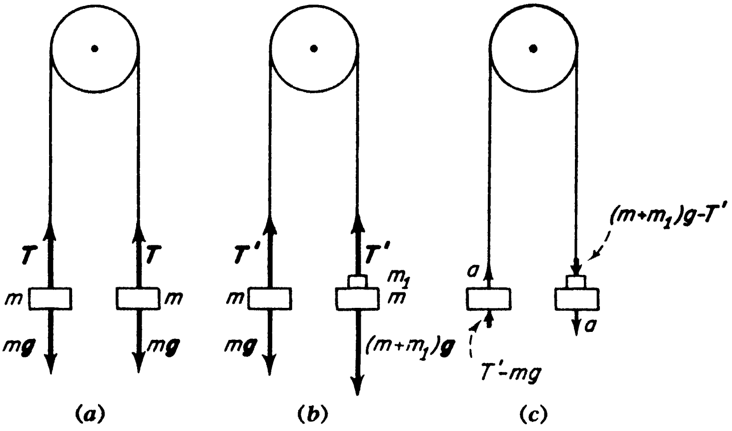

Fig. 8. Atwood's machine. (a) Two equal masses. (b) A 'rider' is added on the right. (c) The resultant forces acting on the masses in (b), and the resulting accelerations.

For the mass on the left, we write

$$T' - mg = ma \tag{12}$$

and, for the masses on the right,

$$(m+m_1)g - T' = (m+m_1)a. \tag{13}$$

Addition of these two equations gives

$$m_1g = (2m+m_1)a, \quad \text{or} \quad a = m_1g/(2m+m_1). \tag{14}$$

Hence m_1g can be considered as the magnitude of the 'resultant force' applied to accelerate a 'system' of total mass $(2m+m_1)$.

In laboratory demonstrations, the rider is of such a small size that its mass m_1 is negligible compared to the mass $2m$ of the original system, so that, although the magnitude of the resultant force is proportional to the mass m_1 of the rider, the total mass of the system experiencing acceleration can be regarded as constant. By adding riders of different mass, it can be shown that the magnitude of the acceleration is proportional to m_1, a result that demonstrates the applicability of Newton's laws to this arrangement.

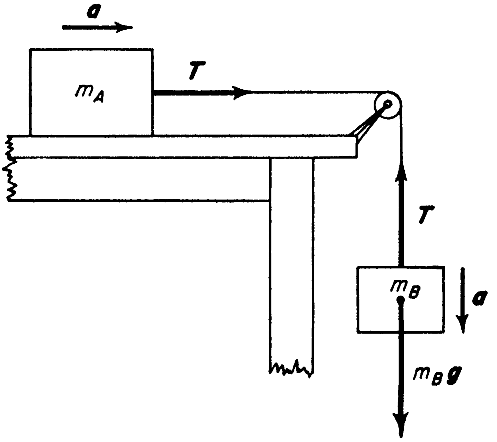

Fig. 9.

One further remark might be made concerning the tension T' in the string. Once a is determined from equation (14), the tension can be determined by substitution in either (12) or (13). Equation (12) shows that $T'=mg+ma$, which indicates that the tension is *greater* than mg, the tension that would be required merely to *support* the weight of the left-hand body, by an amount ma. Equation (13) shows that $T'=(m+m_1)g-(m+m_1)a$ which indicates that the tension is *less* than the tension $(m+m_1)g$ that would be required merely to *support* the weight of the bodies on the right, by an amount $(m+m_1)a$. Thus, we are led to the relation:

$$mg<T'<(m+m_1)g,$$

or weight of body on left < tension in rope < weight of bodies on right.

The reason for this inequality is clear when we remember that the body on the left is accelerated upward while the bodies on the right are accelerated downward.

The next problem we shall consider is that shown in Fig. 9, where a body A of mass m_A lies on a flat, smooth table top and a light string attached to the body passes horizontally to a light, frictionless pulley and thence vertically downward to a body B of mass m_B. The problem involved is the determination of the magnitude of the acceleration a experienced by the two-body system and the tension T in the string. We shall assume that there is no friction between body A and the table top.

Let us write the second law for body A. The resultant force on body A equals the tension T in the string and is exerted horizontally and to the right. The acceleration a of this body has the same direction as the resultant force. Hence, we may write

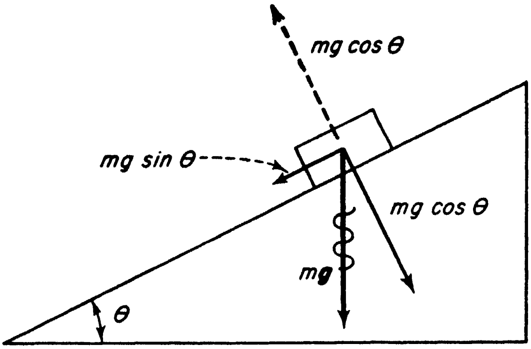

Fig. 10. Forces acting on a body on a frictionless inclined plane.

$$T = m_A a. \tag{15}$$

Now we write the second law for body B. The resultant downward force on body B is $(m_B g - T)$, and B has the same acceleration a downward as A has to the right. The second law gives

$$(m_B g - T) = m_B a,$$

or

$$T = m_B g - m_B a = m_B(g - a). \tag{16}$$

Simultaneous solution of (15) and (16) gives

$$a = \frac{m_B g}{m_A + m_B}; \quad T = \left(\frac{m_B m_A}{m_A + m_B}\right) g. \tag{17}$$

Let us now consider the problem of a body sliding down a frictionless inclined plane. In Fig. 10 a block of mass m is on a frictionless plane inclined at angle θ to the horizontal. The gravitational force on the body is mg, which we can resolve into two components, one parallel to the inclined plane, and the other normal to the plane. The component normal to the plane of the gravitational force *on the body* must be just balanced by the plane's normal reaction force *on the body*, since clearly there is no acceleration component normal to the plane. Hence, the resultant force on the body acts parallel to the plane and has the magnitude $mg\sin\theta$ as shown in Fig. 10. Therefore, the body experiences an acceleration down the plane, of magnitude a given by

$$mg\sin\theta = ma,$$

or

$$a = g\sin\theta. \tag{18}$$

Equation (18) gives the value of a in terms of g for any elevation angle θ of the inclined plane. We may check the results given by (18) for two cases with which we are already familiar. For $\theta = 90°$, equation (18) gives $a = g$. Since, when $\theta = 90°$, the frictionless plane is upright, the body becomes a freely falling body and by definition has acceleration g, we see that the result based on (18) is correct. For $\theta = 0°$, the equation gives $a = 0$. Since, when $\theta = 0$, the plane is horizontal and there is no resultant force on the body, we see that this result is also correct.

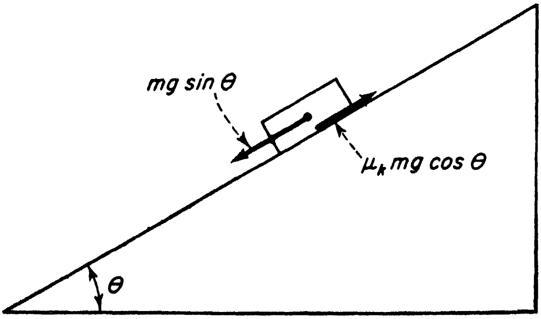

Fig. 11. Forces acting on a body sliding down an inclined plane when there is friction. Only forces *parallel* to the incline are shown; the *normal* component of the resultant force is always zero.

In order to see how frictional forces can be taken into consideration, let us return to the problem of a body sliding down an inclined plane. Figure 11 shows a block moving on an inclined plane where the coefficient of kinetic friction between the block and the plane has a value μ_k. As in the case of the frictionless plane, the component of the gravitational force perpendicular to the plane is just balanced by the reaction force exerted by the plane, because there is no acceleration normal to the plane. Hence we have drawn arrows only for force components acting parallel to the plane. As before, the gravitational force component down the plane is

$mg\sin\theta$. Opposing this force is the frictional force $\mu_k N$, where N is the magnitude of the normal force $mg\cos\theta$. Hence, the resultant force F down the plane is

$$F = mg\sin\theta - \mu_k mg\cos\theta.$$

Thus, by Newton's second law, the acceleration experienced by a block moving down the plane is given by

$$mg\sin\theta - \mu_k mg\cos\theta = ma,$$

or

$$a = g(\sin\theta - \mu_k\cos\theta). \tag{19}$$

If the body is initially moving down the plane with velocity v_0, its velocity will increase provided a is positive, that is, provided

$$\sin\theta > \mu_k\cos\theta.$$

If the acceleration is zero, the velocity will remain unchanged; this condition occurs when

$$\sin\theta = \mu_k\cos\theta,$$

as was pointed out in Chap. 3 in the discussion of kinetic friction. If the acceleration is negative, the velocity will decrease and the body will come to rest and remain at rest; this is the case when

$$\sin\theta < \mu_k\cos\theta.$$

The case of a body moving *up* the plane with initial velocity v_0 is involved in problems 23 and 24 at the end of this section. In working these problems, it should be remembered that whenever the body is at rest, μ_s must be used as the coefficient of friction.

The problems involving freely falling bodies and projectiles which we discussed in detail in Chap. 5 are extremely simple from the standpoint of dynamics, since only constant vertical forces are involved. This leads to a constant vertical component of acceleration and no horizontal component of acceleration, in agreement with our treatment in Chap. 5. If we knew the type of forces produced by air resistance, we could calculate realistic values of horizontal and vertical acceleration components and determine the trajectories of actual projectiles. Determination of the air resistance is made by wind-tunnel and other types of experimental measurements and used in the computation of 'firing tables' for military guns. These computation are extremely complicated. Study of actual trajectories is known as the science of External Ballistics.

One other problem discussed earlier in terms of kinematics merits reconsideration from the dynamical point of view. This is the problem of uniform circular motion. As pointed out in Chap. 5, a body describing uniform circular motion experiences a centripetal acceleration of magnitude v^2/R, where v is the magnitude of the instantaneous velocity of the particle and R is the radius of the circular path. Now, according to Newton's second law, the body must be acted upon by a force $\boldsymbol{f}=m\boldsymbol{a}$ if it is to have this acceleration. This force is called the *centripetal force* and its magnitude is given by

$$f=mv^2/R, \tag{20}$$

where m is the mass of the body. Since the centripetal acceleration is directed toward the center of the circle, the centripetal force is also directed toward the center of the circle as in Fig. 12. In the case of a ball whirled in a horizontal circle at the end of a string, the centripetal force is produced by the tension in the string.

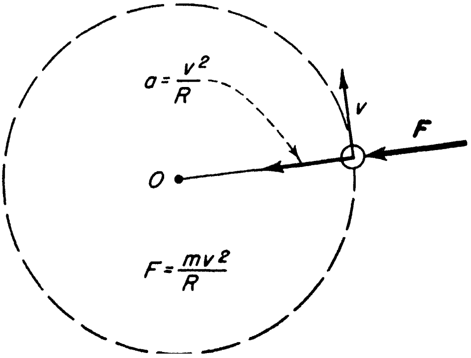

Fig. 12. A constant force of magnitude $F=mv^2/R$ acts at all times on a particle moving with constant speed v in a circular path of radius R. This force is directed toward the center of the circle and is called the *centripetal force.*

A planet moving around the sun moves in an ellipse with the sun at one focus. Actually the path is very close to a circle with the sun at the *center*. The centripetal force on the planet is the gravitational attraction

$$f=G\,m_PM_S/R^2,$$

where m_P is the mass of the planet, M_S is the mass of the sun, and R is the radius of the planetary orbit. By equating this force to $m_Pv_P^2/R$, as given by (20), where v_P is the planet's orbital speed, we may find the mass of the sun. In order to obtain a value for the orbital speed, we note that the planet traverses a path of length $2\pi R$ in the revolution time T; hence,

$v_P = 2\pi R/T$. Using this value for v_P, we may write

$$G\frac{m_P M_S}{R^2} = \frac{m_P}{R}\frac{4\pi^2 R^2}{T^2} = \frac{4\pi^2 R m_P}{T^2}$$

which leads to the value $M_S = 4\pi^2 R^3/GT^2$ (21)

for the mass of the sun. Since T and an approximate value for R are known from astronomical data and G is known from terrestrial experiments, an approximate value for M_S can be determined. The value is not exact because the orbit is not exactly a circle.

Equation (21), written in the form $T^2 = 4\pi^2 R^3/GM_S$, shows that the squares of the periods of revolution of the different planets are proportional to the cubes of their distances from the sun. This law was discovered empirically by Kepler early in the seventeenth century when he was analyzing the extensive planetary observations of Tycho Brahe, and was of assistance to Newton in the development and verification of his theory of universal gravitation.

In general, problems involving the translational motion of bodies are solved by methods similar to those applied in statics, except that the resultant force and the resultant acceleration are not in general zero. Rather, if $\boldsymbol{F}$ is the resultant force acting on any body, the rectangular components of $\boldsymbol{F}$ are related to the rectangular components of $\boldsymbol{a}$ by the equations

$$F_x = ma_x, \quad F_y = ma_y, \quad F_z = ma_z.$$

PROBLEMS

NOTE: Assume the experiments in the following problems are conducted in a location where gravity has its standard value.

1. In an Atwood's machine (see Fig. 8), the mass of each suspended body is 1.000 kg, while the mass of the 'rider' is 20 g. Assuming negligible friction and negligible mass in the pulley wheel, calculate the magnitude of the resulting upward acceleration of the left-hand body and the tension in the string.
Ans: 9.71 cm/sec^2; 9.90 nt.

2. If the rider in Prob. 1 had been a 50-g body, what would have been the magnitude of the acceleration and the tension in the string?

3. In an experiment with an Atwood's machine, the weight of the body on the left in Fig. 8 was 8.0 lbf and the total weight of the bodies on the right was 8.5 lbf. Calculate the magnitude of the resulting acceleration and the tension in the string.
Ans: 0.975 ft/sec^2; 8.24 lbf.

4. In an experiment involving Atwood's machine, the mass of each suspended body was 8.05 lb and the mass of the rider was 8 oz. Calculate the magnitude of the resulting acceleration and also the tension in the string.

5. In an arrangement similar to that shown in Fig. 9, the mass of the body lying on the table top is 64.4 lb and that of the suspended body is 16.1 lb. Find the downward acceleration of the suspended body and the tension in the cord. (Ignore frictional effects.) Ans: 6.44 ft/sec^2; 12.9 lbf.

6. If an additional 5-lb rider were added to the suspended body mentioned in Prob. 5, what would be the magnitude of the resulting acceleration? What would be the tension in the cord?

7. In an arrangement similar to that shown in Fig. 9, the mass of the body on the table top is exactly 4 kg and that of the suspended body is 1 kg. If the coefficient of kinetic friction between the table top and the body lying upon it is 0.2, calculate the tension in the cord and the magnitude of the resulting acceleration once the block is set in motion. Ans: 9.41 nt; 0.392 m/sec^2.

8. If the suspended body in Prob. 7 had a mass of exactly 2 kg, what would be the tension in the string and the magnitude of the acceleration?

9. A 2-kg block is placed on a frictionless board inclined at angle of 30° with the horizontal. What is the magnitude of the acceleration experienced by the body? Ans: 4.90 m/sec^2.

10. A 50-lb box is placed on a frictionless board inclined at an angle of 45° with the horizontal. What will be the acceleration of the body down the board?

11. If the coefficient of friction between the block and the board in Prob. 9 had been 0.2, what would have been the acceleration of the block? Ans: 3.20 m/sec^2.

12. If the coefficient of friction between the box and the board in Prob. 10 had been 0.3, what would have been the acceleration of the box?

13. An 8.05-lb shell is fired from a gun whose barrel is 4 ft in length. The muzzle velocity of the shell is 2000 ft/sec. Assuming that the acceleration of the shell in moving down the barrel is constant, find the average resultant force exerted on the shell while it is in the barrel. Ans: 125,000 lbf.

14. A 2.48-kg shell is fired from a gun whose barrel is 2 m in length. The muzzle velocity is 300 m/sec. Assuming constant acceleration, calculate the average force exerted on the shell while it is in the barrel.

15. A 2-lb ball at the end of a string is swung at a constant speed of 6 ft/sec in a horizontal circle of radius 4 ft. What centripetal force does the string exert on the ball? What angle does the string make with the horizontal? What is the tension in the string? Ans: 0.559 lbf; $74^\circ\!.4$; 2.08 lbf.

16. A 1-kg ball at the end of a cord travels at a constant speed of 3 m/sec in a horizontal circle of radius 1.5 m. What centripetal force is exerted on the ball? What angle does the cord make with the horizontal? What is the tension in the cord?

17. An automobile of 3000-lb mass travels around a curve of 200-ft radius. If the roadway is level and the centripetal force results from friction alone, what is the maximum speed the automobile can have if the coefficient of friction between the tires and the roadway is 0.3? Ans: 44.0 ft/sec.

18. What would have been the maximum speed of the car in Prob. 17 if the radius of the curve had been 150 ft and the coefficient of friction 0.4?

19. An automobile travels around a curve of 200-ft radius at a speed of 60 ft/sec. At what angle θ with the horizontal should the roadway be 'banked' in order for the car to make the curve without depending upon friction between the tires and roadway? Ans: $29^\circ\!.2$.

20. At what angle should the curve in Prob. 19 be banked if the automobile has a speed of 30 mi/hr?

21. Compute the value of the mass of the sun in kg on the assumption that the earth moves in a circular orbit of radius 93,000,000 mi and has a period of revolution of 365 days. Ans: 1.99×10^{30} kg.

22. Compute the value of the earth's mass on the assumption that the moon completes one revolution about the earth every 29½ days in a circular path of radius 240,000 mi. Compare your result with that obtained in Prob. 3, Sec. 6.

23. A body of mass 3 kg is projected at initial speed 5 m/sec up a plane inclined at 20° to the horizontal, on which the coefficients of friction are $\mu_k = 0.2$ and $\mu_s = 0.3$.

(a) What is the deceleration of the block as it moves up?
(b) How far does it move along the plane before coming to rest?
(c) Show that the block will start back down after coming to rest.
(d) What is its acceleration down the plane?
(e) What is its speed when it again reaches the starting point?

Ans: (a) 5.20 m/sec^2; (b) 2.41 m; (d) 1.51 m/sec^2; (e) 2.70 m/sec.

24. A body of mass 6 lb is projected at initial speed 24 ft/sec up a plane, inclined at 30° to the horizontal, on which the coefficients of friction are $\mu_k = 0.4$ and $\mu_s = 0.5$. Answer the same questions as in Prob. 23.

25. A box weighing 100 lbf rests on the flat bed of a truck moving at 40 ft/sec along a level road. Because the bed of the truck is oily, the coefficient of static friction between box and truck bed is only 0.15. How rapidly must the truck decelerate in order for the box to slip forward during the deceleration? Ans: 4.83 ft/sec^2.

26. If the coefficient of kinetic friction in the example of Prob. 25 is 0.12, and the truck accelerates from rest at the uniform rate of 8 ft/sec^2, how far has the truck gone before the box has slipped back 11 ft on the truck body, which is enough to cause it to fall off?

CHAPTER 7

WORK, ENERGY, AND POWER

In this chapter we shall define the quantities *work* and *energy* and show how certain physical phenomena we have discussed previously can be treated conveniently in terms of these quantities. In doing this, we shall develop the principle of *conservation of mechanical energy* for use in situations in which dissipative forces are not involved. In connection with work done against frictional forces, we shall introduce a more general *conservation-of-energy* principle. The introduction of this principle at this point may perhaps seem premature, since the general principle was stated by Helmholtz and Mayer on the basis of the experimental work of Rumford and Joule, which will be discussed in Part II of this book dealing with Heat. However, in view of the fact that conservation of energy is probably the most important principle in physics, we feel justified in introducing it at this point without a thorough discussion of the developments that led to its formulation.

We shall use the energy-conservation principle in discussing simple machines and in defining the ideal and actual mechanical advantage and the efficiency of these devices. In discussing the performance of useful work we shall have occasion to introduce and define *power* and shall discuss the energy relations involved in the operation of several types of *prime movers.*

1. WORK

In nonscientific language, we use the word *work* to denote any type of activity that requires the exertion of muscular or mental effort. In physics, however, the term *work* has a definite technical definition and is used *only* in the restricted sense of this definition. In order for physical work to be done, it is necessary for a force to act on a body and for the body to experience a displacement that has a component parallel to the direction in which the force is acting.

Figure 1 represents a body moving along a horizontal surface in a direction which we shall take as the x-axis. A force $\boldsymbol{F}$ acts on the body at an angle θ with the direction of motion. The work done by this force is *defined* in the following way:

> The work done by a force acting on a body while the body undergoes a displacement is defined as the product of the magnitude of the displacement and the component of the force in the direction of the displacement. Work is a scalar quantity.

Thus, the work dW done by the force in Fig. 1 when the body experiences

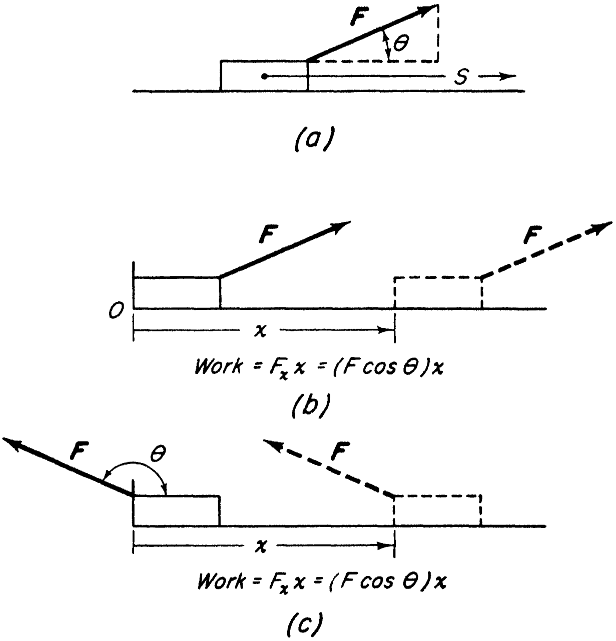
F
θ
S
(a)
F
F
O
x
Work = $F_x x$ = ($F \cos \theta$)x
(b)
F
θ
F
x
Work = $F_x x$ = ($F \cos \theta$)x
(c)

Fig. 1.

an infinitesimal displacement of magnitude dx is

$$dW = (F \cos\theta)\, dx. \tag{1}$$

If the force is constant, the work W done in moving the body from the origin to a point x is

$$W = (F \cos\theta)\, x. \tag{2}$$

In the special case in which the force is constant and has the same direction as the displacement, the work done by the force is equal to the product of the force and the displacement; this is the case when the angle θ in Fig. 1 is zero. If the angle θ between the force vector F and the displacement vector x is greater than 90°, as in Fig. 1(c), $\cos\theta$ is negative and the work is negative. This is the case where a force is applied to retard the motion of a body already moving with a positive velocity in the

x-direction. If θ is 180°, the work is the negative of the product of the force magnitude and the displacement magnitude. When $\theta = 90°$, no work is done.

The quantity called work plays an extremely important role in physics. We note that, regardless of how large a force acts, displacement must occur before work is done. Thus, although opposing teams may exert enormous forces on a rope during a tug of war, neither team performs any work unless the rope moves. Similarly, a heavy man sitting at rest may exert a downward force of 300 lbf on a chair but does no work on the chair. Further, it should be emphasized that even if a body experiences a displacement, no work is done unless a force having a component in the direction of motion acts on the body. Thus, a man does work in lifting a suitcase vertically, since the force he exerts is in the direction of motion. However, the force he exerts on the suitcase does no work when he carries the suitcase horizontally at constant velocity across the waiting-room floor, since this is a *vertical* force and the displacement is *horizontal.*

As noted above, work is defined as the *product* of a force component and the magnitude of a displacement. Therefore, the unit used in measuring work involves the product of a force unit and a length unit. In the British engineering system, the force unit is the *pound-force* (lbf) and the length unit is the *foot* (ft). The work unit is therefore called the *foot. pound-force* (ft·lbf) and is defined as follows:

> One ft·lbf is the work done by a constant force of one pound-force when the body on which the force is exerted moves a distance of one foot in the direction of the force.

Similarly, the work unit in the MKS system is the *meter·newton,* and this unit is defined in terms of the *newton* (nt) and *meter* (m) in the same manner as that used to define the ft·lbf in terms of the foot and the pound-force. This work unit, the *meter·newton,* is called the *joule;* there is no single corresponding word for the ft·lbf.

> One joule is the work done by a constant force of one newton when the body on which the force is exerted moves a distance of one meter in the direction of the force.

The following relations between work units will be found useful:

$$1 \text{ joule} = 1 \text{ m·nt} = 0.7376 \text{ ft·lbf}$$
$$1 \text{ ft·lbf} = 1.356 \text{ joules.}$$

PROBLEMS

1. By exerting a force of 12.0 lbf on a rope attached to a sled, a man pulls the sled across the ice on the surface of a pond. If the rope makes an angle of 45° with the direction of motion, how much work does the man do in moving the sled a distance of 90.0 ft? Ans: 764 ft·lbf.

2. A boy exerts a force of 20 lbf on the handle of a lawn mower. If the line of action of the force makes an angle of 30° with the direction of motion, how much work does he do in moving the lawn mower a distance of 40 ft?

3. An elevator cage has a mass of 1000 kg. If frictional effects are negligible, how much work is done in raising the cage 50 meters at constant speed?
Ans: 4.90×10^5 joules.

4. A 2-kg well bucket containing 8 liters of water is raised at constant speed from the water surface in the well to ground level, 9 m above the water level. How much work is done in this process?

5. A 30-lb sled is dragged at constant velocity across a horizontal surface. If the coefficient of friction between the sled runners and the snow is 0.12, how much work is done in moving the sled a distance of 50 ft if the applied force is in the direction of motion?
Ans: 180 ft·lbf.

6. How much work is done in moving the sled in Prob. 5 if the direction of the applied force is 30° above the horizontal? 30° below the horizontal?

2. ENERGY

Just as in the case of the word *work*, the word *energy* has two somewhat distinct uses. In nonscientific language, the word *energy* is sometimes used to denote activity; for example, if a man is active in social or civic organizations, we are accustomed to describe him as an 'energetic man' or as a man who has 'a great deal of energy.' In mechanics, the term *energy* has a precise technical meaning. The following will serve as a general definition:

> The energy of a body is a measure of the capacity or ability of the body to perform work. It is a scalar quantity and is measured in work units by the work the body is capable of performing.

Using this definition, we can immediately think of many examples of objects having energy because of their *motion*. For example, a moving bullet, a moving automobile, and a rotating flywheel all have the ability to do work during the process of being brought to rest. The energy possessed by a body as a result of its motion is called *kinetic energy*.

A body may also have energy as a result of its position or configuration. For example, a cubic foot of water at the top of a waterfall can perform work in turning a water wheel or turbine as a result of its *position*, while a clockspring, when wound, can do work in operating the clock mechanism as a result of its stressed configuration. Energy possessed by a body as a result of its position or configuration is called *potential energy*. The pound of water has potential energy as a result of its position in the gravitational field of the earth; hence we sometimes refer to its energy as *gravitational potential energy*. In the case of the clock spring, the spring is an elastic body and, when distorted, tends to regain its original shape or configuration; we can therefore think of energy as being imparted to the spring when the clock is wound. This type of energy is referred to as *elastic potential energy.* In Part V we shall also have occasion to discuss electrostatic potential energy.

In both the examples of potential energy we have given, we can think of energy as resulting from the performance of work. In the case of the cubic foot of water, the energy is imparted to the water *when work is done* in raising it to the top of the waterfall; in the case of the spring, the potential energy is produced when *work is done* in distorting the spring. Since work must be done in setting a body in motion, we see immediately that *work is done* in giving kinetic energy to the body. These observations suggest operational definitions of kinetic and potential energy that are more satisfactory than the general definition given earlier, but which we shall show, by means of examples, are equivalent to the general definition:

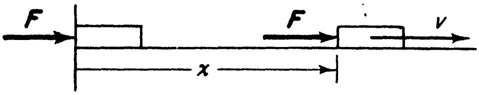

Fig. 2. The external work Fx done in accelerating the body is transformed into the kinetic energy $\frac{1}{2}mv^2$ of the body.

The kinetic energy of a body in motion equals the work done by the resultant force acting on the body in changing the body from a state of rest to the state of motion.

The change in gravitational, elastic, or electrostatic potential energy when a body changes from one position or configuration to another is the work done against (the negative of the work done by) the gravitational, elastic, or electrostatic forces when the body changes its position or configuration.

We shall first derive an expression giving the kinetic energy of a particle in terms of its mass m and its velocity v. In order to obtain this expression, let us compute the work that must be done in accelerating the particle, initially at rest, to velocity $\boldsymbol{v}$. If a constant force $\boldsymbol{F}$ acts on the particle, the acceleration $\boldsymbol{a}$ is constant and the displacement $\boldsymbol{s}$ has the same direction as the force. Therefore, we may choose this direction as the x-axis of a coordinate system in which the particle is initially located at the origin as shown in Fig. 2. The work W done in moving the particle to position x is therefore

$$W = Fx. \tag{3}$$

Let us assume that no frictional forces act on the particle. Since the initial velocity is zero and the acceleration is constant, we may substitute the value $x = \frac{1}{2}at^2$ in (3), and we may write ma for F. Equation (3) then becomes

$$W = (ma)(\tfrac{1}{2}at^2) = \tfrac{1}{2}m(at)^2,$$

or, since $v = at$,

$$W = \tfrac{1}{2}mv^2. \tag{4}$$

Since the kinetic energy of a particle travelling at speed v is defined as the work W required to accelerate the body from rest to speed v, expression (4) is just the kinetic energy of the particle:

$$\text{K.E.} = \tfrac{1}{2}mv^2. \tag{5}$$

Since kinetic energy is measured in the same units as work, *the K.E. in* (5) *is given in joules when m is in kg and v in m/sec, and is given in ft·lbf when m is in slugs and v in ft/sec.*

The above simple derivation of (5) is for the special case in which the particle is accelerated at a constant rate along a straight line. However, it can be readily shown by the methods of integral calculus that the same result is obtained for a particle accelerated in any manner along any type of curve. This proof is given in the footnote.* Expression (5) applies only to the case of a particle or an extended body undergoing pure translation without rotation; it is called the *translational kinetic energy*. For a rotating body there is an additional kinetic energy associated with the work required to set the body into rotation. This *rotational kinetic energy* we shall consider in Chap. 8.

Let us now obtain an expression for the change in gravitational potential energy when a particle changes its position relative to the earth's surface. In doing this, we must find the work done against the force of gravity. The body has potential energy by virtue of its *position*, but, since the pull of gravity is directed *vertically* downward, the gravitational potential energy of the body changes only when the body experiences a *vertical* displacement. Therefore, we need only to find the work

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* Let us consider a particle that changes its speed from v_0 to v_1 as it moves along an arbitrary path under the action of an arbitrarily varying resultant force F. Let s be the distance measured along the path and ds the increment of distance. Then the component of acceleration along the path can be shown to be d^2s/dt^2, and this will be related to the component F_s of force in the direction of motion by $F_s = m\,d^2s/dt^2$. The work done by the resultant force when the particle moves along the path from s_0 to s_1 will then be

$$\begin{aligned} W &= \int_{s_0}^{s_1} F_s\,ds = \int_{s_0}^{s_1} m\frac{d^2s}{dt^2}\,ds = \int_{t_0}^{t_1} m\frac{d^2s}{dt^2}\frac{ds}{dt}\,dt \\ &= \int_{t_0}^{t_1} m\frac{dv}{dt}v\,dt = \int_{t_0}^{t_1} m\frac{d}{dt}(\tfrac{1}{2}v^2)\,dt = \int_{v_0}^{v_1} m\,d(\tfrac{1}{2}v^2) \\ &= \tfrac{1}{2}mv_1^2 - \tfrac{1}{2}mv_0^2. \end{aligned}$$

This expression represents the change in kinetic energy when the particle is accelerated from speed v_0 to speed v_1, and in particular, if $v_0 = 0$, we recover the expression (5) for the kinetic energy, and see that it is entirely independent of the manner in which the particle is accelerated.

done in *raising* the body vertically to a given *height* in order to find its increase in potential energy. For example, let us consider the situation in Fig. 3 which shows a large stone on a cliff at a height h above a lake; we wish to find the gravitational potential energy of the stone relative to the water surface as reference level. We may do this by finding the work done against the earth's gravitational pull when the stone is raised vertically to its position. The force F that must be exerted is equal to the weight mg of the stone and the work done is

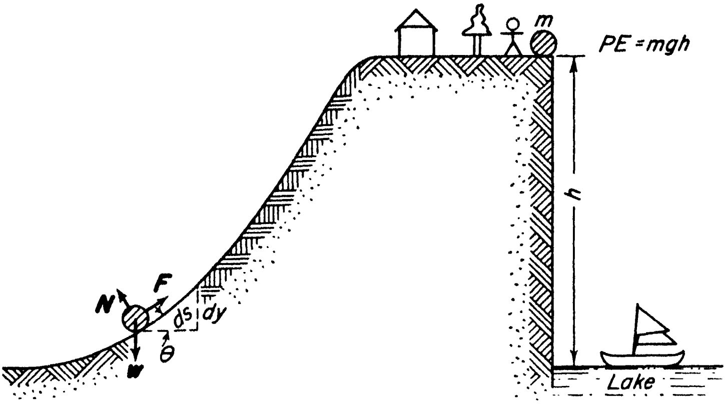

Fig. 3. The stone has potential energy mgh, *with respect to the lake surface.* At the left, w is written for mg.

$$W = Fh = mgh;$$

hence the stone's potential energy relative to its position at water level is equal to the work done in raising the stone:

$$\text{P.E.} = mgh. \tag{6}$$

It will be noted from this example that the potential energy is directly proportional to the height h *above some reference level,* which in this case we have chosen as the surface of the lake. The potential energy given by (6) represents only the ability of the stone to do work as it returns the whole distance to the reference level. For example, if the stone falls over the cliff, it can do work $W = mgh$ on a sailboat on the lake surface, but the stone has *no* ability to work on the objects on the top of the cliff. Thus, the reference level of potential energy is arbitrary; *only differences in potential energy at two different levels have physical significance.*

It is interesting to verify that the work required to pull a body up any type of inclined path is (friction being absent) just the same as the work mgh required to lift the body vertically the same vertical height. Suppose, as at the left of Fig. 3, we pull the body up a frictionless incline by applying a force just large enough to move the body without acceleration. Then, as we found in Chap. 3, we require $F = w \sin\theta$, where θ is the inclination of the hill at any stage. The work done when the body moves a short distance ds is $F\,ds = w\,ds\,\sin\theta$. But $ds\,\sin\theta = dy$, the increment in vertical height. Hence the work required to move the body a vertical height dy is $w\,dy$ independent of the inclination. Thus, the total work required to move the body up the hill of vertical height h is $wh = mgh$, independent of the shape of the incline. If there is friction, more work will be required but, as we shall see in Sec. 4, only the work wh is done against gravity and goes into increase of gravitational potential energy, the additional work done against the force of friction goes into heat.

For the present we shall discuss problems involving only gravitational potential energy, in Chap. 9, Elasticity and Strength of Materials, we shall consider the problem of the potential energy involved in the deformation of elastic bodies and in Part V we shall consider electrostatic potential energy.

PROBLEMS

1. A 3220-lb automobile has an instantaneous velocity of 60 ft/sec southward. What is the kinetic energy of the automobile? If frictional losses were negligible, how much work was done in giving the automobile this velocity?
Ans: 180,000 ft·lbf; 180,000 ft·lbf.

2. A truck weighing 8250 lbf is moving along a highway at a speed of 60 mi/hr. What is the kinetic energy of this truck? Assuming negligible frictional losses, calculate the work done in accelerating the truck.

3. A 7.25-kg shell fired from a gun has a muzzle velocity of magnitude 300 m/sec. What is the shell's kinetic energy? How much work can this shell do against retarding forces as a result of this kinetic energy? Ans: 326,000 joules; 326,000 joules.

4. What is the kinetic energy of a 20-kg shell moving at a speed of 200 m/sec? How much work was done in accelerating this shell? How much work can be done by this shell against retarding forces?

5. A stone with a mass of 1.20 slugs lies at rest at the top of a cliff 360 ft high. What is the potential energy of this stone with respect to the top of the cliff? With respect to the valley floor at the foot of the cliff? Ans: 0; 13,900 ft·lbf.

6. A 161-lb man stands on the roof of a building 86 ft high. What is his potential energy with respect to the sidewalk in front of the building?

7. A 200-kg bomb is released from an airplane flying at an altitude of 225 m. What is the initial potential energy of this bomb with respect to the ground?
Ans: 4.41×10^5 joules.

8. A waterfall is 33 m high. What is the potential energy of one cubic meter of water at the top of the fall with respect to the foot of the waterfall? How much work can this cubic meter of water do in a turbine located at river level below the fall?

3. TRANSFORMATIONS OF MECHANICAL ENERGY

It is sometimes convenient to refer to the kinetic and potential energy of a body as the body's *mechanical energy*. It can be proved from Newton's laws of mechanics that *provided there are no frictional or other dissipative effects, the total mechanical energy of a system of bodies remains constant.* We can easily demonstrate this statement for a ball thrown vertically upward from the earth's surface. Let m be the mass of the ball and $\mathbf{v}_0$ the initial upward velocity of the ball. In the absence of air resistance, the ball will experience a constant downward acceleration $\mathbf{g}$. Let us place the origin of our coordinate system at the point at which the ball is released and consider motion along the vertical axis. Taking the upward direction as positive, the initial upward velocity component is $+v_0$ and the downward acceleration has the upward component $-g$. Therefore, from equation (12) of Chap. 5, p. 105, we may write for the magnitude of the velocity v of the ball at position y:

$$v^2 = v_0^2 - 2gy.$$

By multiplying both sides of this equation by $\frac{1}{2}m$, we obtain

$$\tfrac{1}{2}mv^2 = \tfrac{1}{2}mv_0^2 - mgy,$$

and by transposition we obtain

$$\tfrac{1}{2}mv^2 + mgy = \tfrac{1}{2}mv_0^2, \qquad (7)$$

which is an expression easily interpreted in terms of mechanical energy.

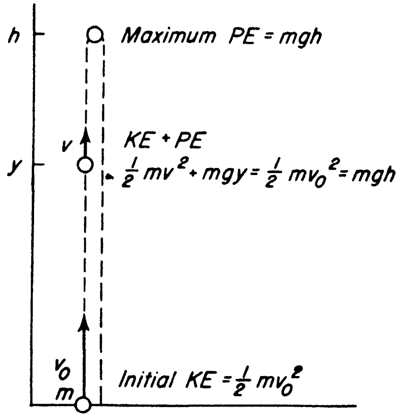
h
Maximum PE = mgh
v
KE + PE
y
$\frac{1}{2}mv^2 + mgy = \frac{1}{2}mv_0^2 = mgh$
v_0
m
Initial KE = $\frac{1}{2}mv_0^2$

Fig. 4. The sum of the kinetic energy and the potential energy of a ball thrown vertically upward from a horizontal plane is at all times constant and equal to the initial kinetic energy of the ball.

The quantities in equation (7) can be interpreted as follows:

$\frac{1}{2}mv^2$ = kinetic energy of the ball at height y,

mgy = potential energy of the ball at height y,

$\frac{1}{2}mv_0^2$ = initial kinetic energy of the ball.

The terms on the left-hand side of equation (7) represent the *total mechanical energy* (kinetic + potential) of the ball at any point in its path, and the term on the right-hand side represents the *total initial mechanical energy,* since we take the potential energy of the ball at the origin is zero. Thus, equation (7) states that at all times *the total mechanical energy of the ball is constant.* Provided there is no air resistance, mechanical energy is *conserved.* Air resistance is of the nature of a frictional force.

It should be pointed out that conservation of mechanical energy is not limited to one-dimensional motion as in the case we have been considering. In order to demonstrate this fact, let us consider the motion of a projectile of mass m fired from a gun with muzzle velocity of magnitude v_0 at an angle θ_0 to the horizontal as shown in Fig. 5.

Let the horizontal and vertical components of the muzzle velocity be v_{0x} and v_{0y}, respectively. In the absence of air resistance, v_x will at all times be equal to v_{0x}:

$$v_x = v_{0x}. \tag{8}$$

The upward acceleration in the coordinate system in Fig. 5 is equal to $-g$.

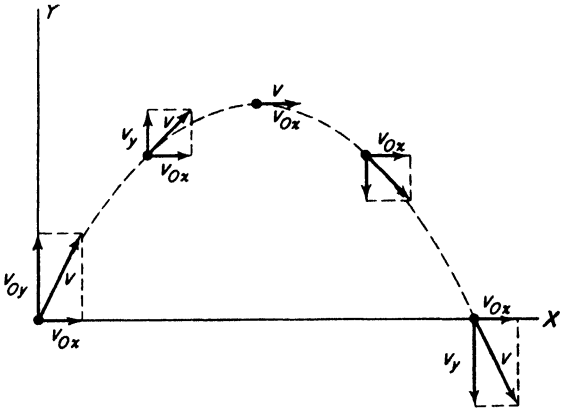

Fig. 5. Path of a projectile with velocities indicated at several points along the trajectory.

Hence, we may write

$$v_y^2 = v_{0y}^2 - 2gy, \quad \text{or} \quad v_y^2 + 2gy = v_{0y}^2. \tag{9}$$

Squaring both sides of equation (8) and adding the resulting equation to equation (9), we obtain

$$v_x^2 + v_y^2 + 2gy = v_{0x}^2 + v_{0y}^2.$$

Multiplication of both sides of this equation by $\frac{1}{2}m$ gives

$$\tfrac{1}{2}m(v_x^2 + v_y^2) + mgy = \tfrac{1}{2}m(v_{0x}^2 + v_{0y}^2),$$

or

$$\tfrac{1}{2}mv^2 + mgy = \tfrac{1}{2}mv_0^2. \tag{10}$$

The terms in equation (10) have the following interpretation:

$\frac{1}{2}mv^2$ = kinetic energy of the projectile at height y,

mgy = potential energy of the projectile at height y,

$\frac{1}{2}mv_0^2$ = initial kinetic energy = total initial mechanical energy.

Hence, we see that equation (10) has the same form and same meaning as equation (7), which was derived for one-dimensional motion.

PROBLEMS

NOTE: In working these problems, assume that frictional effects are negligible.

1. A body of mass 2 slugs is dropped from the top of a building 250 ft above street level. What is the potential energy of the body at the instant it is released? What is the kinetic energy of the body just before it strikes the street? What is the final speed of the body? Ans: 16,100 ft·lbf; 16,100 ft·lbf; 127 ft/sec.

2. A ball of 0.5-slug mass is dropped from a point 81 ft above ground level. What is the potential energy of the ball at the time of release? What is the kinetic energy of the ball just before it strikes the ground? What is the final speed of the ball?

3. A 16-lb body is dropped from a height of 400 ft. What is the kinetic energy of the body just before it strikes the ground? Ans: 6400 ft·lbf.

4. Two 50-lb sleds are initially at rest on a platform at a height of 30 ft above ground level. Sled *A* is pushed over the edge and allowed to fall vertically to the ground; sled *B* is allowed to slide down an inclined plane to the ground. The inclined plane makes an angle of 30° with the horizontal. Find the following quantities: (a) initial potential energy of sled *A*, (b) initial potential energy of sled *B*, (c) final potential energy of sled *A* at ground level, (d) final potential energy of sled *B* at ground level, (e) kinetic energy of sled *A* at the instant it reaches the ground, (f) kinetic energy of sled *B* at the instant it reaches the ground, and (g) the speeds of sleds *A* and *B* just as they reach the ground.

5. A 0.25-kg ball is thrown vertically upward with an initial velocity of 8.0 m/sec. Find (a) the initial kinetic energy of the ball, (b) the kinetic energy of the ball as it reaches its maximum height, (c) the potential energy of the ball at the highest point in its trajectory, and (d) the maximum height attained measured from the point of release. Ans: (a) 8.0 joules; (b) 0; (c) 8.0 joules; (d) 3.26 m.

6. If a 100-g ball were thrown vertically upward and had the same initial kinetic energy as the ball mentioned in Prob. 5, what is the maximum height it would attain? What would be the magnitude of its downward velocity when it returned to its initial point of release?

7. Two $\frac{1}{8}$-lb balls are thrown from the top of a building 328 ft high. Ball *A* is thrown downward with an initial velocity of 72.0 ft/sec and ball *B* is thrown vertically upward with an initial velocity of 72.0 ft/sec. Find the following quantities: (a) the initial potential energies of ball *A* and ball *B*, (b) the initial kinetic energies of ball *A* and ball *B*, (c) the speeds of balls *A* and *B* two seconds after release, and (d) the final speeds of balls *A* and *B* at ground level. Ans: (a) 41.0 ft·lbf and 41.0 ft·lbf; (b) 10.1 ft·lbf and 10.1 ft·lbf; (c) 136 ft/sec and 7.6 ft/sec; (d) 162 ft/sec and 162 ft/sec.

8. A 400-g ball is thrown horizontally from the top of a cliff 120 m high. Find (a) the initial kinetic energy of the ball, (b) the initial potential energy of the ball, (c) the final kinetic energy of the ball just before it strikes the ground, and (d) the final speed of the ball.

9. A 2-slug shell is fired with a muzzle velocity of 1600 ft/sec. If the elevation angle is 60°, find the magnitude of the shell's velocity when it is at altitude of 10,000 ft above the earth's surface. Ans: 1384 ft/sec.

10. What is the speed of the shell in Prob. 9 when it is 6400 ft above the ground? If the elevation angle had been 30°, what would have been the shell's speed at 6400 ft?

4. CONSERVATION OF ENERGY

The examples of the preceding section demonstrated the *principle of conservation of mechanical energy* for a projectile experiencing no frictional resistance. The same principle can readily be demonstrated in the case of a body sliding on a frictionless incline, whether or not the inclined surface is plane. For the work done by the force of gravity is, by definition of potential-energy change, the decrease in potential energy. Since the incline is frictionless, the force of the incline on the body is normal to the direction of motion and does no work. Hence the work done by the force of gravity is the work done by the resultant force and hence by definition is the increase in the kinetic energy. Therefore the increase in kinetic energy equals the decrease in potential energy, and the total mechanical energy is conserved.

The same type of argument can be applied to a system of bodies connected by strings and pulleys, provided there is no friction. The Atwood's machine of Fig. 8, p. 139, and the system of Fig. 9, p. 140 are simple examples. In these cases, the tension in the string, as well as the force of gravity, does work. But the work done by the string tension on one body is equal and opposite to the work done by the same tension on the other body, so the net work done by the string tension is zero. Hence it can be shown that if we consider the two bodies together, the sum of the potential energies plus the sum of the kinetic energies is conserved.

The following theorem* is proved directly from Newton's laws of motion in more advanced treatments of dynamics:

For any system of bodies, connected together in any manner, the total mechanical energy of the system is conserved, provided there are no frictional forces and no collisions between bodies.

In any actual physical situation frictional effects are always present; if these effects are small the above theorem is a very useful approximation. However, *whenever friction is involved, mechanical energy is not conserved;* we say that mechanical energy is *dissipated* in doing work against friction. Mechanical energy is also dissipated in collisions between bodies because of the presence of forces of the nature of internal friction: this will be discussed in detail in Chap. 10.

As an example of the way in which mechanical energy is dissipated against friction, let us consider the situation shown in Fig. 6. A box of mass m initially at rest moves down an inclined plane from a height h above a level floor. The potential energy of the box in the initial position (relative to the bottom of the plane) is given by mgh. If there were no friction between the box and the inclined plane, the kinetic energy $\frac{1}{2}mv^2$ of the box on reaching the lower end of the inclined plane would be exactly equal to the initial potential energy mgh. However, if friction is involved, some of the initial potential energy is dissipated in doing work against friction, and hence the kinetic energy of the box reaching the bottom of the inclined plane is less than its initial potential energy. Let

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* Newton and his contemporaries were aware of this theorem. It might be remarked that the term *energy* was not used in Newton's time. The quantity we call kinetic energy was called by the picturesque term *vis viva,* meaning 'living force.'

μ_k be the coefficient of friction. Since the normal force $N = mg\cos\theta$, the frictional force $\mathfrak{F}$ is given by

$$\mathfrak{F} = \mu_k mg \cos\theta,$$

and hence the work W done by the force of friction as the block moves distance l down the plane is

$$W = -\mathfrak{F}l = -\mu_k mgl \cos\theta. \tag{11}$$

This negative work done by friction is commonly called work done against friction if the sign is changed. We shall show that mechanical energy equal to the work done against friction disappears or is *dissipated.*

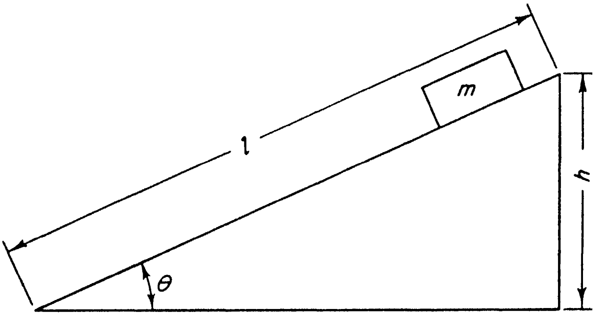

Fig. 6.

To see this, we note that the resultant force down the plane is the difference between the component of the weight and the force of friction, or

$$F = mg \sin\theta - \mathfrak{F}. \tag{12}$$

This causes an acceleration down the plane of magnitude $a = F/m$. The speed at the bottom of the plane is thus given by

$$v^2 = 2al = 2Fl/m.$$

The kinetic energy at the bottom is

$$\tfrac{1}{2}mv^2 = Fl = mgl \sin\theta - \mathfrak{F}l,$$

or

$$\tfrac{1}{2}mv^2 = mgh - \mathfrak{F}l. \tag{13}$$

Thus, the kinetic energy at the bottom of the plane is not so great as the potential energy mgh at the top of the plane; it is less than this potential energy by the work $\mathfrak{F}l$ done against friction. Mechanical energy $\mathfrak{F}l$ disappears during the motion and is said to be *dissipated in friction.*

Now let us consider the question: What becomes of the mechanical energy that is dissipated? It is recognized by everyone who has ever rubbed his hands together to warm them that work done against friction produces *heat.* However, not until more than a hundred years after Newton's death was it clearly understood that the amount of heat produced is directly proportional to the amount of mechanical energy dissipated and that *heat itself is a form of energy.* When work is done against friction, mechanical energy is converted into *heat* or *thermal energy;* in heat engines, thermal energy is converted into mechanical energy.* In both cases, one form of energy is transformed into another form of energy.

Recognition of this fact led to the formulation of one of the most important principles in physical science. This principle of *conservation of energy* can be stated as follows:

> *Energy cannot be created or destroyed, although it can be changed from one form to another.*

In addition to mechanical kinetic and potential energy and heat, there are still other forms of energy, which include chemical energy, electrical energy, magnetic energy, radiant energy, and nuclear energy. An understanding of the meaning of the conservation-of-energy principle and of the ways in which this principle may be applied is all-important in understanding the science of physics itself. The principle finds application in all branches of physics. There is no formal general proof of the principle, but we have no evidence that it is ever violated.

As an example of a process in which several energy transformations occur, let us consider a projectile of mass m fired from a cannon at the top of a cliff as shown in Fig. 7. The shell leaves the barrel, describes a parabolic path, and finally sinks into the mud in the marsh at the foot of the cliff. Since the projectile eventually does work in penetrating the mud in the marsh, let us use the level at which the projectile finally stops as the arbitrary reference level from which heights are to be measured in computing potential energy. Before the shell is fired from the gun, it has potential energy mgh_0, and the propellant has a certain amount of chemical energy. When firing occurs, some of this chemical energy is transformed into thermal energy, and, as a result of the rapid expansion of hot gaseous combustion products, a part of this thermal energy is used in doing work on the shell, which thereby acquires *kinetic energy* $\frac{1}{2}mv_0^2$, where v_0 gives the magnitude of the muzzle velocity. As the shell rises, some of its initial kinetic energy is transformed into potential energy. As the shell starts downward from the highest point in its trajectory, its kinetic energy increases as its potential energy decreases. If there is no air resistance, the total mechanical energy of the shell remains constant from the time it leaves the muzzle of the gun until it strikes the surface of the mud; hence, we might write

---

* The conversion of heat into mechanical energy will be discussed in detail in Chap. 19.

$$\tfrac{1}{2}mv_0^2+mgh_0=\tfrac{1}{2}mv^2+mgh, \tag{14}$$

$$\left\{\begin{matrix}\text{initial mechanical energy}\\ \text{of the projectile}\end{matrix}\right\}=\left\{\begin{matrix}\text{mechanical energy of the}\\ \text{projectile at any height } h\end{matrix}\right\}$$

where v is the speed of the shell at any height h above the reference level. Equation (14) applies at all times when the shell is in free flight. However, once the shell strikes the surface of the marsh, it begins dissipating its mechanical energy in doing work against frictional forces. As the shell plows its way into the mud, its speed gradually decreases and it eventually comes to rest at the reference level. The total work done against friction is equal to the total mechanical energy associated with the projectile at the moment it enters the mud. If $\bar{\mathfrak{F}}$ is the average value of the frictional force encountered by the shell in moving a vertical distance h_1 through the mud, we may write for the work done against friction,

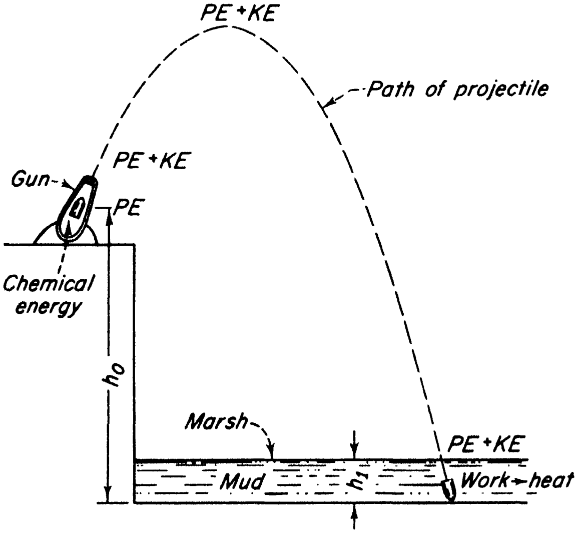

Fig. 7. Energy transformations.

$$\bar{\mathfrak{F}}h_1=\tfrac{1}{2}mv_1^2+mgh_1 \tag{15}$$

where v_1 is the speed of the projectile at height h_1 above the reference level, that is, the speed of the shell at the instant it enters the mud. However, from equation (14), the term on the right side of the equation (15) is equal to the initial mechanical energy of the projectile leaving the muzzle of the gun. Hence, the entire initial mechanical energy of the shell is dissipated as heat when the shell penetrates the mud. The resulting thermal energy is evidenced by an increase in the temperature of the shell and the surrounding mud.

The heat produced in processes similar to the one just described is sometimes readily observable. For example, a nail becomes noticeably warmer when driven into or suddenly withdrawn from a board. Similarly, when a bullet enters a piece of wood, the walls of the resulting hole sometimes become charred as a result of the high temperatures attained, and a lead bullet sometimes melts. It is also observed that fast-moving meteors are heated to incandescence on entering the earth's atmosphere.

In closing the present discussion, it may be well to emphasize once more that the *principle of conservation of energy* is applicable in all branches of physics and there is no evidence that there is ever any violation of this principle. The *total* energy in the *universe* seems to be constant! In the chapters that follow, we shall have many occasions to apply this principle.

PROBLEMS

1. A sled weighing 40 lbf is at the top of a hill at a height of 60 ft above a level field. The sled slides down a runway to the foot of the hill and its final speed at the end of the runway is 50 ft/sec. How much mechanical energy was dissipated during the sled's descent? Ans: 848 ft·lbf.

2. If the final speed of the sled in Prob. 1 had been 45 ft/sec, how much mechanical energy would have been dissipated?

3. A 150-g ball with a rough cover is thrown vertically upward with an initial velocity of 20 m/sec. The ball rises to a height of 15 m. How much mechanical energy is dissipated during the upward motion of the ball? Ans: 7.94 joules.

4. If the final velocity of the ball in Prob. 3 when it returns to the ground is 15 m/sec, how much mechanical energy was dissipated during its downward passage?

5. An automobile weighing 3220 lbf moves at a speed of 60 ft/sec along a level roadway. When the car reaches a certain hill, the driver allows the car to coast uphill until it comes to rest. If the final position of the car on the hill is 45 ft above the level of the approaching roadway, how much mechanical energy was dissipated during the car's ascent of the hill? Ans: 35,100 ft·lbf.

6. If the car in Prob. 5 were allowed to roll backward down the hill and the same amount of mechanical energy were lost during descent as during ascent, what would be the speed of the car when it reached the foot of the hill?

7. A large crate weighing 400 lbf is permitted to slide down a ramp inclined at an angle of 30° to the horizontal. The length of the ramp is 50 ft. If the magnitude of the acceleration of the crate is 4 ft/sec^2, how much mechanical energy is dissipated during descent? Ans: 7510 ft·lbf.

8. What was the velocity of the crate in Prob. 7 when it reached the foot of the ramp? What was the kinetic energy? What was the coefficient of friction between the crate and the ramp?

9. The 1-ton hammer of a pile driver drops from a vertical height of 10 ft, strikes a pile, and drives the pile 4 inches into the ground. Assuming that all the energy of the pile driver is communicated to the pile, compute the average force resisting the motion of the pile. Ans: 60,000 lbf.

10. If a 2-ton hammer had been used in the pile driver in Prob. 9 and the average force resisting the motion of the pile had been the same, how far would the pile have been driven into the ground?

5. SIMPLE MACHINES

From the energy-conservation principle, we are led to the conclusion that the performance of a given amount of useful work requires the expenditure of an equal amount of energy. At first glance, it might also appear that the choice of a method or procedure for performing work is immaterial. However, further consideration reveals that the question of *convenience* should be considered. For example, since

$$\text{work} = Fs \cos\theta,$$

there is an immediately apparent choice of exerting a large force to produce a small displacement or a small force to produce a large displacement, as well as a choice of the direction θ in which the force is to be applied with respect to the displacement. *A machine is a device that enables us to do work more conveniently than would otherwise be possible.*

In performing a given amount of work, for example in raising a large body of weight w to a height above its original position, it is *usually more convenient* for the operator to exert a small force through a large distance rather than a large force through a small distance. The operator uses a machine designed in such a manner that when he applies a small force $\boldsymbol{f}$ to the machine, the machine exerts a larger force on the load. When the small applied force produces a large displacement of some part of the machine, the machine produces a small displacement of the large load. In order to have some way of comparing the relative merits of various machines, it is desirable for us to define certain pertinent quantities.

As mentioned above, one usually wishes to employ a machine that applies a large force to the load when a smaller force is applied to the machine. The quantity that is used to describe this property is called the *mechanical advantage* of the machine.

> The mechanical advantage of a machine is the ratio of the force exerted *by* the machine to the force applied *to* the machine:
>
> $$\text{mechanical advantage} = \frac{\text{force exerted by the machine}}{\text{force applied to the machine}}.$$

The ratio may be either greater than unity or less than unity. For example, a crowbar as normally used has a mechanical advantage greater than unity, whereas the oar of a rowboat has a mechanical advantage less than unity.

A second quantity of importance in describing the operations of machines is *efficiency*. Since there is always some friction in every machine, the work done *by* the machine—the *work output*—is always less than the work done *on* the machine—the *work input*. This statement is in accord with the conservation-of-energy principle, since any work done against friction is converted into *heat*.

> The efficiency of a machine is the ratio of the work output to the work input:
>
> $$\text{efficiency} = \frac{\text{work output}}{\text{work input}}.$$

It is *usually* desirable for the efficiency of a machine to be as near unity (100%) as possible. If there is a large amount of energy involved, a low efficiency is objectionable from two points of view: first, the dissipation of large quantities of energy is usually expensive; and, second, the accompanying production of large amounts of heat may result in damage to the machine itself. For example, when large amounts of heat are produced in a bearing, the bearing may 'burn out' or, in the case of a railroad car, a 'hotbox' may result. There are some special cases, however, in which low efficiency is desirable; for example, the jackscrew described below *must* have a low efficiency for successful operation.

We shall mention one other quantity that is sometimes of use in describing machines. This quantity is called the *ideal mechanical advantage.*

> The ideal mechanical advantage of a machine is the mechanical advantage the machine would have if there were no frictional effects.

This quantity can be determined in terms of the distances through which the force applied to the machine and the force applied by the machine to the load move. If l is the distance moved by the point of application of the force F applied to the machine and h the distance a load w is raised, we may write the following equation for an ideal machine:

$$\text{work input} = \text{work output},$$

$$Fl = wh, \tag{16}$$

where w is the weight of the load and F is the force that must be applied to the ideal machine. Now the mechanical advantage of this ideal machine is w/F, and an expression for this ratio can be obtained from equation (16):

$$\text{ideal mechanical advantage} = w/F = l/h. \tag{17}$$

Thus, we see that the ideal mechanical advantage of the machine can be expressed as the ratio l/h. There is a useful relation between the mechanical advantage, the ideal mechanical advantage, and the efficiency of a machine, which we shall derive in connection with one of the simple machines to be described.

Although some machines used in engineering and industrial work are highly complicated, the principles involved in their construction and operation can be described in terms of a few devices called *simple machines,* such as the inclined plane, the lever, and the simple pulley. In order to illustrate the ways in which the quantities *mechanical advantage* and *efficiency* can be used, we shall now discuss several of these devices and leave the treatment of others as problems at the end of this section.

First, let us consider the inclined plane as a machine. When employed as a *ramp* in a railroad station, the inclined plane permits the raising of large pieces of baggage or freight by the application of smaller forces than would have to be employed in raising them vertically. Figure 8 shows such an inclined plane along which a trunk of mass m is being moved at constant speed. By applying a force f parallel to the ramp in the manner shown, the desired effect of raising the trunk to the height h is accomplished. Since the effect accomplished is to raise the weight $w = mg$ against gravity, the mechanical advantage of the ramp is mg/f. Since f is the force required to maintain constant velocity along the incline, it is equal to the sum of the component $mg\ \sin\theta$ of the weight which acts down the plane and the frictional force $\mu_k mg\cos\theta$ which opposes the motion, or

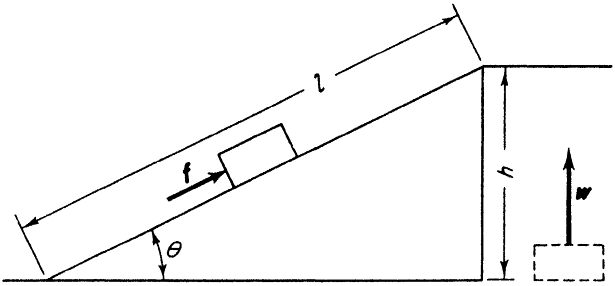

Fig. 8. The inclined plane as a simple machine.

$$\text{mechanical advantage} = \frac{mg}{mg\sin\theta + \mu_k mg\cos\theta} = \frac{1}{\sin\theta + \mu_k\cos\theta}.$$

The ideal mechanical advantage of the ramp is the mechanical advantage the ramp would have if $\mu_k = 0$, or

$$\text{ideal mechanical advantage} = mg/(mg\sin\theta) = 1/\sin\theta.$$

Since $\sin\theta = h/l$, the ideal mechanical advantage may be written

$$\text{ideal mechanical advantage} = l/h,$$

where l is the distance moved by the point at which f is applied and h is the vertical displacement of the load; this result is in accord with equation (17), which was obtained from energy considerations.

Now let us write an expression for the efficiency of the ramp. Since the work done on the machine by the operator is fl, we may write

$$\text{work input} = fl = (mg\sin\theta + \mu_k mg\cos\theta)l.$$

The useful work performed by the machine is simply the product of the trunk's weight and its vertical displacement, or

$$\text{work output} = mgh.$$

Hence, the efficiency of the ramp is given by

$$\text{efficiency} = \frac{mgh}{(mg\sin\theta + \mu_k mg\cos\theta)l}.$$

By rearrangement of this expression,

$$\text{efficiency} = \frac{1/(\sin\theta + \mu_k\cos\theta)}{l/h},$$

where the numerator is recognizable as the mechanical advantage and the denominator as the ideal mechanical advantage of the machine.

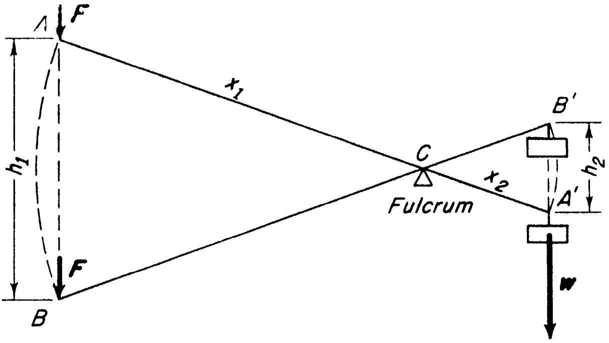

Fig. 9. The lever.

Hence, we may write the relation

$$\text{efficiency} = \frac{\text{mechanical advantage}}{\text{ideal mechanical advantage}}.$$

This result is usually applicable to problems involving machines but, as indicated below, should be used with discretion.

Let us now consider a simple lever used for raising a load of weight w in the manner indicated in Fig. 9. We shall consider the lever as balanced at the fulcrum, so its weight need not be considered. Line AA' represents the initial position of the lever, from one end of which is suspended the load w to be lifted. Vertical force $\boldsymbol{F}$ is applied to its left end, to rotate the lever about a horizontal axis at the fulcrum C until the lever is in position BB'. This rotation gives the load an upward displacement h_2. The downward displacement of the point at which vertical force $\boldsymbol{F}$ is applied is h_1. The mechanical advantage is given by the ratio w/F, which is always less than the ideal mechanical advantage h_1/h_2 because of frictional effects at the fulcrum and elsewhere. The efficiency of the lever is given by $wh_2/Fh_1 = (w/F)/(h_1/h_2)$. Sometimes it is more convenient to measure the ideal mechanical advantage as x_1/x_2, where x_1 and x_2 are the readily measurable distances between the points of application of F and w, respectively, from the fulcrum; that $x_1/x_2 = h_1/h_2$ is seen from the similarity of triangles ABC and $A'B'C$.

In the above discussion, we see that the efficiency is the ratio of the mechanical advantage to the ideal mechanical advantage. However, if the weight of the lever itself acting at its center of gravity also exerts a torque that tends to raise the load, we might compute an efficiency of more than 100 per cent from the ratio of mechanical advantage to ideal mechanical advantage. A little reflection reveals that in this case work must be done on the lever itself in giving it its original position AA', and this preliminary work should be considered in calculating the efficiency by its definition as the ratio of output work to input work.

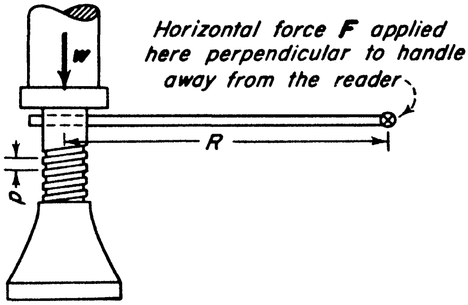
Horizontal force F applied here perpendicular to handle away from the reader
w
R
P

Fig. 10. The jackscrew. When the point of application of force F moves distance $2\pi R$, the weight w is raised a vertical distance P equal to the distance between adjacent threads on the screw.

The jackscrew is a simple machine used to lift large loads by applying relatively small forces. In the jackscrew shown in Fig. 10, one turn gives the load a vertical displacement equal to the pitch of the screw, that is, the distance P between adjacent threads. The work done is wP, where w is the weight of the load. In order to produce this vertical displacement of the load, a force F is applied near the end of the handle in a direction perpendicular to the handle in the indicated manner. The point of application of the applied force must move around the circumference of the circle whose radius R is the length of the lever arm. The work done by this applied force for one turn is therefore $2\pi RF$. If there were no friction, we would have $2\pi RF = wP$, and the ideal mechanical advantage, defined as w/F in the frictionless case, has the value

$$\text{ideal mechanical advantage} = 2\pi R/P.$$

In the jackscrew the mechanical advantage is always much smaller than this, since in a screw there is always a great deal of friction. Here this property is desirable, since a useful jack must support the load after the load has been raised; a frictionless jackscrew would unwind as soon as the applied force F ceased to act and thereby would allow the load to descend.

The pulley wheel is another simple machine. A single pulley wheel such as the one shown in Fig. 11(a) has an ideal mechanical advantage of unity and is used merely as a convenience in permitting the application of a force in a desired direction. Thus, the weight w in Fig. 11(a) could be lifted by applying force F *upward*, but, since it is in many cases 'easier' (that is, more convenient) to pull *downward*, the single pulley is a useful device. The system involving two pulley wheels shown in Fig. 11(b) has an ideal mechanical advantage of 2. The ideal mechanical advantage can be determined by measuring the distance moved by the point at which force F is applied and the corresponding vertical displacement of the load. An alternative method of determining the ideal mechanical advantage of this pulley system involves the fact that the tension at all points in the long rope is constant, provided there is no friction in the pulleys. In the arrangement shown in Fig. 11(b), the magnitude of the tension is at all points equal to F, the applied force. If the weight of the pulleys can be ignored, the tension in the short rope supporting the top

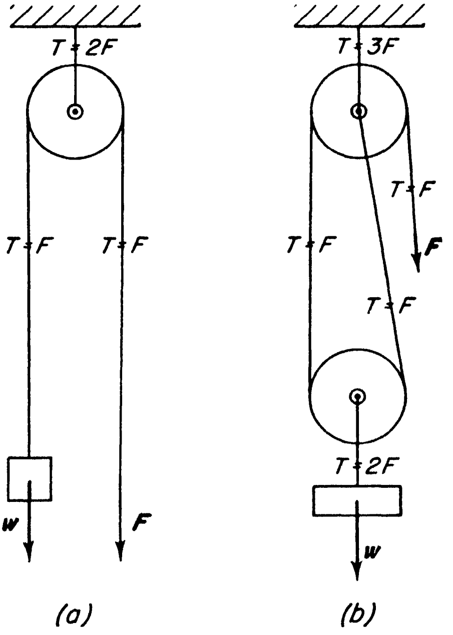

Fig. 11. Pulley systems.

pulley will be $3F$ and the tension in the short rope supporting the load will be $2F$, so $w = 2F$ provided there is no friction.

PROBLEMS

1. A ramp inclined at angle of 30° with the horizontal is used in loading crates on a railroad car. If there were no friction involved, what would be the mechanical advantage of this ramp? The coefficient of friction between the crates and the ramp is actually 0.2. What is the actual mechanical advantage of the ramp? What is the efficiency of the ramp? If the ramp were used to raise a 1000-lb crate a vertical distance of 5 ft, how much mechanical energy would be dissipated?
Ans: 2; 1.49; 74.3%; 1730 ft·lbf.

2. The ramp in Prob. 1 is modified so that the angle of inclination is 15°. For this new ramp answer all questions asked in Prob. 1.

3. A lever of uniform cross section is used as in Fig. 9 to raise a load of 200 lbf through a vertical distance of 1.5 ft. The lever is 10 ft in length and has a weight of 40 lbf. If the fulcrum is placed 3 ft from the load end and there is no friction, what must be the value of the force F? What is the mechanical advantage of the machine as measured by the ratio h_1/h_2? by the ratio w/F? Explain any discrepancy. How much work is done by the force F as shown in the figure? How much work is done on the load? How much preliminary work must be done in raising the lever to its initial position? Ans: 74.3 lbf; 2.33; 2.69; 260 ft·lbf; 300 ft·lbf; 40 ft·lbf.

4. The fulcrum for the lever mentioned in Prob. 3 is moved to a position 2 ft from the load end of the lever and the load is raised 1 ft. For this new arrangement, answer all questions asked in Prob. 3.

5. For the jackscrew shown in Fig. 10, the pitch P of the screw is 0.50 in and the force perpendicular to the lever is applied at a point 4 ft from the axis of rotation of the screw. What is the ideal mechanical advantage of the jackscrew? When the jackscrew is used to raise a weight $w = 5000$ lbf, a force $F = 50$ lbf must be applied in a direction perpendicular to the lever. What is the actual mechanical advantage of the jackscrew? The screw mechanism itself including the lever weighs 200 lbf. How much work is done against frictional forces in raising the load a distance of 10 in? How much useful work was done in raising the load? How much useless work was done in raising the screw?
Ans: 603; 100; 20,700 ft·lbf; 4170 ft·lbf; 167 ft·lbf.

6. Suppose that the force on the lever in Prob. 5 had been applied at a point 5 ft from the axis of rotation of the screw. What would have been the ideal mechanical advantage of the jackscrew? the actual mechanical advantage? How would the total work done in raising the load be changed?

7. What is the ideal mechanical advantage of the pulley system shown in Fig. 11(a)? If the force F needed to raise a load of 100 lbf is actually 110 lbf, what is the actual mechanical advantage of this pulley? its efficiency?
Ans: 1.000; 0.909; 90.9%.

8. What is the ideal mechanical advantage of the pulley system shown in Fig. 11(b)? In lifting a weight of 200 lbf, a force F of 133.3 lbf is actually needed. What is the actual mechanical advantage of this pulley system? its efficiency?

9. The lower pulley block in the system of Prob. 8 weighs 20 lbf. When the load of 200 lbf is raised through a vertical distance of 2 ft, how much useful work is done in raising the load? How much useless work is done in raising the lower pulley block? How much work is done against frictional forces?
Ans: 400 ft·lbf; 40 ft·lbf; 93 ft·lbf.

10. Assuming that the frictional losses are proportional to the total weight (load plus weight of lower block) supported by the upper pulley block, calculate the total work that must be done in raising a 400-lbf load a vertical distance of 3 ft with the pulley arrangement discussed in Prob. 9.

11. In a wedge such as that shown in Fig. 12, the critical dimensions are $a = 1$ in, $b = 6$ in. What is the ideal mechanical advantage of this device? Draw a diagram showing the forces exerted on the wedge. Ans: 6.

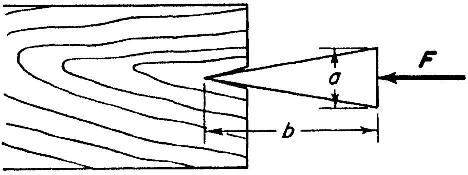

Fig. 12. The wedge used in splitting logs. By applying a small longitudinal force F as shown, one may produce large transverse forces at right angles to F, which are effective in splitting the log. (The artist was apparently not a woodsman—a log splits *along* the grain.)

12. Frictional forces exerted parallel to the slanting faces of the wedge decrease the mechanical advantage of the wedge. However, these frictional forces serve a useful purpose. If they were absent, what would happen if force F were removed after the wedge had been driven part way into a log?

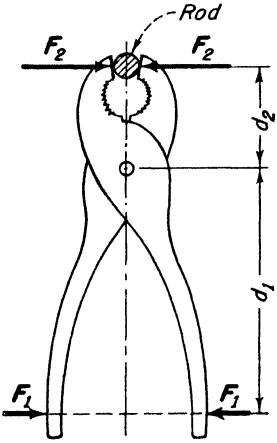

Fig. 13. Pliers. By applying small forces F_1 to the handles one may produce large forces F_2 at the rod.

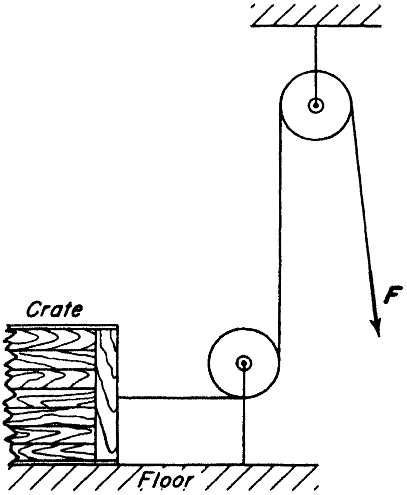

Fig. 14. A system of stationary pulleys used for pulling crates across rough floors.

13. In the pliers shown in Fig. 13, $d_1 = 4.5$ in and $d_2 = 1.5$ in. When the forces $F_1 = 20$ lbf are applied to the handles, what forces F_2 would be applied by the jaws of the pliers? Ans: 60 lbf.

14. If the rod being held by the pliers in Fig. 13 were moved in toward the axis to a point where $d_2 = 1$ in, what forces would be exerted on the rod when $F_1 = 20$ lbf?

15. What is the ideal mechanical advantage of the pulley system shown in Fig. 14? Is anything to be gained by using such a pulley system? Ans: 1.

16. In moving a 200-lb crate across a floor at constant speed, a force $F = 40$ lbf must be applied to the end of the rope shown in Fig. 14. What is the value of the coefficient of friction between the crate and the floor if the pulleys are frictionless?

17. What is the ideal mechanical advantage of the pulley system shown in Fig. 15? If a force of 60 lbf is required to raise a load $w = 200$ lbf at constant velocity, what is the actual mechanical advantage of the pulley system? the efficiency? Ans: 4; 3.33; 83.3%.

18. If the pulley system of Prob. 17 is used to raise the 200-lbf load 2 ft, how much work is done by the force $\boldsymbol{F}$? How much work is done on the load $\boldsymbol{w}$? If the lower pulley block weighs 20 lbf, how much work is dissipated in heat?

19. In the hydraulic press shown in Fig. 16, the area a_1 of the small piston is 1 in², and the area a_2 of the large piston is 120 in². A force $F_1 = 2.5$ lbf is required to raise a load $w = 200$ lbf. From these data, calculate (a) the ideal mechanical advantage, (b) the actual mechanical advantage, and (c) the efficiency of this device. Ans: (a) 120; (b) 80; (c) 66.7%.

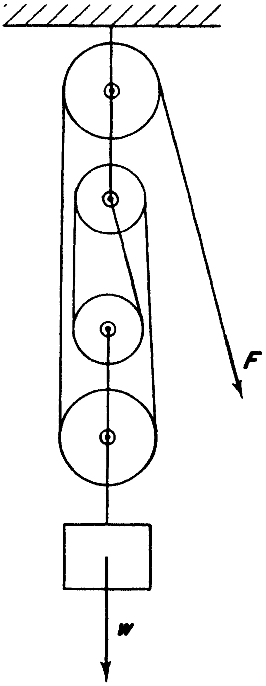

Fig. 15. A pulley system.

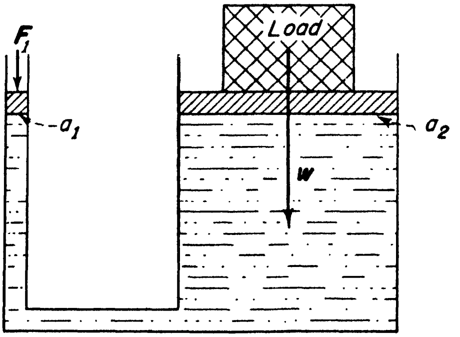

Fig. 16. Hydraulic press.

20. How much work must be done by force $\boldsymbol{F}_1$ in order to raise the 200-lbf load in Prob. 19 a distance of 2 in? Through what total distance must the small piston move?

6. POWER

Time is not involved in any way in the definition of *work*. The same amount of work is done in raising a given weight through a given vertical distance regardless of the time required. For example, the work done in raising a trunk to a loading platform at a railroad station is the same whether the work is done in one second, one day, or one month. However, the *time rate* at which work is done is an important quantity called power:

> Power is the time rate at which work is done. Power is a scalar quantity.

If W represents the work performed in a time interval (t_1-t_0), the average power $\bar{P}$ is defined by the relation

$$\bar{P}=W/(t_1-t_0).$$

The power at any instant is given by the relation

$$P=dW/dt,$$

where this derivative (the limit of $\bar{P}$ as $(t_1-t_0)\to 0$) is evaluated at the instant in question.

The unit used for the measurement of power involves the ratio of a work unit to a time unit. Thus, in the British engineering system, we can use the ft·lbf/sec as a power unit. In the MKS system, the corresponding unit is the *joule/sec*, which is given a special name: the *watt* (w).

$$1 \text{ watt}=1 \text{ joule/sec}.$$

The watt and the ft·lbf/sec are both inconveniently small units for many practical power measurements, and hence it has been found desirable to define larger units. In the MKS system two multiples of the watt are commonly used: the *kilowatt* (1 kw = 1000 watts) and the *megawatt* (1 megawatt = 1000 kw = 1,000,000 watts). In the British engineering system, the commonly used power unit is called the *horsepower* (hp), defined by

$$1 \text{ hp}=550 \text{ ft·lbf/sec}=33{,}000 \text{ ft·lbf/min}.$$

From the relations between the newton, pound-force, meter, and foot, it can easily be shown that

$$1 \text{ hp}=745.7 \text{ watts}=0.7457 \text{ kw}.$$

It is convenient in making rough calculations to remember that 1 horsepower is about ¾ kilowatt.

We shall now consider two examples that illustrate the way in which the concept of *power* is applied and the way in which relations between power units are used. Suppose that a 200-lb man climbs up a long stairway and, at the end of 50 seconds, reaches a point 40 ft above the level at which he started. Let us find the average power developed by the man in this process if no frictional effects are involved. First, to find the work done, we note that the man's vertical displacement h is accomplished as a result of work done against the pull of gravity on the man; thus we may write

$$\text{work}=wh=200 \text{ lbf}\cdot 40 \text{ ft}=8000 \text{ ft·lbf}.$$

Since the time required for this work was 50 sec, the man's average power output is

$$\text{average power} = \frac{\text{work done}}{\text{elapsed time}} = \frac{8000 \text{ ft·lbf}}{50 \text{ sec}} = 160 \text{ ft·lbf/sec}.$$

To convert to horsepower, we substitute 1 ft·lbf/sec = $\frac{1}{550}$ hp, to obtain

$$\text{average power} = 160\ (\tfrac{1}{550} \text{ hp}) = 0.29 \text{ hp}.$$

To convert to kilowatts, we use the relation 1 hp = 0.746 kw to obtain

$$\text{average power} = 0.29\ (0.746 \text{ kw}) = 0.22 \text{ kw}.$$

It should be noted that the average power we have calculated is the power involved in giving the man a certain vertical displacement in a given time interval. The man undoubtedly expends energy (obtained from the chemical energy in food) at a greater rate because some of the energy is wasted as heat in the muscles and elsewhere.

As a second example, let us find the minimum time in which a $\frac{1}{10}$-hp motor could raise a 2-ton packing case from the ground to a loading platform 5 ft above ground level. We shall ignore the effects of friction. The total amount of work needed is given by

$$W = wh = 4000 \text{ lbf} \cdot 5 \text{ ft} = 20{,}000 \text{ ft·lbf}.$$

The power output P of the $\frac{1}{10}$-hp motor is

$$P = \tfrac{1}{10} \cdot 550 \text{ ft·lbf/sec} = 55.0 \text{ ft·lbf/sec}.$$

Therefore, the time t required for performing the work is given by

$$t = \frac{W}{P} = \frac{20{,}000 \text{ ft·lbf}}{55 \text{ ft·lbf/sec}} = 363 \text{ sec}.$$

Thus, it will require the small $\frac{1}{10}$-hp motor at least 363 sec, or 6 min 3 sec, to raise the load. If the job must be accomplished in a shorter time, a larger, more powerful motor must be used.

Having defined the conveniently large *power units*, the horsepower and the kilowatt, we may now define two additional conveniently large *work units*, the *horsepower·hour* (hp·hr) and the *kilowatt·hour* (kwh).

> One horsepower·hour is the work done in one hour by a device working at constant rate of one horsepower.

Since such a device does 33,000 ft·lbf of work per minute, the work done in one hour is 60 min·33,000 ft·lbf/min = 1,980,000 ft·lbf, or

$$1 \text{ horsepower·hour} = 1.98 \times 10^6 \text{ ft·lbf}.$$

> One kilowatt·hour is the work done in one hour by a device working at a constant rate of one kilowatt.

Since 1 kw = 1000 w = 1000 joules/sec, and 1 hr = 3600 sec,

$$1 \text{ kwh} = (1 \text{ kw}) \times (1 \text{ hr}) = (1000 \text{ joules/sec})\ (3600 \text{ sec}) = 3{,}600{,}000 \text{ joules},$$

or,

$$1 \text{ kwh} = 3.6 \times 10^6 \text{ joules}.$$

It should be noted that the horsepower·hour and the kilowatt·hour are units of *work* or *energy*, *not power*. Thus, when an electric 'power company' presents a bill to a customer, charges are actually made for *energy*, not power. In the form of electrical energy, a kilowatt·hour can be purchased at a price that varies from a few tenths of a cent to a few cents, depending upon the locality of the consumer and on the quantity of energy he purchases. Since input power requirements of electrical appliances are stated in *watts* and electric utility rates are stated in terms of *kilowatt·hours*, there is a popular misconception that there is something 'electrical' about a watt, a kilowatt, and a kilowatt·hour. This is not the case. These particular units occur because all electrical units and the whole of electrical engineering are based on the *metric* system, as distinguished from mechanical engineering, which in the United States generally employs British system units such as horsepower. Although electrical power is commonly measured in watts and kilowatts and electrical energy is usually measured in kilowatt·hours, it would be quite correct to state the power rating of incandescent lamps in horsepower and to measure electrical energy in horsepower·hours. Similarly, it would be entirely correct to rate automobile engines in kilowatts.

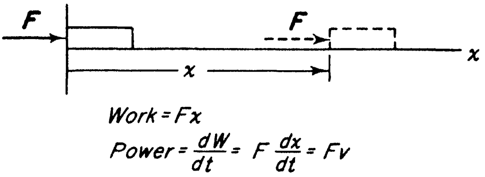

Fig. 17. A constant force parallel to the x-axis producing a displacement parallel to the x-axis. The work done is directly proportional to the displacement; the power supplied is directly proportional to the speed.

Let us now compute the power expended by a force acting on a moving body, that is, the *rate* at which the force does work. Suppose a constant force of magnitude F acting in the x-direction is exerted on a body as shown in Fig. 17 while the body undergoes a displacement x in time t. Then the work done is $W = Fx$, and the power developed is

$$P = W/t = Fx/t = Fv.$$

If $\boldsymbol{F}$ is not constant, but does work $F_{av}\,\Delta x$ while the body moves a distance Δx in the time Δt, the average power is

$$\bar{P} = \Delta W/\Delta t = F_{av}\,\Delta x/\Delta t = F_{av}\bar{v}$$

In the limit, as $\Delta t \to 0$, this equation becomes

$$P = Fv, \tag{18}$$

where v is the instantaneous value of the velocity of the body on which the force F acts.

The derivation of an expression for the power P that must be developed to give a body of mass m a constant acceleration a is not difficult. If there is no friction, the constant force $F = ma$ will be required, and (18) gives

$$P = (ma)v.$$

If the body starts from rest at $t = 0$ and experiences constant acceleration, $v = at$: hence

$$P = ma^2t. \tag{19}$$

Thus, if constant acceleration a is to be maintained, the power output of the device causing the acceleration must increase with time. Equations (18) and (19) apply only to the case of *rectilinear motion.*

One further comment on the use of the power concept might be worth while. It is common practice in certain types of work to express the efficiency of a machine as the ratio: *power output/power input.* This practice is justified when no energy is stored in the machine, as can be seen from the following argument: Let the input power P_i and the output power P_o be constant. Then, in time Δt, the input work $W_i = P_i\,\Delta t$ and the output work $W_o = P_o\,\Delta t$. We may write

$$\text{efficiency} = W_o/W_i = (P_o\,\Delta t)/(P_i\,\Delta t) = P_o/P_i,$$

provided there is no *storage* of energy, that is, if all the energy supplied *to* the machine in Δt is expended *by* the machine in the same time interval Δt. However, if there is storage of energy, the ratio P_o/P_i does not give the efficiency. For example, in the case of a clock spring, work done to the spring in winding the clock may be accomplished in 15 sec, but this energy is expended over a 24-hr period. Hence, the power ratio gives an incorrect measure of efficiency, and we must use the ratio W_o/W_i, where W_i is the work done in winding the spring and W_o is the work done by the spring in moving the hands and other parts of the clock.

PROBLEMS

1. A man weighing 200 lbf climbs a stairway in 2 min. If his upward displacement is 60 ft, how much work does he do against the pull of gravity? What is his average useful power output? Express this power output in hp.
Ans: 12,000 ft·lbf; 100 ft·lbf/sec; 0.182 hp.

2. A 2-ton elevator experiences a vertical displacement of 60 ft in 45 sec. Find the total work done, and the useful power output of the motor operating the elevator.

3. A 3220-lb automobile starts from rest and attains a speed of 80 ft/sec in 40 sec. Find the total useful work done on the car and the average power output of the automobile engine that was effective in accelerating the car. Express the average power in hp units. Ans: 320,000 ft·lbf; 8000 ft·lbf/sec; 14.5 hp.

4. A 4000-lb automobile moving at a speed of 30 mi/hr is brought to rest in 10 sec. What is the average braking power involved in this process?

5. A 5-kg projectile is fired with a muzzle velocity of 400 m/sec from a gun barrel 4 m long. Assuming uniform acceleration of the projectile in the gun barrel, calculate the average and the maximum power involved in acceleration of the projectile. Ans: 20 megawatts, 40 megawatts.

6. A 1000-kg automobile starting from rest acquires a speed of 36 km/hr in 20 sec. What is the average effective power output of the automobile engine?

7. A 100-kg packing box is moved 15 m across a rough floor in 1 min. If the coefficient of friction between the box and the floor is 0.4, what power is involved in moving the box? Ans: 98.1 watts.

8. How long would be required to move the box in Prob. 7 across the floor if a motor of ½-kw output could be used in doing the job?

9. Water flows over a 50-ft waterfall at the rate of 40 ft³/min. What would be the maximum power output of a water wheel operated at this waterfall if all the energy of the falling water could be utilized? Express this power in hp and in kw. Ans: 2080 ft·lbf/sec; 3.78 hp; 2.82 kw.

10. Find the power available from a waterfall 120 ft high if the flow of water is 1500 ft³/min. Express the result in hp and kw.

11. A motor of 2-hp output is used to raise a 1-ton elevator cage. How far can this motor raise the cage in one minute? (Assume that the cage moves with constant velocity.) Ans: 33 ft.

12. What horsepower is required to raise a 2-ton elevator cage at the constant speed of 15 ft/sec?

13. A locomotive exerts a constant drawbar pull of 4 tons-force (8000 lbf) while pulling a freight train at 45 mi/hr on a level track. What is the useful power developed by this locomotive? Ans: 5.28×10^5 ft·lbf/sec = 960 hp.

14. If the locomotive of Prob. 13 is pulling 1000 tons of cars up a 1-percent grade at 45 mi/hr, what horsepower is the engine delivering if it is assumed that 4 tons-force are still required to overcome friction?

15. A constant horizontal force of 8 lbf is applied to a block of mass 2 slugs initially at rest on a frictionless horizontal surface. What instantaneous power is being developed at the end of the third second? at the end of the fifth second? What is the average power developed during the first five seconds?
Ans: 96 ft·lbf/sec; 160 ft·lbf/sec; 80 ft·lbf/sec.

16. Show that the power required to give a block of mass m an acceleration a when it is moving at velocity v up a frictionless inclined plane making angle θ with the horizontal is

$$P = mva + mgv \sin\theta.$$

Derive this expression in two ways:

(a) by computing the force required and using (18);

(b) by determining the time rate of change of mechanical energy by differentiation of $\frac{1}{2}mv^2 + mgh$.

17. A ½-hp electric motor operates continuously at rated load for 5 hr. How much work is done by this motor? Express your answer in hp·hr, ft·lbf, joules, and kwh. Ans: 2.5 hp·hr; 4.95×10^6 ft·lbf; 6.71×10^6 joules, 1.86 kwh.

18. If a 50-kw electric generator operates continuously at full capacity for 8 hours, how much energy is delivered? Express your answer in hp·hr, ft·lbf, joules, and kwh.

19. An electric power company sells energy at 6 cents per kwh. How many joules can be purchased for 10 cents? Ans: 6×10^6.

20. Buying energy from the electric company at 6 cents per kwh, a customer operates a 2-hp motor at rated load for two hours. Assuming negligible losses in the motor, calculate the cost to the customer. What is the company's charge for 1 hp·hr?

7. PRIME MOVERS

In our discussion of machines, we pointed out that *energy must be supplied to a machine in order for the machine to be used for the performance of work.* However, in the section on *power,* we spoke of power being *developed* by an agent or device; in one of the illustrative examples, we calculated the *power developed by a man* in climbing a stairway. Furthermore, it is quite common to speak of *power production* in connection with hydroelectric plants. These terms, *power development* and *power production,* may at first seem confusing in view of our earlier discussion of the conservation of energy, and hence it may be well to consider the energy transformations involved in power production.

First of all, let us start with the conservation-of-energy principle. If we desire to perform some operation involving work, we must draw upon some environmental supply of energy. We supply some type of environmental energy—for instance, the mechanical energy of wind or water or the chemical energy of coal or petroleum—to some device that transforms the supplied energy into a form that can be used directly or indirectly in performing the desired operation. Since such a device usually transforms environmental energy into useful mechanical energy of motion, it is called a *prime mover.* In more general terms a prime mover can be defined as follows:

> A prime mover is a device that transforms environmental energy into a form of energy that can be used in doing useful work.

The distinction between a *prime mover* and a *machine* as we have defined it is important. A machine is a device used in performing work conveniently, but a prime mover must be used to supply energy to the machine in a suitable form. Many examples can be enumerated: a plow is a useful machine only when mechanical energy is supplied to it by a prime mover such as a horse, mule, or tractor; a water pump is a useful machine only when mechanical energy is supplied to it directly by a prime mover such as a man, a windmill, or a gasoline engine, or by some other machine such as an electric motor supplied with electrical energy from an electric generator which in turn is operated by mechanical energy from a prime mover such as a waterwheel, water turbine, or steam engine; a lawn mower is a useful machine only when mechanical energy is supplied to it by a prime mover such as a man or a gasoline engine. From these examples, we see immediately that prime movers can be divided into two classes: *animate* and *inanimate.*

The animate prime movers mentioned include the horse, the mule, and man himself. Environmental energy is supplied to these animate prime movers in the form of *food;* some of the chemical energy in the food is ultimately transformed into mechanical energy that can be used in doing such work as pulling a plow or pushing a lawn mower. In the earliest stages of civilization, all useful work was performed by animate prime movers working directly on the job at hand or on crude machines which performed the desired work more conveniently.

The inanimate prime movers mentioned can in turn be divided into two classes. The windmill, the water wheel, and the water turbine transform the *environmental mechanical energy* associated with the motion or position of air and water into useful mechanical energy that can be supplied to machines. Prime movers depending on environmental mechanical energy were the earliest *inanimate* prime movers used by man; the sail was perhaps the earliest, but the water wheel and windmill have also been used for many centuries. The advantage of these devices over animate prime movers lies in their ability to transform large amounts of environmental mechanical energy into useful mechanical energy in a comparatively short time; for example, when the wind is favorable a man can cross a lake much more rapidly if he uses a sail as a prime mover than if he himself acts as the prime mover and moves the boat across the lake by means of oars. In other words, environmental energy can be converted into useful mechanical energy more rapidly by the sail than by the man. This statement is true provided abundant environmental energy is available to both; the wind must be blowing in order for the sail to operate and an ample supply of food must be available to the man. The transformation of environmental energy by a prime mover into a form of energy that can be used in doing useful work is the process known as *power production;* greater amounts of power can be obtained from inanimate prime movers such as the sail, the water wheel, and the windmill than from animate prime movers such as the horse, the mule, and man.

The remaining inanimate prime movers mentioned above, the gasoline engine and the steam engine, both belong to a class of prime movers known as *heat engines.* The principles involved in the operation of heat engines will be discussed in some detail in Chap. 19. The environmental energy supplied to modern heat engines is usually the *chemical energy* of fuels such as coal and petroleum.* By the process of combustion the chemical energy of the fuel is converted into heat, and a portion of this heat can be converted into mechanical energy by allowing heated gases to expand and move pistons, or to do mechanical work in some other way. The large-scale use of heat engines as prime movers is relatively recent; steam engines have been used only since the latter part of the eighteenth century and internal-combustion engines have come into wide use only since the end of the nineteenth century. However, the impact these devices have had upon society is enormous. The Industrial Revolution occurred when heat engines were substituted for animate prime movers. This substitution put the laborer in position to control and employ tremendously greater quantities of energy than he could himself supply.

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* Power production involving atomic energy will be discussed in Chap. 44.

It might be worth while to look further into the nature of the environmental energy used by prime movers. In nearly every case, the environmental energy used by prime movers can be attributed directly or indirectly to radiant energy reaching the earth from the sun.* The availability of environmental mechanical energy can be attributed to the heating effects of solar radiation; wind is caused by unequal heating of different parts of the earth's surface and the adjacent atmosphere, whereas water power results from evaporation of water by solar heat, transportation of water vapor by winds, and finally precipitation.

Another way in which the absorption of solar radiation results in an increase in environmental energy is the process of photosynthesis that takes place in the green leaves of plants. In the presence of chlorophyll, solar radiation causes water and carbon dioxide to combine to form simple sugars, which have more chemical energy than either carbon dioxide or water. Thus, *the process of photosynthesis involves the transformation of radiant energy from the sun into chemical energy.* Once photosynthesis has taken place, the resulting compounds can take part in chemical reactions that result in the production of more complex organic compounds, which serve as food for animals and therefore can be used as a source of environmental energy for *animate prime movers.*

It should be noted that the chemical energy used by animate prime movers is derived from *recently formed* organic compounds. The chemical energy used by heat engines is usually derived from fuels like petroleum and coal, which were formed from organic materials produced in the distant past by processes ultimately involving photosynthesis. Thus, modern heat engines utilize chemical energy from fuel stocks that are not being appreciably replenished. When these fuel stocks are exhausted, the use of conventional heat engines will no longer be possible unless suitable fuels can be produced from current supplies of radiant energy from the sun or unless prime movers utilizing the direct heating effects of solar radiation can be developed. Research work along both lines is being carried on; studies of photosynthesis are being made in the hope of developing new methods of transforming radiant energy into chemical energy, and investigations of the possibilities of the construction of practical 'solar-heat engines' are also in progress. In closing the discussion, it should be pointed out that although there are indications that the known petroleum reserves may be exhausted within a century, the world's known coal reserves should last for several thousand years. Hence, the study of conventional heat engines is still worth while!

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* Atomic energy and tidal energy are exceptions.

CHAPTER 8

ROTATIONAL MOTION

In earlier chapters we have pointed out that any motion of a rigid body can be described as a combination of *translation* and *rotation*. But so far, in our treatments of kinematics and dynamics, we have confined our attention to particles and to rigid bodies moving without change in spatial orientation. Thus we have discussed the kinematics and dynamics of translational motion in considerable detail, but we have not discussed rotational motion.

A rigid body is essentially a collection of particles constrained by forces that keep the particles in place in the body. Since the dynamical behavior of the rigid body can be deduced from the dynamical behavior of the particles of which it is composed, it is desirable first to study the dynamics of particles.

In the present chapter we shall begin the study of the kinematics and dynamics of the rotational motion of rigid bodies. *We shall restrict our treatment to the case of rotational motion about an axis that has a fixed direction, though not necessarily a fixed position, in space.* This restricted treatment is closely analogous to the treatment of one-dimensional translational motion; for this reason we shall find that we are able to carry it through very easily.

A brief discussion of the much more difficult case of rotational motion when the axis of rotation does not have a fixed direction in space (ordinarily called 'gyroscopic' motion) will be given in Chap. 13.

1. KINEMATICS OF PURE ROTATION: ANGULAR DISPLACEMENT, VELOCITY, AND ACCELERATION

In this section we shall discuss the kinematics of the rotation of a rigid body about a fixed axis. Rotation about a *fixed* axis is known as *pure rotation*, since there is no translation of the body.

Let us consider Fig. 1, which shows the cross section of a cylinder that can rotate about a fixed axis through O perpendicular to the plane of the diagram. The cylinder might be a grinding wheel or the flywheel of some piece of machinery with the axis through O as the center line of the shaft on which the wheel is mounted. All points on the axis remain stationary during motion of the wheel, while all other points in the body move in circles with centers located on the axis. The position of every point in a body having this type of motion can be determined in terms of the angular position of any reference radius Oa of the body relative to some fixed direction Ox. In other words, the angle between Oa and Ox can be used in determining the positions of all points in a rotating rigid body, since the *relative* positions of the points in the body do not change.

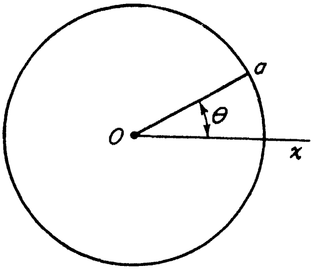


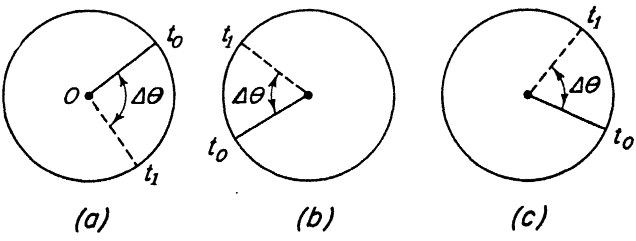
Fig. 1. Angular displacement. Ox is a fixed direction in space; Oa a radial line 'painted' on the wheel. A fixed axis of rotation, perpendicular to the diagram, passes through point O.

The angle between Ox and Oa can be used to describe the *orientation* of the rigid body as a whole. If Oa initially coincides with Ox and later has turned through angle θ, we can think of θ as the *angular displacement* of the body.

> Angular displacement of a rigid body about a fixed axis is defined as the angle through which any radius of the body turns, taken as positive for a counterclockwise rotation and negative for a clockwise rotation.

Angular displacement in pure rotation is a scalar quantity. It is useful to think of it as analogous to the x-component of linear displacement in the case of linear motion of a particle along the x-axis.

Fig. 2 represents certain angular displacements in which a radius turns through angle $\Delta\theta$ between time t_0 and time t_1. The angular



(a) (b) (c)

Fig. 2. In (a), $\Delta\theta = -1.6$ rad; in (b), $\Delta\theta = -1.2$ rad; in (c), $\Delta\theta = +1.4$ rad.

displacement is negative in parts (a) and (b) of the figure and positive in part (c). It should be noted that although we have used a *particular* radius Oa in measuring displacements, we could have used *any* radius, since *all* radii drawn from the axis of rotation turn through the *same* angle $\Delta\theta$ during rotation of a rigid body. Of course whether an angular displacement is clockwise or counterclockwise depends on whether one looks at the wheel from 'before' or from 'behind,' but in any given problem we should draw a diagram and adhere to the convention based on the point of view of this diagram. A similar choice of positive sense of the x-axis must be made in discussing the linear motion of a particle.

Now that we have defined angular displacement θ, we may proceed at once to the formal definition of angular velocity ω by means of the equation

$$\omega = d\theta/dt. \qquad (1)$$

If θ is measured in radians and t in seconds, ω is expressed in radians per second (rad/sec).

> The angular velocity of a rigid body in pure rotation about a fixed axis is the time rate of change of its angular displacement. It is a scalar quantity, positive values representing counterclockwise rotations and negative values representing clockwise rotations.

The angular velocity in the case of pure rotations is analogous to the x-component of linear velocity in the case of linear motion along the x-axis.

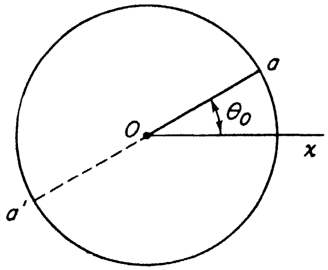


Fig. 3.

When the angular velocity is constant, the angular displacement from the initial position of the body at the end of Δt sec is given by

$$\Delta\theta = \omega\Delta t.$$

If the initial orientation of the body in a given coordinate system is θ_0 at time $t=0$, the orientation θ at any future time t is

$$\theta = \theta_0 + \Delta\theta = \theta_0 + \omega t \qquad (2)$$

if ω is constant. Equation (2) is closely analogous to the equation

$$x = x_0 + v_x t,$$

which gives the x-coordinate of a particle moving with constant velocity component v_x along the x-axis.

In order to show how equation (2) can be applied, let us consider the grinding wheel shown in cross section in Fig. 3. In this figure, Oa represents some reference radius, which might be indicated by a chalk mark on the wheel. Suppose the initial angle between Oa and Ox is 30° or $\frac{1}{6}\pi$ rad at time $t=0$ and that the wheel has a constant counterclockwise angular velocity of $\frac{1}{2}$ rev/sec or π rad/sec. Let us find the orientation or angular position of the wheel at the end of 15 sec. The initial orientation

θ_0 is $+\frac{1}{6}\pi$ rad and $\Delta\theta$, the magnitude of the angular displacement during 15 sec is $+15\pi$ rad. Hence, the position θ of the wheel at the end of 15 sec as given by equation (2) is

$$\theta = \tfrac{1}{6}\pi + 15\pi = {}^{91}\!/_{6}\,\pi \text{ rad.}$$

Since there are 2π radians in one complete revolution, the wheel has turned counterclockwise through $15\pi/2\pi$ or $7\frac{1}{2}$ complete revolutions. Hence, the final position of our reference radius will be Oa', just opposite its initial position Oa.

A new observer looking at the wheel after the angular displacement had been completed would give the position θ of the wheel as $+\frac{7}{6}\pi$ rad or $-\frac{5}{6}\pi$ rad instead of ${}^{91}\!/_{6}\,\pi$ rad, the value we have obtained. Hence, it would appear that angular *position* θ is not unique but depends upon the observer's decision. This is correct, but it should be emphasized that $\Delta\theta$, the angular *displacement*, does have real significance in any case. In any given problem it is important to have some single consistent method for stating angular position θ. Thus, in our original problem the final value of $\theta = {}^{91}\!/_{6}\,\pi$ rad *is* unique.

We can now proceed to a definition of angular acceleration α by means of the equation

$$\alpha = d\omega/dt.$$

If ω is measured in radians per second and time in seconds, angular acceleration α will be expressed in radians per second per second (rad/sec^2).

> Angular acceleration of a rigid body about a fixed axis is defined as the time rate of change of angular velocity.

Angular acceleration in the case of pure rotation is a scalar quantity that is analogous to the x-component of linear acceleration in the case of linear motion along the x-axis. From the relation

$$d\omega = \alpha\, dt,$$

we see that positive angular acceleration produces a positive change in angular velocity. If the body is initially moving counterclockwise and has a positive angular acceleration, its speed of rotation is increasing; if it is initially moving clockwise and has a positive angular acceleration, its rotation is slowing down.

We may now write equations describing the motion of a body experiencing *constant* angular acceleration α about a fixed axis. If the angular velocity of the body is ω_0 at time $t=0$, its angular velocity ω at the end of t sec can be written as

$$\omega = \omega_0 + \Delta\omega,$$

where $\Delta\omega$ is the change in angular velocity during the t-sec interval.

Since for constant angular acceleration $\Delta\omega = \alpha t$, this expression for ω becomes

$$\omega = \omega_0 + \alpha t. \qquad (3)$$

In order to find the angular displacement $\Delta\theta$ experienced by the body during the t-sec interval, we write

$$\Delta\theta = \bar{\omega} t, \qquad (4)$$

where $\bar{\omega}$ is the average angular velocity. Just as in the case of uniformly accelerated linear motion, we may express $\bar{\omega}$ in the form

$$\bar{\omega} = \tfrac{1}{2}(\omega_0 + \omega), \qquad (5)$$

where ω is the final velocity as given by equation (3). Substitution of (3) in (5) leads to

$$\bar{\omega} = \omega_0 + \tfrac{1}{2}\alpha t. \qquad (6)$$

Substitution of this value of $\bar{\omega}$ in (4) gives

$$\Delta\theta = \omega_0 t + \tfrac{1}{2}\alpha t^2 \qquad (7)$$

for the magnitude of the angular displacement during the t-sec interval. To obtain an expression for the angular position of the body, we write

$$\theta = \theta_0 + \Delta\theta,$$

or

$$\theta = \theta_0 + \omega_0 t + \tfrac{1}{2}\alpha t^2, \qquad (8)$$

where θ_0 gives the initial value of the angular position (at $t = 0$).

By elimination of t from equations (3) and (7), we obtain the relation

$$\omega^2 = \omega_0^2 + 2\alpha\,\Delta\theta, \qquad (9)$$

analogous to a corresponding relation in linear motion.

In order to show how these equations are applied, let us consider a motor-driven grinding wheel that starts from rest and receives a constant counterclockwise angular acceleration of 3 rad/sec^2 for 12 sec. From (3), the angular velocity of the wheel at the end of 12 sec is +36 rad/sec. The average angular velocity of the wheel during the first 12 sec is +18 rad/sec, as given by (5). The total angular displacement of the wheel can be obtained from (4):

$$\Delta\theta = (18\ \text{rad/sec})(12\ \text{sec}) = 216\ \text{rad},$$

or from (7):

$$\Delta\theta = \tfrac{1}{2}(3\ \text{rad/sec}^2)(12\ \text{sec})^2 = 216\ \text{rad}.$$

Since there are 2π radians in one complete revolution, the wheel turns through $216/2\pi$ revolutions during the first 12 sec of motion.

We note that there is an exact correspondence between rotational motion about a fixed axis and translational motion along a straight line (the x-axis) if we associate θ with x, ω with v_x, and α with a_x. The relations we have found between θ, ω, and α are exactly the same as the relations we found in Chap. 5 between x, v_x, and a_x. As we go into the dynamics of rotational motion and enlarge this list of associated quantities, we shall find essentially that we shall not need to remember any new analytical relations if we remember the analytical relations governing translational motion along a straight line and the fairly obvious system of correspondence between rotational and translational quantities.

PROBLEMS

1. What angular displacement in radians is experienced by the minute hand of a clock during a 15-min interval? What is the angular displacement of the hour hand during this interval? Ans: $-\frac{1}{2}\pi$ rad; $-\frac{1}{24}\pi$ rad.

2. What angular displacement is experienced by the second hand of a watch during 1 hr? As viewed by an observer above the North Pole, what angular displacement arising from rotation about its axis does the earth experience during 12 hr? during 18 hr? during 24 hr?

3. The shaft of a certain motor makes 3600 rev/min and turns in the clockwise direction as observed by the operator. What is the angular velocity of this motor shaft in rad/sec? What angular displacement does the shaft experience in 40 sec? Ans: -120π rad/sec; -4800π rad.

4. Compute the magnitude of the average angular velocity in rad/sec of a large flywheel that makes 1200 revolutions in 3 min. Through what angle in radians does this wheel rotate in 15 sec if its angular velocity is constant?

5. A grinding wheel is mounted on the shaft of a motor that makes 1800 rev/min when moving at full speed. The wheel attains full speed 15 sec after the switch is closed. What is the magnitude of the average angular acceleration experienced by the wheel? If the acceleration was constant, find the magnitude of the average angular velocity of the wheel. Assuming constant acceleration, find the angular displacement of the wheel during this 15-sec interval. Ans: 4π rad/sec^2, 30π rad/sec, 450π rad.

6. If the switch is turned off after the wheel in Prob. 5 has attained full speed, the wheel comes to rest after 3 min. Find the magnitude of the angular deceleration of the wheel. Assuming constant deceleration, find the magnitude of the average angular velocity of the wheel and the angular displacement experienced during this 3-min interval.

7. A flywheel initially at rest experiences constant angular acceleration. In 9 sec this flywheel is observed to turn through an angle of 450 radians in the counterclockwise direction. What is the average angular velocity of the wheel? What is the final angular velocity of the wheel? What is the angular acceleration of the wheel? Ans: +50 rad/sec; +100 rad/sec; +11.1 rad/sec^2.

8. Starting from rest and experiencing constant angular acceleration, a flywheel makes 4000 clockwise revolutions in $1\frac{1}{2}$ min. Find its angular acceleration.

9. A flywheel experiencing constant angular acceleration makes 10 complete counterclockwise rotations in 4 sec. If the angular velocity of the wheel at the end of this 4-sec interval is 20 rad/sec, find the initial angular velocity and the angular acceleration of the wheel. Ans: 11.4 rad/sec; 2.15 rad/sec^2.

10. A flywheel experiencing constant angular acceleration makes 12 complete counterclockwise rotations in 5 sec. If the angular velocity of the wheel at the beginning of this 5-sec interval is 4 rad/sec, find the angular acceleration and the final angular velocity of the wheel.

11. At what times between noon and 1 P.M. is the angle between the minute hand and the hour hand of a clock equal to 1 radian? Ans: 12:10.4 and 12:55.0.

12. At what times between six and seven o'clock is the angle between the minute hand and the hour hand of a clock equal to 1 rad?

2. THE MOTION OF A POINT IN A RIGID BODY IN PURE ROTATION

The angular displacement, angular velocity, and angular acceleration of a rigid body rotating about an axis are characteristic of the motion of the rigid body *as a whole*. Let us now consider the relations between these quantities and the displacement, velocity, and acceleration of a single point in the rotating body. Every point in a body rotating about a fixed axis moves in a circle whose center is on the axis of rotation. Let the broken circle in Fig. 4 represent the path of a point P in a body rotating about an axis through O. The point is at distance r from the axis of rotation. When the radius OP makes angle θ radians with a fixed reference line Ox, the arc length from P measured counterclockwise from the reference line will be the length l in Fig. 4, where

$$l = r\theta. \quad \checkmark \quad (10)$$

If θ is given as a function of t, this relation will determine l as a function of t. Since r is a constant, we can differentiate the relation (10) with respect to t to obtain

$$dl/dt = r\, d\theta/dt. \quad \checkmark$$

But dl/dt is the magnitude of the velocity of the point P along its circular path, which we denote by v, and $d\theta/dt$ is the angular velocity ω, so

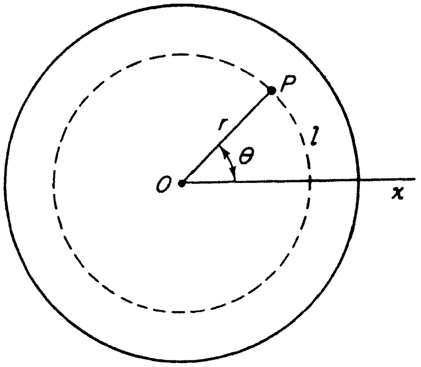


Fig. 4.

$$v = r\omega. \qquad (11) \ \checkmark$$

This relation gives the speed and direction of motion of P in terms of the angular velocity of the wheel. We note that in this relation, v comes out positive for counterclockwise rotation (ω positive) and negative for clockwise rotation (ω negative). For a given angular velocity, the speed varies directly as the distance r from the axis, as is reasonable.

We now differentiate equation (11) with respect to t. This gives

$$dv/dt = r\, d\omega/dt.$$

The expression dv/dt is *not* the acceleration of P since the velocity is continuously changing direction; it is the rate of change of speed rather than the rate of change of velocity. But it is, as remarked at the end of Chap. 5, one component of the acceleration of P: the component *tangent* to the circular path of P, which we shall denote by a_T. We shall give a formal proof of this below. Since $d\omega/dt$ is the angular acceleration α, the above relation becomes

$$a_T = r\alpha. \tag{12}$$

The radial component of the acceleration of P, directed toward the center, is, as in Chap. 5,

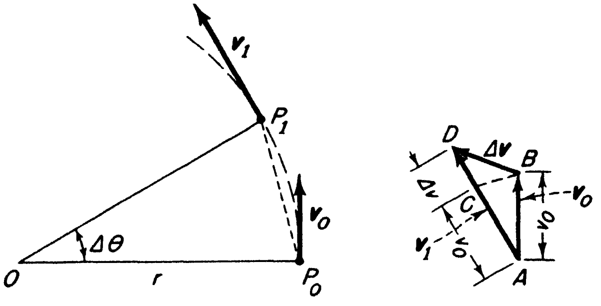
$$a_R = v^2/r. \tag{13}$$

This relation obtains whether or not the speed v is constant, as we shall show. From (11), we get the useful alternative expressions

$$a_R = v\omega = \omega^2 r. \tag{13$'$}$$

This completes the kinematical description (except for certain proofs) of the motion of P in terms of the angular motion of the rotating body of which P is a point. The position l of P, measured along the arc, its speed v, and the components of its acceleration a are given by

$$l = r\theta; \quad v = r\omega; \quad a_T = r\alpha, \quad a_R = v^2/r = v\omega = \omega^2 r. \tag{14}$$



(a) (b)

Fig. 5.

These relations hold only when the angular unit in θ, ω, α is the radian. In checking the dimensions of these equations, note that the radian is a dimensionless quantity defined as the ratio of two lengths (arc/radius) measured in the same length units.

The velocity of P is a vector tangent to the path, of magnitude given by the magnitude of v and direction determined by the sign of v. The acceleration vector has a component a_T tangent to the path, pointing in the sense of increasing θ when a_T is positive and in the sense of decreasing θ when a_T is negative. The other rectangular component of the acceleration vector has magnitude a_R and is directed toward the axis of rotation (see Fig. 6).

We shall now extend the derivation of Sec. 7 of Chap. 5 to obtain the acceleration of a point moving in a circle at variable speed. In Fig. 5, let the velocity of P be $\boldsymbol{v}_0$ at time t_0 and $\boldsymbol{v}_1$ at time t_1. Let $t_1 - t_0 = \Delta t$, and $\boldsymbol{v}_1 - \boldsymbol{v}_0$ be $\Delta\boldsymbol{v}$. Let the change in *speed* (change in *length* of the velocity vector) be Δv. Here Δv is *not* the magnitude of $\Delta\boldsymbol{v}$ but is the quantity indicated in Fig. 5(b). In Fig. 5, the triangle ABC is similar to the triangle OP_0P_1. The vector change in velocity $\Delta\boldsymbol{v}$ can then be resolved into two components: one of magnitude BC and a direction normal to P_0P_1, the other of magnitude $CD = \Delta v$ and directed tangent to the circle at P_1. As we go to the limit $\Delta t \rightarrow 0$, in which the point P_1 approaches coincidence with P_0, we have seen in Chap. 5 that the ratio $BC/\Delta t \rightarrow v^2/r$ and that the limiting direction of BC is radial. This component of $\Delta\boldsymbol{v}$ thus gives the centripetal acceleration v^2/r. Correspondingly, in the limit $\Delta t \rightarrow 0$, $CD/\Delta t = \Delta v/\Delta t \rightarrow dv/dt$, the rate of change of speed. In the limit, the direction of CD is the direction of the tangent to the circle at P_0, so this component of $\Delta\boldsymbol{v}$ gives a tangential component of acceleration $a_T = dv/dt$. This is the relation we used to derive (12). These acceleration components are illustrated in Fig. 6.

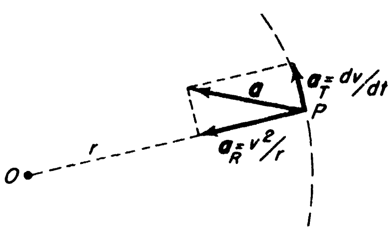


Fig. 6. Radial and tangential acceleration components. The tangential component has the direction indicated if dv/dt is positive; the opposite direction if dv/dt is negative.

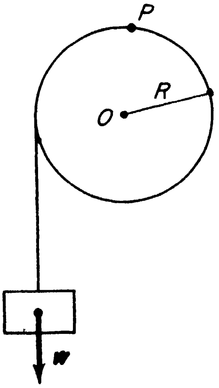


Fig. 7.

Let us now consider a numerical problem. Suppose that a small weight is attached to the end of a cord that is wound around a cylinder free to turn about a horizontal axis O as shown in Fig. 7. If the diameter of the cylinder is 18 inches and if the weight moves down 4 feet with constant acceleration in the first 4 seconds after being released, let us find the total angle through which the cylinder has rotated, the final angular velocity of the cylinder, the angular acceleration of the cylinder, and the components of the linear acceleration of points on the outer surface of the cylinder.

The radius R of the wheel is 9 in, or $\frac{3}{4}$ ft. When the weight moves downward a distance of $h = 4$ ft, all points such as P on the surface of the cylinder also move a total distance $l = 4$ ft. Hence, we may immediately write down the value for the total angle θ moved by the cylinder

$$\theta = l/R = (4\text{ ft})/(\tfrac{3}{4}\text{ ft}) = \tfrac{16}{3}\text{ rad.}$$

The average downward velocity of the weight is

$$\bar{v} = 4\text{ ft}/4\text{ sec} = 1\text{ ft/sec}.$$

Since the downward acceleration of the weight is constant and its initial velocity is zero, its final downward velocity v is

$$v = 2\text{ ft/sec}.$$

This is also the value of the final speed of point P; hence the final angular velocity of the cylinder is

$$\omega = v/R = (2\text{ ft/sec})/(\tfrac{3}{4}\text{ ft}) = \tfrac{8}{3}\text{ rad/sec}.$$

Since the downward acceleration a of the weight is constant and is equal to the tangential acceleration a_T of point P, we may obtain a_T from

$$h = \tfrac{1}{2}at^2, \quad \text{which gives} \quad a_T = a = 2h/t^2 = 8\text{ ft}/16\text{ sec}^2 = \tfrac{1}{2}\text{ ft/sec}^2.$$

Using this value, we obtain

$$\alpha = a_T/R = (\tfrac{1}{2}\text{ ft/sec}^2)/(\tfrac{3}{4}\text{ ft}) = \tfrac{2}{3}\text{ rad/sec}^2.$$

We may check this result by noting that the angular acceleration α is also constant and is therefore given by

$$\alpha = \omega/t = (\tfrac{8}{3}\text{ rad/sec})/(4\text{ sec}) = \tfrac{2}{3}\text{ rad/sec}^2.$$

Finally, we may obtain a_R, the radial or centripetal component of the final acceleration of point P, by using equation (14):

$$a_R = \omega^2 R = (\tfrac{8}{3}\text{ rad/sec})^2(\tfrac{3}{4}\text{ ft}) = \tfrac{16}{3}\text{ ft/sec}^2 = 5.33\text{ ft/sec}^2.$$

It should be noted that although the tangential component a_T of the acceleration is constant, the radial component a_R, is *not* constant. Since $\omega = \alpha t$, we see that a_R increases as the square of the time t after the start of the motion.

PROBLEMS

1. A grinding wheel 12 inches in diameter is mounted on the shaft of a motor, which makes 1800 rev/min. Find the magnitude of the velocity of a point on the rim of the wheel. Ans: 30π ft/sec.

2. A flywheel 5 ft in diameter is rotating at a speed of 450 rev/min. Find the magnitude of the velocity of a point on the rim of the flywheel.

3. Find the centripetal acceleration of a point on the rim of the grinding wheel in Prob. 1. Express this acceleration in terms of g_0, the standard gravitational acceleration. Ans: 17,800 ft/sec^2; 552 g_0.

4. Find the centripetal acceleration in ft/sec^2 of a point on the rim of the flywheel mentioned in Prob. 2. Express this in terms of g_0, the standard gravitational acceleration.

5. A flywheel 4 ft in diameter experiences an angular acceleration of 5 rad/sec^2. Find the tangential acceleration of a point on the rim of this flywheel. Ans: 10 ft/sec^2.

6. A grinding wheel 10 inches in diameter experiences an angular acceleration of 3 rad/sec². Find the tangential acceleration of a point on the rim of the wheel.

7. In an arrangement similar to that shown in Fig. 7, the weight *w* has a constant downward acceleration of 1.5 ft/sec² and the radius of the cylinder is 2 ft. If the weight starts from rest, find the tangential velocity and acceleration of a point such as *P* ten seconds after motion has begun. What is the angular acceleration of the cylinder at this time? the angular velocity? the radial acceleration of point *P*?
Ans: 15 ft/sec; 1.5 ft/sec²; 0.75 rad/sec²; 7.5 rad/sec; 112.5 ft/sec².

8. Referring to Prob. 7, find the tangential velocity and the tangential and radial components of the acceleration of point *P* at 2, 4, and 6 sec after motion has started. What can be concluded concerning the time variations of the radial and tangential components of the acceleration of points on the rim?

9. A particle of mass 2 kg rests without slipping on a horizontal turntable at a distance of 40 cm from the vertical axis of rotation. Draw a diagram showing the magnitudes and directions of all forces acting on the particle when the turntable is rotating at 4 rad/sec and accelerating at 3 rad/sec². What is the magnitude of the force of friction and the angle it makes with the radius? Ans: 13.0 nt; 10°.6.

3. DYNAMICS OF ROTATIONAL MOTION ABOUT A FIXED AXIS

In beginning our discussion of the dynamics of rotational motion, let us recall that in Chap. 2 we introduced *torque* L as the measurement of the tendency of a force to produce changes in rotational motion. When a body undergoes a change in rotational motion, it experiences an angular acceleration. We should like to find a relation between torque L and angular acceleration α that is analogous to the Newtonian relation $F = ma$, which has proved so useful in problems involving linear translational motion.

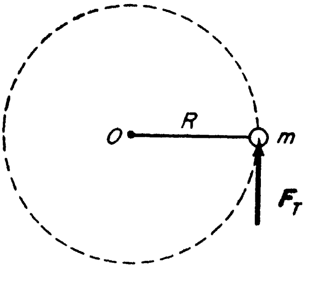


Fig. 8.

We may *derive* a relation between L and α by applying Newton's laws to a simple example of rotational motion such as that shown in Fig. 8, which shows a particle of mass m that is constrained to move in a circular path of radius R about an axis through O. We might approximate this situation by attaching a small steel ball of mass m to one end of a slender wooden rod of negligible mass and attaching the other end of the wooden rod to a bearing that is free to rotate about a vertical axis so that the ball moves in a horizontal circle. Or, m might be a particle resting on a frictionless horizontal table and attached by a string of length R to a point O on the table. Now let us apply a tangential force of magnitude F_T to the particle of mass m. The particle will then experience a tangential acceleration of magnitude a_T given by Newton's second law,

$$F_T = ma_T,$$

since F_T is the only force component tangent to the circle.

Now let us consider the rotational acceleration of the particle about the axis through O; by use of equation (12) we may write

$$F_T = mR\alpha, \tag{15}$$

where α is in rad/sec^2. This equation states that the angular acceleration of the particle about the axis through O is proportional to the magnitude of the tangential force. Since the force is tangential, its lever arm is simply the radius R of the circular path of the particle; thus, the torque L of this force about the axis through O is $F_T R$. Hence, by multiplying both sides of (15) by R, we obtain

$$L = [mR^2]\,\alpha, \tag{16}$$

which gives us our desired relationship between L and α.

We note immediately the similarity between equation (16) and the equation for Newton's second law. The role of *torque* L in rotational motion is analogous to the role of *force* F in translational motion; the role of angular acceleration α in rotational motion is analogous to that of linear acceleration a in translational motion. Hence, we see that in (16) the term $[mR^2]$ is analogous to mass m in Newton's law

$$F = ma$$

for rectilinear translational motion. Recalling that the mass m of a body is a measure of the *inertia* of the body, we see that *the quantity* $[mR^2]$ *is a measure of the rotational inertia of the particle of mass m located at a distance R from the axis.* The rotational inertia term $[mR^2]$ is called the *moment of inertia* of the particle about the axis through O and is denoted by I. Hence, we may rewrite equation (16) in the form

$$L = I\alpha. \tag{17}$$

Equation (17) is sometimes called 'Newton's second law for rotational motion'; this nomenclature is somewhat misleading, since we *derived* (17) by applying Newton's second law to a particular problem. Hence, the rotational equation (17) is *not* a fundamental law of mechanics but may quite properly be called the *rotational analogue of the second law.*

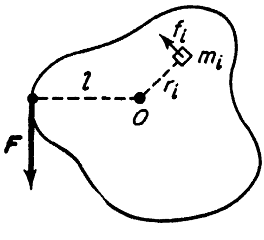


Fig. 9.

Now let us consider a rigid body of any shape mounted so that it can rotate about an axis through O as in Fig. 9. The body can be considered as divided up into small pieces, of which the ith typical piece, of mass m_i, is shown in the figure. If the rigid body is to have angular acceleration α, this typical piece must have acting on it a resultant tangential force given by (15) as $f_i = m_i r_i \alpha$, corresponding to a torque about O of $L_i = [m_i r_i^2]\alpha$. The torque required to cause the angular acceleration must be supplied by some external force $\boldsymbol{F}$, with lever arm l, giving torque $L=Fl$. *It can be rigorously proved* (but we do not give the rigorous proof here) *that the total external torque needed to give the rigid body the angular acceleration* α *is just the sum over the pieces of the body of the torques* $L_i=[m_i r_i^2]\alpha$. Thus,

$$L=\sum_i L_i=\sum_i [m_i r_i^2]\,\alpha.$$

Since α is a common factor in all the terms on the right, we can write this equation as

$$L=[\sum_i m_i r_i^2]\,\alpha=I\alpha, \tag{18}$$

where

$$I=\sum_i m_i r_i^2 \tag{19}$$

is called *the moment of inertia of the rigid body about the axis through O.*

If we can learn to compute the moment of inertia of a rigid body (this problem is considered in detail in the next section), then the equation $L=I\alpha$ gives the resultant torque about O necessary to give the body an angular acceleration α. If α is positive, the torque is positive (counterclockwise); if α is negative, the torque is negative (clockwise). Note that (18) holds only when absolute units are used. One of the following combinations will be employed in our work:

L	I	α
lbf·ft	slug·ft^2	rad/sec^2
nt·m	kg·m^2	rad/sec^2

4. THE MOMENT OF INERTIA OF A RIGID BODY

Before turning to problems in rotational dynamics, let us consider how the moment of inertia of a rigid body can be computed. We may consider a rigid body as being made up of a large number of rigidly connected particles or point masses. The moment of inertia I of the body in Fig. 9 is given by

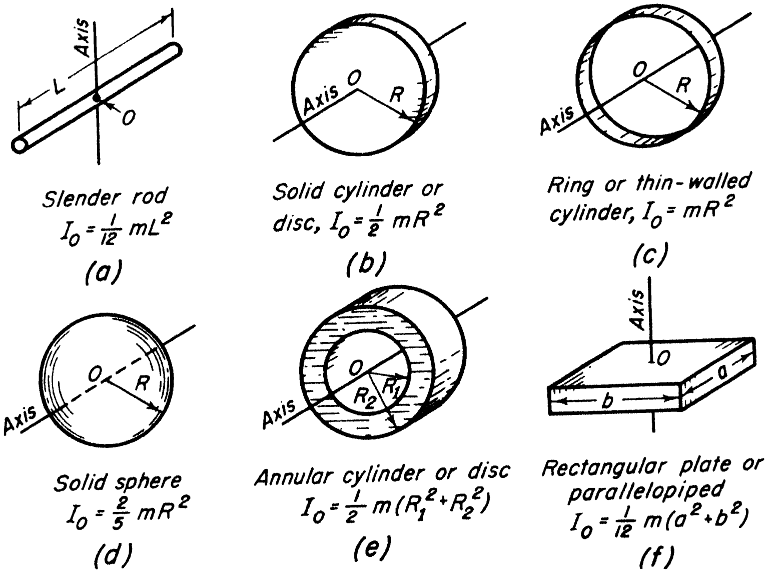
$$I=\sum_i m_i r_i^2, \tag{19}$$

where the summation is over all particles in the body.

Equation (19) conveys the basic idea involved in the definition of the moment of inertia of a rigid body but is inconvenient to use in actual computation, since we are immediately faced with the question of what particles in a rigid body we should consider as approximating point masses. We might consider the atoms in the body as point masses and attempt to carry out the summation indicated in (19) but would be confronted with the difficulty of determining r_i for each atom and then carrying out the summation over enormous numbers of atoms. This method of calculation is obviously impractical if we wish to calculate the moment of inertia of a flywheel or some other large object. The method actually used involves the assumption that solid bodies are *continuous* rather than being composed of discrete particles. If we make this assumption, we may replace the summation in (19) by the integral

$$I = \int r^2\, dm, \tag{19'}$$

where r is the distance of the mass element dm from the axis of rotation; the integral is taken over all mass elements in the body. This integration can be performed quite easily for most regular solids. Since these integrations are presented as problems in most elementary calculus texts, we shall merely list here the moments of inertia for several regular solids about axes through their centers of gravity. These moments of inertia are given in Fig. 10. The units in which we shall measure moments of inertia are kg·m² in the MKS system and slug·ft² in the British engineering system.



Slender rod $I_0 = \frac{1}{12} mL^2$ (a)

Solid cylinder or disc, $I_0 = \frac{1}{2} mR^2$ (b)

Ring or thin-walled cylinder, $I_0 = mR^2$ (c)

Solid sphere $I_0 = \frac{2}{5} mR^2$ (d)

Annular cylinder or disc $I_0 = \frac{1}{2} m(R_1^2 + R_2^2)$ (e)

Rectangular plate or parallelopiped $I_0 = \frac{1}{12} m(a^2 + b^2)$ (f)

Fig. 10. Moments of inertia of various rigid bodies of mass m about axes through their centers of gravity.

From the values given in Fig. 10 and from (19) and (19′), it can be seen that the moment of inertia of a body depends upon two factors: (1) *the mass of the body* and (2) *the distribution of the mass about the axis of rotation.* For a given axis of rotation passing through the center of gravity of the body, the second factor is determined by the *shape* of the body. Whatever the shape of a body, it is always possible to find a radial distance from any given axis at which the mass of a body could be concentrated without altering the moment of inertia of the body about that axis. This radial distance is called the *radius of gyration* of the body about the given axis and is usually denoted by k. Thus, if the mass m of the body were located at this distance k from the axis, the moment of inertia I would be mk^2. Since, by definition of k, this is equal to the actual moment of inertia, we have

$$mk^2 = I, \quad \text{or} \quad k = \sqrt{I/m}.$$

As an example of the computation of the radius of gyration, let us determine the value of k for a slender rod of mass m and length L about a transverse axis through its center of mass. The value of the moment of inertia of this body about the axis in question is given in Fig. 10 as $\frac{1}{12}mL^2$. Hence,

$$mk^2 = \tfrac{1}{12}mL^2, \quad \text{and} \quad k = \sqrt{\tfrac{1}{12}}\,L = 0.289\,L.$$

Thus, so far as rotation about the axis in question is concerned, we could replace the rod by a ring of radius k or a point mass at distance k from the axis as in Fig. 11. The rotational inertia of an irregularly shaped wheel, such as a flywheel, is frequently specified by giving its mass and its radius of gyration.

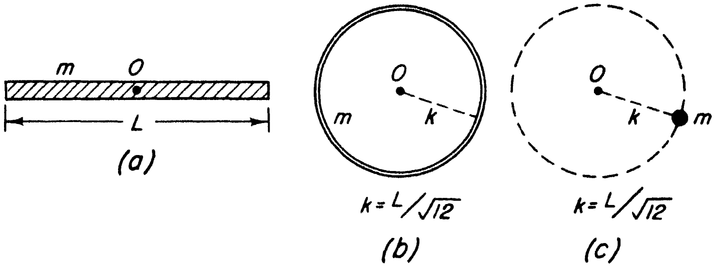


Fig. 11. The slender rod in (a) has the same moment of inertia as the ring shown in (b) and the particle in (c) provided the masses of all three are equal. The radius k is called the *radius of gyration.*

The moments of inertia given in Fig. 10 are for rotation about axes passing through the centers of gravity of the objects shown. The *parallel-axis theorem* is a useful relation which enables us to find the moment of inertia of a body about *any* axis, provided that we know the moment of inertia about the parallel axis through the center of gravity. This theorem states:

The moment of inertia of a body about any axis is equal to its moment of inertia about a parallel axis through the center of gravity plus the product of the mass of the body and the square of the perpendicular distance between the two axes.

Let us prove this theorem by making use of Fig. 12. We wish to calculate the moment of inertia of a body of arbitrary shape about an axis perpendicular to the plane of the figure and passing through O'. We have taken the origin of our coordinate system O at the center of gravity of the body. Consider a particle of mass m_i. The distance of this particle from axis O' is r'_i and hence the moment of inertia I'_i of this particle about axis O' is

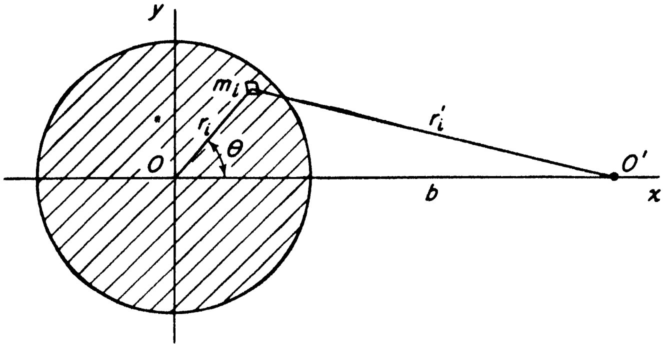


Fig. 12. The moment of inertia I' about the axis of rotation O' is equal to the moment of inertia I_O about a parallel axis through the center of gravity plus the product of the mass of the body and the square of the perpendicular distance between the axes:

$$I' = I_O + mb^2.$$

$$I'_i = m_i r'^2_i.$$

However, by the law of cosines,

$$r'^2_i = r^2_i + b^2 - 2br_i \cos\theta.$$

Furthermore, $$r_i \cos\theta = x_i,$$

where x_i is the x-coordinate of the typical particle. Hence, we may write

$$I'_i = m_i r^2_i + m_i b^2 - 2m_i b x_i.$$

Since the moment of inertia I' of the body about axis O' is given by

$$I' = \Sigma\, I'_i,$$

we have $$I' = \Sigma\, m_i r^2_i + b^2\, \Sigma\, m_i - 2b\, \Sigma\, m_i x_i.$$

The first term, $\Sigma\, m_i r^2_i$, is simply I_O, the moment of inertia of the body about an axis perpendicular to the plane of the diagram and passing through O, the center of gravity. In the second term, $\Sigma\, m_i = m$, the mass of the disk. The third term is zero, since, by definition of the center of gravity, $\Sigma\, m_i g x_i = 0$. Therefore, $\Sigma\, m_i x_i = 0$. Hence,

$$I' = I_O + mb^2, \tag{20}$$

where I_O is the moment of inertia of the body about the axis through the center of gravity, m is the mass of the body, and b is the perpendicular distance of the axis through O' from the axis through O. Thus, *the parallel-axis theorem* is proved.

For the case of a disk or cylinder of radius R, as actually drawn in Fig. 12, we have

$$I' = \tfrac{1}{2}mR^2 + mb^2.$$

As a further illustration of the application the parallel-axis theorem, let us calculate the moment of inertia I' of a slender rod about a transverse axis passing through O' at one end of the rod as shown in Fig. 13. According to the parallel-axis theorem, the moment of inertia I' is given by the relation (20). The moment of inertia I_O about axis O through the center of gravity is $\tfrac{1}{12}mL^2$, as given in Fig. 10, and the value of b in Fig. 13 is $\tfrac{1}{2}L$. Hence,

$$I' = \tfrac{1}{12}mL^2 + \tfrac{1}{4}mL^2 = \tfrac{1}{3}mL^2.$$

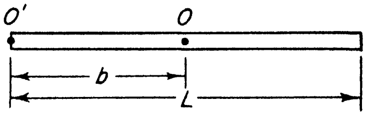


Fig. 13.

It must be emphasized that (20) holds only when I_O is the moment of inertia about an axis *through the center of gravity*, since we have used the 'balancing' property of the center of gravity explicitly in the derivation.

PROBLEMS

1. A string is wound around the rim of a cylindrical grindstone of mass 2 slugs. By pulling on the end of the string, a man exerts a constant tangential force of 3 lbf. If the radius of the grindstone is 8 in, find the magnitude of the torque applied to the grindstone. If its bearings are frictionless, find the magnitude of the angular acceleration of the grindstone and its angular velocity at the end of 6 sec.
Ans: 2 lbf·ft; 4.5 rad/sec^2; 27 rad/sec.

2. What torque must be exerted in order to give the grindstone in Prob. 1 an angular acceleration of 11 rad/sec^2? If this torque were produced by pulling on the string as described, what should be the applied force?

3. A resultant torque of 6 nt·m must be applied to a 5-kg wheel in order to give the wheel an angular acceleration of 2 rad/sec^2. Find the moment of inertia of the wheel and its radius of gyration. Ans: 3 kg·m^2; 77.5 cm.

4. When a torque of 6 nt·m is applied to a wheel identical in geometrical shape and in dimensions with the one described in Prob. 3, the resulting angular acceleration is 2.33 rad/sec^2. What is the mass of this wheel?

5. What torque must be applied to a 64.4-lb solid cylinder 18 inches in diameter in order to produce an angular acceleration of 3 rad/sec^2 about its own axis? about an axis parallel to the axis of the cylinder but passing through the rim of the cylinder?
Ans: 1.69 lbf·ft; 5.06 lbf·ft.

6. Answer the questions in Prob. 5 for a sphere of the same mass and diameter as the cylinder.

7. In the arrangement shown in Fig. 7, the mass of the cylinder is 12 kg, the radius of the cylinder is 50 cm, and the suspended body weighs 9.81 nt. Find the downward acceleration of the body, the angular acceleration of the cylinder, and the tension in the string.
Ans: 1.40 m/sec^2; 2.80 rad/sec^2; 8.40 nt.

8. Solve Prob. 7 for a case in which the mass of the suspended body is 2.5 kg.

9. The pendulum in a certain clock consists of a long slender rod of mass m and length l and a flat disk of mass M as shown in Fig. 14. By the parallel-axis theorem, show that the moment of inertia of this pendulum about an axis through O' is given by the expression

$$I' = \tfrac{1}{3}ml^2 + \tfrac{3}{2}MR^2 + 2MlR + Ml^2.$$

10. Find an expression for the moment of inertia of the pendulum in Prob. 9 when the disk is replaced by a sphere of radius R and mass M. What would be the moment of inertia of this pendulum about an axis through O''?

11. A 10-lb mass is joined to a 5-lb mass by a string that passes over a pulley of 3-lb mass. The pulley is in the shape of a solid cylinder of 3-in radius. If there is no friction at the bearings and no slipping of the string on the cylinder, find the acceleration of the masses and the tensions in the two parts of the string. Ans: 9.76 ft/sec^2; 6.97, 6.52 lbf.

12. A 10-kg mass is joined to a 5-kg mass by a string that passes over a pulley of mass 7 kg, radius 14 cm, and radius of gyration 11 cm. If there is no friction at the bearings and no slipping of the string, find the acceleration of the masses and the tensions in the two parts of the string.

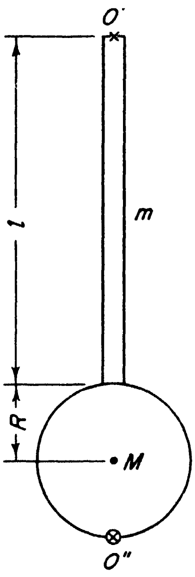


Fig. 14.

5. WORK, POWER, AND KINETIC ENERGY IN ROTATIONAL MOTION

Consider a wheel on which a constant torque is acting, for example the system of Fig. 15 in which a constant force F exerts a constant torque $L = FR$. If the string in Fig. 15 moves a distance s, work $W = Fs$ is done by the force F. During this motion, the wheel turns through an angle $\theta = s/R$. We desire to express the work in terms of torque and angle instead of force and distance. We can do this by substituting $F = L/R$ and $s = R\theta$ in the expression $W = Fs$ to obtain

$$W = L\theta. \tag{21}$$

This useful equation states that *work is equal to the product of torque by angular displacement* if the torque is constant. If the torque is not constant, we can show in any case that the element of work done by the torque L in the small angular displacement $d\theta$ is given by

$$dW = L\,d\theta.$$

If we divide this last equation through by the element of time dt during which the angular displacement $d\theta$ takes place, we obtain

$$dW/dt = L\, d\theta/dt.$$

In this equation, dW/dt is the rate at which the torque L does work, that is, the *power;* and $d\theta/dt$ is the angular velocity ω. Hence the power

$$\mathcal{P} = L\omega. \tag{22}$$

Power is given by the product of torque and angular velocity.

Now let us consider the kinetic energy of a rotating body. That a rotating body has energy by virtue of its motion is apparent if we try to stop a large flywheel by applying a brake shoe. The brake shoe becomes very hot as the rotational kinetic energy is transformed into heat. It is also well known that considerable work must be done to set a large wheel into rotation even when frictional effects are negligible.

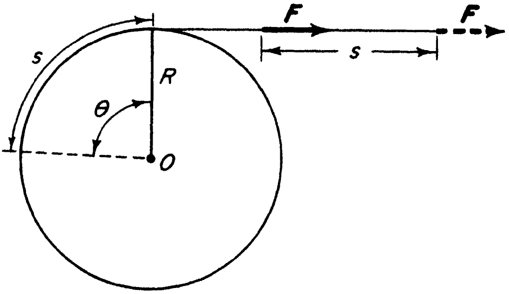


Fig. 15.

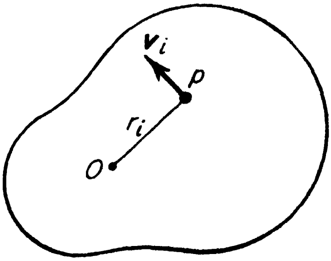


Fig. 16.

The kinetic energy of a rotating body is the sum of the kinetic energies of the particles of which it is composed. This kinetic energy can be expressed in terms of the moment of inertia of the body and its angular velocity. To obtain the expression, let us consider the body in Fig. 16, which is rotating at angular velocity ω about a stationary axis through O. Point p represents a typical particle of mass m_i. The kinetic energy of this particle is $\frac{1}{2}m_i v_i^2$, where v_i is the magnitude of the velocity of the particle in its circular path of radius r_i. Since $v_i = r_i\omega$, we may write the kinetic energy of the particle in the form

$$\tfrac{1}{2}m_i r_i^2 \omega^2.$$

Therefore, the total kinetic energy of the body is the sum

$$\text{K.E.} = \Sigma\, \tfrac{1}{2}m_i r_i^2 \omega^2.$$

Since the angular velocity ω of all particles in the body is the same, we can factor out $\frac{1}{2}\omega$ and write

$$\text{K.E.} = \tfrac{1}{2}\omega^2 \sum m_i r_i^2.$$

The sum $\sum m_i r_i^2$ is recognized as the moment of inertia of the wheel about the axis through O. Therefore, we may write

$$\text{K.E.} = \tfrac{1}{2} I \omega^2 \tag{23}$$

for a body of moment of inertia I rotating with angular velocity ω. This formula applies whether or not the axis of rotation passes through the center of gravity of the body; for example it would give the kinetic energy of the pendulum of Fig. 14 when rotating about O', provided I is taken as the moment of inertia about the axis of rotation O'.

We can verify the correctness of (23) by calculating the rotational kinetic energy in terms of the work done in imparting the kinetic energy to the rotating body. For example, consider the wheel shown in Fig. 15. The wheel, initially at rest, is set in rotation about the axis through O by a force F exerted on a cord wound around the periphery. When the wheel has turned through angle θ, the torque $L = FR$ has done work $W = L\theta$, by (21). If there are no frictional effects, all this work is converted into rotational kinetic energy, so the kinetic energy when the wheel has turned through angle θ should be

$$\text{K.E.} = W = L\theta.$$

However, by (18), $L = I\alpha$; and by (8), $\theta = \frac{1}{2}\alpha t^2$, since $\omega_0 = 0$. Therefore,

$$\text{K.E.} = \tfrac{1}{2} I \alpha^2 t^2;$$

and since, by (3), $\omega = \alpha t$,

$$\text{K.E.} = \tfrac{1}{2} I \omega^2,$$

which is the result given in (23).

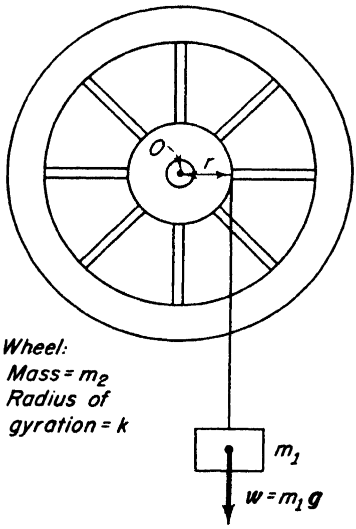


Fig. 17.

The principle of conservation of energy is very useful in solving problems involving rotational motion. As an example, let us consider the frictionless arrangement shown in Fig. 17. A weight of 16.1 lbf, suspended from a cord wrapped around a hub of radius 4 inches, is released and allowed to descend, thereby setting the large wheel into rotation about a fixed axis O. The wheel has a mass of 2 slugs and a radius of gyration of 15 inches. Let us determine the linear velocity of the suspended body and the angular velocity of the wheel after the suspended body has descended a distance of 6 ft. We note immediately that the final kinetic energy of the system,

$$\tfrac{1}{2}I\omega^2+\tfrac{1}{2}m_1v^2,$$

will be equal to the decrease in the potential energy of the weight:

$$wh=16.1\text{ lbf}\cdot 6\text{ ft}=96.6\text{ ft}\cdot\text{lbf}.$$

In evaluating the final kinetic energy, it is convenient to use the relation $v=r\omega$ between the downward linear velocity v of the weight, the angular velocity ω of the wheel, and the radius r of the hub. Substituting this value for v, and equating the final kinetic energy to the decrease in potential energy, we obtain

$$\tfrac{1}{2}m_2k^2\omega^2+\tfrac{1}{2}m_1r^2\omega^2=wh.$$

This gives

$$\omega^2=\frac{2wh}{m_2k^2+m_1r^2}.$$

By substitution of the values $m_2=2$ slugs, $k=\tfrac{5}{4}$ ft, $m_1=\tfrac{1}{2}$ slug, and $r=\tfrac{1}{3}$ ft, we obtain

$$\omega^2=\frac{2\times 96.6\text{ lbf}\cdot\text{ft}}{2\text{ slug}\cdot\tfrac{25}{16}\text{ ft}^2+\tfrac{1}{2}\text{ slug}\cdot\tfrac{1}{9}\text{ ft}^2}=60.7\frac{\text{lbf}}{\text{slug}\cdot\text{ft}}.$$

To interpret this last expression, we remember that 1 lbf = 1 slug·ft/sec², in accordance with Newton's second law. If we make this substitution we find $\omega^2=60.7/\text{sec}^2$, which we can write more intelligibly as 60.7 rad²/sec², since the radian is a dimensionless unit. The square root of this value gives

$$\omega=7.79\text{ rad/sec}.$$

The final downward velocity of the weight is therefore

$$v=r\omega=\tfrac{1}{3}\text{ ft}\cdot 7.79\text{ rad/sec}=2.60\text{ ft/sec}.$$

It is of interest to check this result by the methods used earlier in this chapter. In doing this we note that the resultant downward force acting on the suspended body is $(w-T)$, where T is the tension in the cord, and that the resultant clockwise torque acting on the wheel is $L=Tr$. We may write the two equations of motion as

$$L=I\alpha \quad\text{or}\quad Tr=I\alpha,$$

and

$$w-T=m_1a \quad\text{or}\quad w-T=m_1r\alpha,$$

since $a=r\alpha$. From these equations, we may determine α and thereafter determine ω in terms of α and the angle $\theta=h/r$ through which the wheel has turned. Completion of this check is left as an exercise.

We can now discuss further the close analogy between the physical quantities and relations involved in rotational motion about a fixed axis and those involved in translational motion along a straight line. Table I gives a summary of the more important analogies. Study of this table should prove useful.

TABLE I

QUANTITIES AND RELATIONS USED IN DESCRIBING RECTILINEAR MOTION AND THEIR ROTATIONAL ANALOGUES

Quantity or relation	Rectilinear motion	Rotational motion
Displacement	s (linear)	θ (angular)
Velocity	$v = ds/dt$ (linear)	$\omega = d\theta/dt$ (angular)
Acceleration	$a = dv/dt$ (linear)	$\alpha = d\omega/dt$ (angular)
Inertia	m (mass)	I (moment of inertia)
Force, torque	F (force)	L (torque)
Newton's second law	$F = ma$	$L = I\alpha$
Element of work	$F\,ds$	$L\,d\theta$
Kinetic energy	$\frac{1}{2}mv^2$	$\frac{1}{2}I\omega^2$
Power	Fv	$L\omega$
Momentum*	mv (linear)	$I\omega$ (angular)
Impulse*	Ft (linear)	Lt (angular)

* These quantities will be discussed in Chap. 9

PROBLEMS

NOTE: Do the following problems by considering work, energy, and power rather than by employing Newton's second law.

1. A cylindrical grinding wheel that weighs 32.2 lbf is rotating at an angular velocity of 80 rad/sec. The diameter of the wheel is 18 inches. What is the kinetic energy of this wheel? Ans: [illegible] ft·lbf.
1800

2. Find the kinetic energy of a 2.5-slug cylindrical grinding wheel 15 inches in diameter when the wheel is rotating at a speed of 600 rev/min.

3. A solid cylinder similar to that shown in Fig. 15 is set in motion by the method shown in the figure. The weight of this cylinder is 16.1 lbf and its radius is 6 in. If the force acting on the cord is 20.0 lbf, find the angular velocity of the cylinder when the cord has experienced a 10-ft displacement as shown in the figure. Ans: 80 rad/sec.

4. Find the angular velocity of the cylinder described in Prob. 3 when a cord exerting 15 lbf undergoes a 5-ft displacement.

5. A 64.4-lb flywheel has a radius of gyration of 18 in. What is the angular velocity of this flywheel when its rotational kinetic energy is equal to that of a 4-lb shell moving at a velocity of 1000 ft/sec? Ans: 166 rad/sec.

6. What angular velocity does the flywheel in Prob. 5 have when its kinetic energy is equal to that of a 2-ton automobile moving at a speed of 30 mi/hr?

7. A man shuts off the motor operating the grinding wheel mentioned in Prob. 1 and the wheel comes to rest after making 300 revolutions. What torque (assumed constant) is associated with the frictional forces acting on the wheel? Ans: 0.478 lbf·ft.

8. The operator turns off the motor driving the wheel described in Prob. 2. The wheel comes to rest after making 450 revolutions. What torque (assumed constant) is associated with the frictional forces acting on this wheel?

9. Suppose that the mass of the wheel shown in Fig. 17 is 12 kg and that its radius of gyration is 120 cm. The mass of the suspended body is 3 kg and the radius of the hub is 40 cm. Find the rotational kinetic energy of the wheel and the translational kinetic energy of the suspended body if the body starts from rest and moves down a distance of 5 m. What is the angular velocity of the wheel? the linear velocity of the suspended body? the tension in the cord? (Assume that friction is negligible.)
Ans: 143 joules, 3.97 joules; 4.07 rad/sec; 1.63 m/sec; 28.6 nt.

10. Solve Prob. 9 for a case in which the mass of the suspended body is 15 kg.

11. A flywheel of mass 12 kg and radius of gyration 1.2 m, with a hub of radius 0.4 m, is rotating at a speed of 15 rev/sec when a brake shoe is applied to the hub, the coefficient of friction between shoe and hub being 0.3. How large a normal force must be exerted on the hub by the shoe if the wheel comes to rest after making 50 revolutions? What becomes of the initial kinetic energy of the wheel? Ans: 2,040 nt.

12. If the force applied by the brake shoe to the hub of the wheel described in Prob. 11 were 100 nt, how many revolutions would the wheel make before coming to rest?

13. A Prony brake is a water-cooled brake used for loading a motor or an engine in tests of its power output. The brake is so arranged that the torque that the motor is exerting on the brake can be measured by means of a lever arm and a scale.

(a) Draw a sketch showing a suitable arrangement for making this torque measurement.

(b) The torque exerted by an electric motor turning at 2000 rev/min is measured as 60 lbf·ft. What horsepower is being delivered by the motor?

(c) The electrical input to this motor is measured by electrical instruments as 19.2 kw. What is the efficiency of the motor? Ans: (b) 22.8 hp; (c) 88.5%.

14. A rope brake is used to load a motor in a test of its power output. The motor is equipped with a flanged water-cooled pulley, and 1½ turns of rope are taken around the pulley. The two ends of the rope are brought vertically down to the floor where they are attached to spring balances. A suitable arrangement is provided to tighten the rope in order to load the motor.

(a) Draw a sketch of this arrangement.

(b) If the centers of the two vertical ropes are 18 inches apart and the tensions are 125 and 325 lbf when the motor is turning at 480 rev/min, what horsepower is being delivered by the motor?

(c) If the electrical input to the motor is measured by electrical instruments as 12.2 kw, what is the efficiency of the motor?

15. A flywheel whose moment of inertia is 150 kg·m^2 will come to rest in 300 sec from an angular velocity of 600 rev/min owing to the constant frictional torque of the bearings. How much power must be expended against this frictional torque to maintain the speed at 600 rev/min? Ans: 1.97 kw.

16. If the frictional torque on the wheel of a gyroscope (moment of inertia 2200 g·cm^2) is so small that it requires a power of only 9 milliwatts to keep it running at 6000 rev/min, how long will the wheel run before coming to rest after the supply of power is cut off, the frictional torque being assumed constant?

17. For the system of Prob. 11, p. 198, determine from conservation of energy the speed of the masses when they have started from rest and travelled 9 ft.
Ans: 13.2 ft/sec.

18. For the system of Prob. 12, p. 198, determine from conservation of energy the speed of the masses when they have started from rest and travelled 11 ft.

6. DYNAMICS OF ROTATION AND TRANSLATION IN A PLANE

We have noted in Chap. 2 that *any* motion of a rigid body can be considered as a combination of *translation* and *rotation.* Now that we have discussed the dynamics of translation and rotation separately, let us consider the dynamics of motion involving both translation and rotation of a rigid body. We shall confine our attention to the case where the motion takes place in a plane, say the xy-plane. By this we mean that the velocity vector of each point of the body lies in (or parallel to) the xy-plane, the angular velocity is about an axis perpendicular to this plane, and all the force vectors lie in this plane. A wheel rolling down an inclined plane is a good example to keep in mind; the xy-plane is the plane of the diagram in which we draw the wheel (see Fig. 22).

In discussing such motion, it turns out that the center of gravity of the body plays a very important role. We cannot give rigorous proofs at this stage, but we shall state and use two theorems that can be rigorously proved. In Fig. 18, let axes x and y be fixed in space. Let C be the center of gravity of the body. Let CX be a direction parallel to Ox. Let CA be a line 'painted' on the body, making angle θ with CX. As the body moves, the center of gravity C moves and the line CX moves but always remains parallel to Ox. The angle θ changes with time, and, by definition,

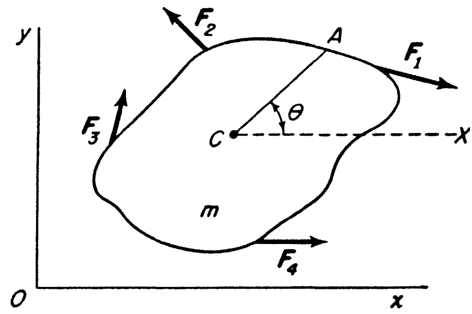


Fig. 18. A body undergoing translation and rotation in a plane. C is the center of gravity, and lies in the xy-plane. CX remains parallel to Ox. CA is a line 'painted' on the body (or, rather, *in* the body, since CA is in the xy-plane and probably buried in the body).

$$\text{angular velocity of rotation} = \omega = d\theta/dt,$$
$$\text{angular acceleration} = \alpha = d\omega/dt = d^2\theta/dt^2.$$

It is important to note that no point or axis need be specified when angular velocity or angular acceleration of a rigid body are given. Every *line* *painted on the body turns at the same rate ω and has the same angular acceleration α.*

As indicated in Fig. 18, the system of forces acting on the body lies entirely in the xy-plane. In the case of the forces on the rim of a rolling wheel, the forces are distributed across the width of the rim. This is permissible, provided the forces are distributed symmetrically with respect to the xy-plane. Under these conditions, if the body is initially moving in this plane, it will remain in this plane.* For such motion in a plane, the following theorems are true:

The resultant force equals the mass of the body times the acceleration of the center of gravity. (This theorem is true for the most general three-dimensional motion, and will be proved in Chap. 10.)

The resultant torque about the center of gravity equals the moment of inertia about an axis through the center of gravity and perpendicular to the xy-plane times the angular acceleration.

In symbols we can say that

$$\Sigma\, F_x = ma_x, \quad \Sigma\, F_y = ma_y, \quad \Sigma\, L = I\alpha, \tag{24}$$

where $\Sigma\, F_x$ and $\Sigma\, F_y$ are the sum of the x- and y-components of all of the applied forces, $\Sigma\, L$ is the sum of the torques about the center of gravity of all of the applied forces; m is the total mass of the body, I its moment of inertia about the axis through C; a_x and a_y the components of the acceleration of the point C, and α the angular acceleration.

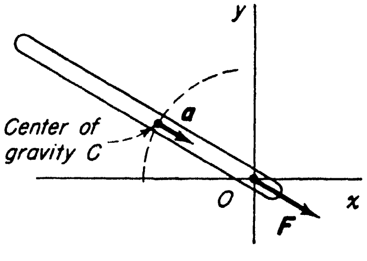


Fig. 19. A horizontal rod rotating about a vertical axis at O.

As a first example of the application of these theorems, let us consider a horizontal rod rotating at constant angular velocity about a fixed vertical axis near one end. We desire to know what force $\boldsymbol{F}$ the bearing must exert to keep the rod from flying out. This force is the only force on the rod having components in the xy-plane of Fig. 19, so it must equal $m\boldsymbol{a}$, where m is the total mass of the rod and $\boldsymbol{a}$ is the acceleration *of the center of gravity.* The acceleration of the center of gravity is directed toward the axis and is equal to $\omega^2 r$, where r is the distance CO. Hence, $\boldsymbol{F}$ has the constant magnitude $m\omega^2 r$ and a direction that rotates with the rod as indicated in Fig. 19.

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* This statement is not true unless the axis through C perpendicular to the xy-plane is what is known as a *principal axis of inertia.* This further condition is satisfied if the xy-plane is a plane of symmetry of the body, with the mass distribution the same in front of and behind this plane, or if the axis through C is an axis of symmetry. All of these provisos may serve as an indication of the extreme complexity of the general subject of the dynamics of rotation of a rigid body.

As a second example, consider the motion of a body projected or thrown into the air, with air resistance neglected. We have seen that the trajectory of the body is a parabola if the body can be treated as a particle. But suppose we throw a *chair* into the air. Which point of the chair describes the parabola? The answer is that the center of gravity does, because *the motion of the center of gravity is just like the motion of a particle acted on by the total force of gravity on the rigid body.* This statement follows from the first of the theorems given above. In spite of the fact that gravity exerts no torque about the center of gravity, the rotational motion can be very complex unless the rather exacting conditions we have laid down for two-dimensional motion are satisfied. In this two-dimensional case, the angular velocity, which must be about an axis perpendicular to the plane of the trajectory, is constant. One can illustrate these statements by throwing a tennis racket or a book-shaped wooden block with various types of initial rotation.

The interesting example of a rolling body will be considered in the next section. In preparation, we need to learn to compute the kinetic and potential energies of a rigid body undergoing translation and rotation in two dimensions. *The whole motion can be considered as a superposition of a translation of the whole body with the velocity of the center of gravity, plus a rotation of the body about the axis through the center of gravity, considered as a fixed axis.* The velocity of any point in the body can be obtained by vector addition of the velocities associated with these two motions. Also, it can be shown that the kinetic energy is the sum of the kinetic energies associated with these two motions individually, that is,

$$\text{K.E.} = \tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2, \tag{25}$$

where v is the speed of the center of gravity, I is the moment of inertia about the axis through the center of gravity, and ω is the angular velocity.

In the expression for the potential energy of a body, the center of gravity again plays a unique role. The potential energy is given by

$$\text{P.E.} = mgh, \tag{26}$$

where m is the total mass of the body and h is the *height of the center of gravity* above the reference level to which zero potential energy is assigned. We derive this expression by considering the total potential energy as the sum of the potential energies of the particles of which the body is composed. Let the ith particle have mass m_i and be at height h_i above the reference level. Its potential energy is then $m_i g h_i$. The whole potential energy will be

$$\text{P.E.} = \Sigma\ m_i g h_i = g\ \Sigma\ m_i h_i = gmh,$$

by definition of the height of the center of gravity.

Thus in the solution of problems involving the rotation of rigid bodies by the principle of conservation of energy, *the change in potential energy of a body is determined entirely by the change in elevation of the center of gravity.*

The student will find that he can solve problems by conservation of energy that he would find entirely untractable by direct application of Newton's laws. Examples occur in the following set.

PROBLEMS

1. A uniform iron rod of length 10 ft and mass 3 slugs lies at rest on the surface of an ice-covered pond. A man applies a horizontal force of 12 lbf to one end in a direction perpendicular to the axis of the rod. Find the magnitudes of the initial linear acceleration of the center of the rod and the initial angular acceleration of the rod. Friction between the rod and the ice is negligible. Ans: 4 ft/sec^2; 2.4 rad/sec^2.

2. If the man had applied the transverse force at a point 3 ft from one end of the rod described in Prob. 1, what would have been the magnitudes of the initial accelerations? Where should the man apply the transverse force in order to produce maximum initial linear acceleration of the center? maximum angular acceleration?

3. A 200-g meter stick lies at rest on the smooth surface of a table. Two horizontal transverse forces of equal magnitude and opposite direction are applied to the stick. The magnitude of each force is 0.3 nt. Find the magnitude of the angular acceleration of the stick (a) when one force acts at the 0-cm mark and the other at the 20-cm mark, (b) when one of the forces acts at the 40-cm mark and the other at the 60-cm mark, and (c) when one of the forces acts at the 60-cm mark and the other at the 80-cm mark. Find the linear acceleration of the center of mass for each of the above cases. Ans: 3.6 rad/sec^2, 0 m/sec^2, in every case.

4. Two applied forces, equal in magnitude but opposite in direction, are said to constitute a *couple*. In the case of plane motion, show that when a couple is applied to a body at rest as in the example of Prob. 3 above:

(a) it tends to induce pure rotation about the center of gravity;

(b) the moment of the couple about the center of gravity equals the product of the magnitude of one of the forces and the perpendicular distance between the lines of action of the two forces, and is otherwise independent of where the couple is applied to the body.

5. A uniform rod is 10 ft long and has a mass of 3 slugs. This rod is suspended in a horizontal position by two vertical wires attached to the ends of the rod. If the wire at one end suddenly breaks, what is the initial downward acceleration of the center of gravity? What is the initial angular acceleration of the rod? What is the initial tension in the other wire? Ans: 24.1 ft/sec^2; 4.83 rad/sec^2; 24.1 lbf.

6. If a man weighing 129 lbf were hanging by one hand at a point 2 ft from one of the wires supporting the rod mentioned in Prob. 5, what would be his initial downward acceleration if the wire closer to him were to break suddenly? if the wire at the other end were to break suddenly?

7. A uniform rod of length 3 ft and mass 7 lb is free to rotate in a vertical plane about a fixed horizontal axis at one end. If the rod is released from rest when it is standing straight up from the pivot, what is its angular velocity when it passes through the straight-down position? when it is horizontal?

Ans: 8.02 rad/sec; 5.67 rad/sec.

8. When the rod of Prob. 7 hangs straight down at rest, it is struck a blow that gives the lower end an initial horizontal velocity of 18 ft/sec. How high does the rod swing before coming to rest?

9. If a stick is stood on end on a perfectly frictionless horizontal plane and allowed to fall from rest, describe the path of the center of gravity.

10. A trapeze artist is swinging in a vertical circle on a rigid trapeze. If his angular velocity is 2 rad/sec at the top of the swing, what is his angular velocity at the bottom of the swing? Take the center of gravity as 12 ft from the axis of rotation, and the radius of gyration as 13 ft.

7. ROLLING BODIES

We wish now to consider the case of bodies that roll on a surface without slipping. We shall confine our attention to the two-dimensional case in which we can use the theorems of the previous section. Furthermore we shall consider only symmetrical bodies such as wheels, cylinders, and spheres, in which the center of gravity is at the center of the object.

The condition that the body roll without slipping imposes a definite relationship between the linear motion and the angular motion of the body. Figure 20 shows a cross section of a wheel rolling along a surface (not necessarily horizontal). Let the initial position of the axis through the center O pass through the origin of the coordinate system as shown. If the wheel rotates in a clockwise direction until one complete clockwise rotation has taken place, the center experiences a displacement $2\pi R$ in the positive x-direction; that is, the wheel must make one complete revolution in the time that its center advances a distance equal to the circumference of the wheel. Thus, when the clockwise angular displacement θ is 2π, the linear displacement of the center in the x-direction is $2\pi R$. Hence, for the rolling wheel, we see that

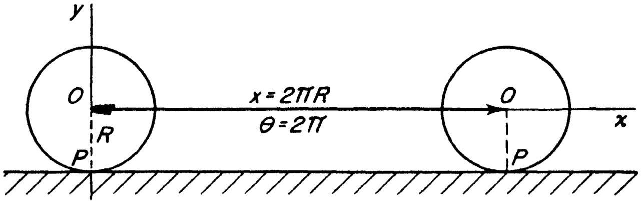

Fig. 20. When the rolling cylinder turns through an angle of 2π radians, the center O experiences a displacement $2\pi R$ in the x-direction.

$$x = R\theta. \tag{27}$$

Taking the derivative of both sides of this equation with respect to time, we obtain

$$dx/dt = R\,d\theta/dt, \quad \text{or} \quad v = R\omega, \tag{28}$$

where v is the x-component of the velocity of the center of the wheel and ω is the angular velocity. Taking the time derivative of both sides of

(28), we obtain

$$dv/dt = R\,d\omega/dt \quad \text{or} \quad a = R\alpha \tag{29}$$

where a is the x-component of the acceleration of the center and α is the angular acceleration. Note that θ, ω, and α are positive in a clockwise sense in these equations, rather than in the usual counterclockwise sense. This change of convention is useful for problems of rolling bodies.

With relation (29) in mind, let us find the linear acceleration of the center of gravity that results when a force $\boldsymbol{F}$ acts through the center of gravity of the solid cylinder of radius R shown in Fig. 21. A force of friction $\mathfrak{F}$ will be required to prevent slipping. We shall be able to solve this problem completely by employing equations (24). We first note that $w = N$, since the center of gravity has no acceleration in the y-direction. The equation of motion of the center of gravity in the x-direction is

$$F - \mathfrak{F} = ma;$$

and the clockwise torque about the center determines the angular acceleration:

$$\mathfrak{F}R = I\alpha,$$

where I is the moment of inertia about the center of gravity. Remembering that $I = \frac{1}{2}mR^2$ for a solid cylinder, and that $\alpha = a/R$, we may rewrite the second equation in the form

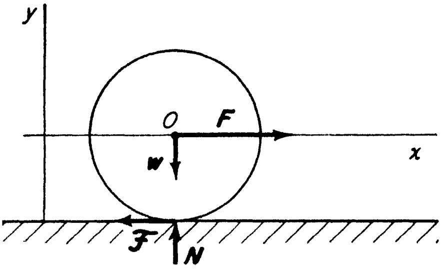

Fig. 21. A solid cylinder that rolls without slipping. The resultant of $\boldsymbol{F}$ and $\mathfrak{F}$ causes a linear acceleration of the center of gravity O. The moment of the frictional force $\mathfrak{F}$, about O, causes an angular acceleration.

$$\mathfrak{F}R = (\tfrac{1}{2}mR^2)(a/R), \quad \text{or} \quad \mathfrak{F} = \tfrac{1}{2}ma.$$

Adding this equation to the first equation and solving the resulting equation for a, we obtain

$$a = \tfrac{2}{3}\,F/m. \tag{30}$$

The rolling cylinder accelerates two-thirds as fast as a particle having the same applied force. The angular acceleration is

$$\alpha = a/R = \tfrac{2}{3}\,F/mR.$$

The force of friction is

$$\mathfrak{F} = \tfrac{1}{2}ma = \tfrac{1}{3}\,F.$$

Now let us consider the same cylinder rolling under gravity down the inclined plane of Fig. 22. The forces acting on the cylinder will be the force $\boldsymbol{w} = m\boldsymbol{g}$ of gravity, the normal reaction $\boldsymbol{N}$ of the plane, and whatever frictional force $\mathfrak{F}$ is required to prevent slipping. The forces normal to the plane will balance, since there is no acceleration of the center of gravity in this direction. Therefore,

$$N = mg \cos\theta. \tag{31}$$

The net force down the plane will produce an acceleration a of the center of gravity:

$$mg \sin\theta - \mathfrak{F} = ma. \tag{32}$$

The clockwise torque about the center of gravity will equal the moment of inertia times the clockwise angular acceleration:

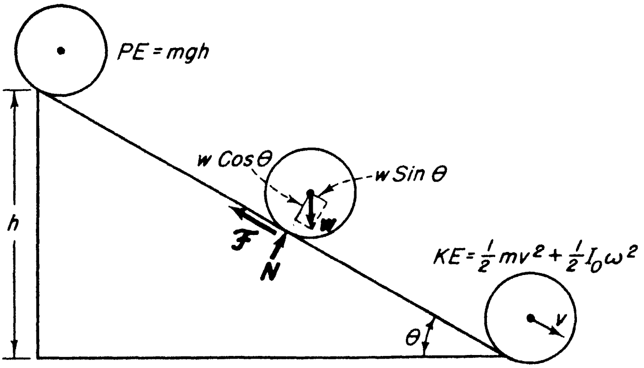

Fig. 22. A cylinder that rolls without slipping down an inclined plane.

$$\mathfrak{F}R = I\alpha. \tag{33}$$

As before, $\alpha = a/R$, and $I = \frac{1}{2}mR^2$, so (33) gives $\mathfrak{F} = \frac{1}{2}ma$. When this value is substituted in (32), we find that

$$a = \tfrac{2}{3}g \sin\theta, \tag{34}$$

which is a constant acceleration two-thirds as great as the acceleration of a particle sliding down a *smooth* plane of the same inclination. We also find that

$$\mathfrak{F} = \tfrac{1}{3}mg \sin\theta,$$

so that we need a coefficient of *static* friction of at least

$$\mathfrak{F}/N = \tfrac{1}{3} \tan\theta$$

to prevent slipping.

If the cylinder in Fig. 22 starts from rest and rolls down through a vertical height h (slant height $l = h/\sin\theta$), we can obtain its velocity at the bottom from the kinematical laws of uniform acceleration. The center of gravity moves a distance l with acceleration $a = \frac{2}{3}g \sin\theta$. It therefore acquires velocity given by

$$v^2 = 2al = \tfrac{4}{3}gl\sin\theta = \tfrac{4}{3}gh,$$

or

$$v = 2\sqrt{\tfrac{1}{3}gh}. \tag{35}$$

Curiously enough, in spite of the fact that we need a large force of friction in order to make possible the motion we have described, energy is conserved in the case of pure rolling and we can derive this velocity from application of conservation of energy. A force of *kinetic* friction acts on a point of the body that is moving. Furthermore, it acts in a direction opposite to the velocity, so that it does negative work, which represents a loss of mechanical energy and generation of heat energy. But a force of *static* friction never does work, because it acts on a point of the body that is *at rest.* In the case of pure rolling without slipping, the friction is *static* friction, the force of friction acting on a point of the body that is instantaneously at rest. Hence there is no energy loss owing to friction in pure rolling. This is the ideal case; actually, a wheel rolling on a horizontal surface will gradually slow up—but *much* less rapidly than a sliding body—because there is some *distortion* of the surfaces of contact so that the force system is not *exactly* as we have described it. The wheel is generally regarded as man's greatest mechanical invention. It was discovered very early in Europe or Asia but was never discovered on the American continent, even by civilizations so advanced in other respects as that of the Mayas.

We can consider that the cylinder in Fig. 22, at rest at the top of the hill, has potential energy mgh. At the bottom of the hill this has been converted into translational kinetic energy $\tfrac{1}{2}mv^2$ plus rotational kinetic energy $\tfrac{1}{2}I\omega^2$:

$$mgh = \tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2.$$

Remembering that $I = \tfrac{1}{2}mR^2$ and $\omega = v/R$, we rewrite this energy equation as

$$mgh = \tfrac{1}{2}mv^2 + \tfrac{1}{4}mv^2 = \tfrac{3}{4}mv^2,$$

which gives the value

$$v = 2\sqrt{\tfrac{1}{3}gh}$$

for the magnitude of the final translational velocity of the center of mass. This equation is identical with (35). It will be noted that the principle of conservation of energy is a powerful tool that has enabled us to obtain this equation with much less work than is involved in the direct application of the laws of motion.

Now, let us consider the instantaneous velocity of various points on a wheel rolling in the x-direction as shown in Fig. 23. We can regard the rolling as a combination of *translation* and *rotation about the central axis through O.* If we consider *translation only,* all points in the wheel have the same velocity v as the center; this is shown in Fig. 23(a). If we consider *rotation only*, the center is at rest, whereas point P at the top of the cylinder has x-velocity $+R\omega$ and point O' at the bottom of the cylinder has x-velocity $-R\omega$; these are indicated in Fig. 23(b). Combining these x-components of velocity, we have

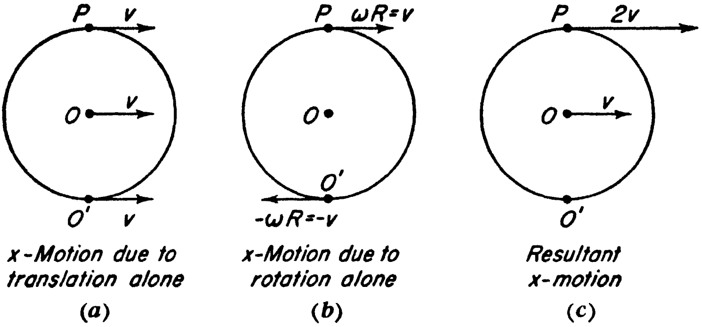
P
v
O
v
O'
v
x-Motion due to translation alone
(a)
P
ωR=v
O
O'
-ωR=-v
x-Motion due to rotation alone
(b)
P
2v
O
v
O'
Resultant x-motion
(c)

Fig. 23. A rolling wheel. The combined effects of translation and rotation about an axis through O give the same result as a pure rotation with the same angular velocity about O'.

$$\text{for point } P: \quad v_x = v + R\omega,$$
$$\text{for point } O: \quad v_x = v,$$
$$\text{for point } O': \quad v_x = v - R\omega.$$

However, by (28), $R\omega = v$, and therefore the resultant x-velocities of the three points are

$$\text{for point } P: \quad v_x = 2v,$$
$$\text{for point } O: \quad v_x = v,$$
$$\text{for point } O': \quad v_x = 0.$$

These results are shown schematically in Fig. 23(c).

We note that the point that at any instant is at the bottom of the wheel is at rest. This must be so because this point is in contact with, and not slipping on, a surface at rest. Since the wheel is a rigid body, the velocities given in Fig. 23(c) are sufficient to determine that at any instant the whole velocity pattern is one of pure rotation about the point of contact with angular velocity v/R, the same as the angular velocity in the rotation about O.

Although the velocities of the points in the rolling wheel are correctly represented by the picture of a pure rotation about O', the accelerations are not given correctly by this picture. Hence, the axis through O' cannot in general be used like the fixed axis of Sec. 3 for dynamical considerations. The equation $L = I\alpha$ holds about a fixed axis. It also holds about an axis through the center of gravity when our stated conditions for two-dimensional motion are satisfied. *It does not, in general, hold about an instantaneous axis of rotation.**

PROBLEMS

1. An automobile is equipped with wheels of diameter 2 ft. At a certain time this automobile is moving without skidding at a velocity of 30 mi/hr eastward. Find the instantaneous linear velocity of a point at the top of one of the wheels (a) with respect to the car and (b) with respect to the road. Find the instantaneous linear velocity of a point at the bottom of one of the wheels (c) with respect to the car and (d) with respect to the road.
Ans: (a) 44 ft/sec, eastward; (b) 88 ft/sec, eastward; (c) 44 ft/sec, westward; (d) 0.

2. An automobile is equipped with wheels 80 cm in diameter. The car is moving without skidding northward at a speed of 36 km/hr. Find the instantaneous linear velocity of a point at the top of one of the wheels (a) with respect to the automobile and (b) with respect to the road. Find the instantaneous linear velocity of a point at the bottom of one of the wheels (c) with respect to the car and (d) with respect to the road.

3. What is the magnitude of the instantaneous angular velocity of the wheel in Prob. **1**? Ans: 44 rad/sec.

4. What is the magnitude of the instantaneous angular velocity of the wheel in Prob. 2?

5. A lawn roller consists of a large solid cylinder 2 ft in diameter with a handle attached at the axle. The mass of the roller is 4 slugs. By pulling on the handle a boy exerts a horizontal force of 50 lbf in the manner indicated in Fig. 21. Find the linear acceleration of the center of gravity and the angular acceleration of the cylinder.
Ans: 8.33 ft/sec^2; 8.33 rad/sec^2.

6. A boy exerts a force of 4 lbf on a metal hoop 3 ft in diameter. The mass of the hoop is 16.1 lb. If the force is horizontal and its line of action passes through the center of mass as indicated in Fig. 21, find the acceleration of the center of mass.

7. The effective coefficient of friction between the ground and the lawn roller described in Prob. 5 is 0.4. What is the largest horizontal acceleration that can be imparted to the center of mass without causing the roller to slide? What horizontal force must be applied in the manner shown in Fig. 21 in order to produce this acceleration? Ans: 25.8 ft/sec^2; 155 lbf.

8. If the effective coefficient of friction between the sidewalk and the hoop described in Prob. 6 is 0.2, what is the maximum horizontal acceleration that can be imparted to the hoop without causing it to slide?

9. A solid sphere of 1-ft radius and 40-lb mass rolls without slipping down an inclined plane inclined at an angle of 30° to the horizontal. Find the linear acceleration of the center of gravity of this sphere. Solve this problem for a sphere of radius 2 ft and mass 250 lb. Ans: 11.5 ft/sec^2; 11.5 ft/sec^2.

10. A hoop 3.5 ft in diameter and mass 27.2 lb rolls without slipping down a plane inclined at an angle of 30° to the horizontal. Find the linear acceleration of the center of gravity of this hoop.

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* The contrary is erroneously stated in many places because $L = I\alpha$ *happens* to hold about the instantaneous axis *when* the center of gravity lies at the center of the wheel. It does not hold if the center of gravity is *not* at the center of the wheel. Failure to realize this has caused much confusion, even in advanced textbooks. We do not recommend use of this equation about the instantaneous axis, even in cases where it gives the right answer, because it is of the nature of a special trick rather than a general principle.

11. A solid cylinder of diameter 2 ft and mass 32.2 lb rolls down an inclined plane. The initial position of the cylinder was 20 ft vertically above its final position at the foot of the slope. What is the final total kinetic energy of the cylinder at the foot of the inclined plane? If we choose to consider the motion of the cylinder as a translation with the velocity of the center of gravity plus a rotation about an axis through the center of gravity, how much kinetic energy is associated with translational motion and how much kinetic energy is associated with rotational motion?

Ans: 644 ft·lbf; 429 ft·lbf; 215 ft·lbf.

12. Solve Prob. 11 for a hollow cylinder (hoop) with mass and diameter equal to that of the solid cylinder.

13. Derive an expression for the final translational velocity of the center of gravity of a hollow cylinder (hoop) that has rolled without slipping down an inclined plane from an initial height h. Ans: $v = \sqrt{gh}$.

14. Derive an expression for the final translational velocity of the center of mass of a solid cylinder that has rolled without slipping down an inclined plane from an initial height h. Make a similar derivation for a solid sphere and for a wheel of radius of gyration k. Show that the equation for the wheel also applies to the other bodies considered when the proper values of k are used.

15. A 'yo-yo' has the string wrapped around a reduced section of radius 0.5 cm, but the radius of gyration is 3 cm. The string is held in the hand and the yo-yo with its axis horizontal is allowed to unwind the string as it falls vertically. What fraction of the kinetic energy it acquires is rotational? Ans: $^{36}/_{37}$.

16. A spool of thread has rim diameter 4 cm, but the thread unwinds from a layer 2 cm in diameter. The radius of gyration is 1 cm. The spool is placed with its axis horizontal on a horizontal surface, with the end of the thread leading out from the *under* side of the spool. The thread end is held horizontally and pulled in a direction perpendicular to the spool axis. Show that if the rims of the spool do not slip, the thread will wind itself up! Determine the relation between the acceleration of the spool, the tension in the thread, and the mass of the spool.

CHAPTER 9

ELASTICITY AND STRENGTH OF MATERIALS

In our treatment of mechanics thus far, we have assumed that the application of forces to bodies produces no deformation of the bodies on which the forces act. Bodies have been assumed to be rigid, strings to be inextensible, and liquids to be incompressible. These assumptions are not strictly valid in any real physical situation, but can in many cases be justified when deformations are small or when the effects of the deformations are unimportant to the main problem being considered. For example, if a body is to be supported by means of wires of large diameter, the wires may experience only slight elongations that can be ignored in the computation of the forces to be expected in this system. If, however, the body is to be supported by fine wires, the wires may be elongated so much that the geometry of the system, and therefore the forces involved, may be quite different from what would be expected on the assumption that the wires are inextensible. In an extreme case the supporting wires might even break. The same type of considerations apply to beams and other structural members. Since the engineer must deal with real rather than with ideal materials in designing structures and machines, it is important that the elastic properties and strengths of materials be taken into consideration. In the present chapter, we shall consider some of the more important elastic properties of materials. The treatment we shall give will be an extremely elementary theory of elasticity. We shall try to obtain an understanding of the elastic properties of materials in terms of forces between the atoms or molecules comprising the lattice structure of the solid material.

1. HOOKE'S LAW, ELASTIC POTENTIAL ENERGY

Everyone has observed the bending of a piece of wood such as a diving board when a load is added; when the load is removed, the board regains its original shape. Likewise, a helical spring increases in length when a small load is added but regains its original length when the load is removed. These observed effects are examples of the *elasticity* of matter.

An *elastic body* is one that experiences a change in volume or shape when the deforming forces act upon it but resumes its original size or shape when the deforming forces cease to act. The recovery of the original configuration is practically perfect for bodies composed of a great many kinds of materials, provided the distorting forces are not too great. If the distorting forces are too great, the *elastic limit* is exceeded and the recovery of the original configuration is incomplete. In this case, the body is said to have acquired a *permanent set* or permanent deformation. For example, if too heavy a load is supported by a helical spring, the spring may be permanently stretched. The limiting load that can be supported by a spring without permanent deformation depends upon the size of the wire used, the way in which the spring is wound, and the material of which the wire is made. For example, for springs of the same shape and wire size, a spring made of lead wire would be permanently deformed by a much smaller load than that required to produce permanent deformation of a spring made of steel wire. As we shall see later, the term *elastic limit* can be defined in such a way that it has a definite value for every *material.*

If the distorting forces acting on a body are made sufficiently great, the *breaking strength* will be reached and the body will rupture.

Materials for which the elastic limit is extremely small are called *inelastic materials;* for example, dough, putty, and lead solder are inelastic materials, since bodies composed of these materials are permanently deformed when acted upon by relatively small forces. Steel is a highly elastic material, since relatively enormous forces are required to produce permanent distortion of bodies composed of steel.

Nearly three centuries ago, the foundation of the theory of elasticity was laid by the discovery of the fundamental relationship between the force applied to an elastic body and the deformation produced. This relationship is known as *Hooke's law;** this law, as Hooke originally expressed it in concise Latin, can be translated literally: 'As the deformation, so the force.' In other words:

> *The deformation of an elastic body is directly proportional to the magnitude of the applied force, provided the elastic limit is not exceeded.*

The correctness of Hooke's law in a special case can be verified by adding weights to a helical spring in the manner indicated in Fig. 1. Before weight $\boldsymbol{W}$ is added, the length of the spring is l_0. The added weight of magnitude W produces elongation y.

At the observed elongation, the spring exerts an upward restoring force $\boldsymbol{F}$ on the suspended body; this force is equal in magnitude and opposite in direction to the weight of the body, since the suspended body is in equilibrium. We shall call this force $\boldsymbol{F}$ the *elastic force,* since it is exerted by the elastic spring by virtue of its deformation.

*** Formulated by the English experimental physicist Robert Hooke (1635–1703).**

By adding different weights and noting the elongation, we obtain data that can be plotted in the manner shown in Fig. 2. The resulting straight-line relationship indicates that the observed elongation y is proportional to the deforming force W and consequently to the elastic force F. As the elongation y is a measure of the distortion, we see that Hooke's law is verified for the spring in question; the relationships between the deforming force W, the elastic force F, and the elongation y can be written

$$W = ky, \quad F = -ky, \tag{1}$$

where k is called the *force constant* of the spring. The force constant k gives the force per unit displacement and is therefore measured in units like lbf/ft or nt/m depending upon the units in which F and y are measured. Its magnitude is a measure of the 'stiffness' of the spring. The negative sign in (1) is used to indicate that the elastic force $\boldsymbol{F}$ is exerted in a direction opposite to the displacement y. Thus, if the displacement is downward as indicated in Fig. 1, the elastic force $\boldsymbol{F}$ is directed upward. If an external force $\boldsymbol{F}_{ex}$ acts as shown in Fig. 3 in such a manner as to produce a negative elongation (compression), the elastic force $\boldsymbol{F}$ exerted by the spring acts downward and according to our sign convention is regarded as positive.

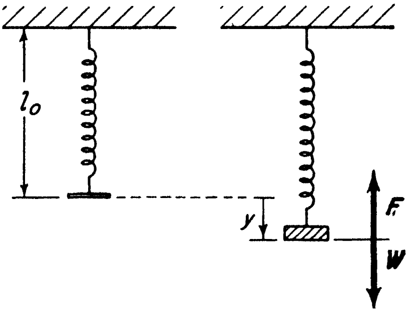

Fig. 1. The force of the weight on the spring is $\boldsymbol{W}$. The equal and opposite reaction of the spring on the weight is $\boldsymbol{F}$.

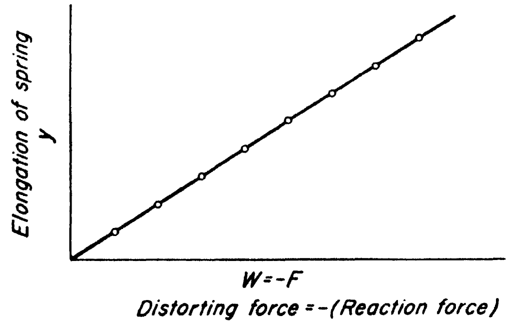

Fig. 2.

We can now compute the potential energy stored in a stretched or compressed spring. We shall first define elastic potential energy in a manner analogous to that in which gravitational potential energy was defined earlier. In the elastic case, changes in potential energy are associated with changes in configuration of the elastic body. It is only these changes in potential energy that have physical significance. However, if we are interested in the potential energy when the body is distorted by application of a particular external force or torque, it is convenient to take the potential energy as zero in the configuration the body has when this particular external force or torque is removed, other external forces being unchanged. Thus we shall call the potential energy of the spring of Fig. 1 zero when the weight W is removed and the spring has length l_0, although other forces (of gravity and the supporting beam) still act; if these other forces could be removed the spring would have a still different length. With this understanding of the meaning of the configuration of zero potential energy, we may define *elastic potential energy* as follows:

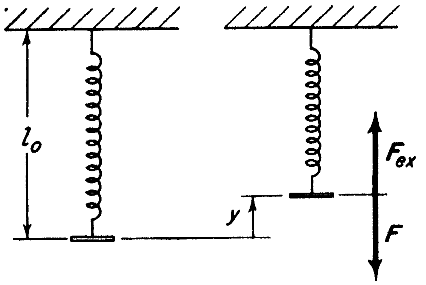

Fig. 3. F_{ex} is the external force applied to the spring. F is the equal and opposite reaction exerted by the spring.

> The elastic potential energy of a deformed body is the work that must be done *against* the elastic forces in effecting the deformation, starting from the configuration of zero potential energy. This is the same as the work done *by* the external forces required to effect the deformation, provided these forces are increased *gradually* from zero to their final value.

The external forces must be increased slowly so that the deformation will take place slowly and the elastic body will acquire no appreciable kinetic energy. If the deformation takes place rapidly the work of the external forces will have to supply not only the elastic potential energy but the kinetic energy associated with the particles in the elastic body.

We see that the above definition is consistent with the general definition of energy given on p. 151, because the elastic potential energy of a deformed body equals the work that could be done *by the elastic forces* as the body returns from its deformed configuration to the configuration of zero potential energy.

The potential energy of the spring we have been considering, when it has elongation y, is equal to the work done *against* the elastic force F of the spring as the elongation changes from 0 to y. This is the negative of the work done *by* F, and can be written as

$$\text{P.E.} = -(\text{average elastic force}) \times (\text{displacement}) = -F_{av}\, y.$$

The average elastic force F_{av} to be used in this relation is an *average with respect to distance*. Since the elastic force is zero at the start of the elongation and has the magnitude $-ky$ when the elongation is y, and varies linearly between these two values as the elongation changes from 0 to y, the required average is

$$F_{av} = \tfrac{1}{2}(0 - ky) = -\tfrac{1}{2}ky.$$

Therefore,
$$\text{P.E.} = -(-\tfrac{1}{2}ky)y = \tfrac{1}{2}ky^2. \tag{2}$$

Thus, we see that the elastic potential energy of the spring is directly proportional to the force constant and to the square of the elongation. The potential energy is seen to have the same value for a compression as for an extension of the same magnitude, since in either case the same work must be done against the elastic force to effect the deformation.

Hooke's law can also be easily verified for many other simple cases. Two examples are shown in Fig. 4. In part (a) of this figure a thin piece of steel such as a hack-saw blade is clamped at one end and deformed by suspending weights W from the free end. Experiment shows that within the elastic limit the distortion y is proportional to the magnitude W.

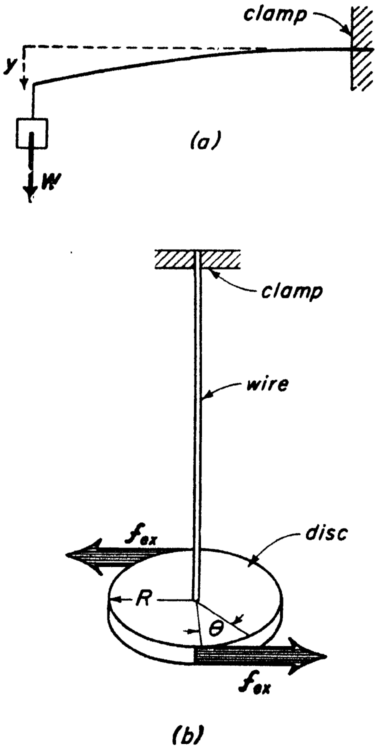

Fig. 4.

Part (b) of Fig. 4 shows a wire clamped at the upper end. To the lower end is attached a disk to which forces f_{ex} can be applied in such a manner as to exert a torque $L_{ex} = 2f_{ex}R$ which tends to twist the wire. The angle of twist can be taken as a measure of the deformation produced in the wire. Experiment shows that the deformation θ is directly proportional to the applied torque. In a manner similar to that used for the helical spring, we may write

$$L_{ex} = c\theta. \tag{3a}$$

The elastic reaction of the wire will exert an equal and opposite torque

$$L = -c\theta. \tag{3b}$$

By an argument similar to that given previously, we find the expression

$$\text{P.E.} = \tfrac{1}{2}c\theta^2 \qquad (4)$$

for the elastic potential energy stored in the twisted wire. The proportionality constant c in these equations is called the *torsion constant* of the wire. It is measured in torque units per unit angular displacement, for example, nt·m/rad or lbf·ft/rad.

It might be well to mention at this point that, whereas Hooke's law holds well for most bodies composed of metal, wood, glass, and many other common materials, provided deformations are not too large, there are some materials for which it does not hold at all. Rubber is such a material. For example, let us take a rubber band of initial length l_0 and deform the band by suspending weights W from the lower end in the manner indicated in Fig. 5(a). If various weights are added and the resulting elongations y are noted, a plot of the data has the form shown in Fig. 5(b). The curve shown in this figure bears little resemblance to the straight line to be expected on the basis of Hooke's law. We shall not attempt to give an adequate explanation of the peculiar behavior of plastic materials like rubber. It should be remembered that *elastic* elongations of five or six times the original length can easily be obtained with a rubber band, whereas materials like metal, wood, and glass can only be stretched *elastically* a small fraction of their original length before

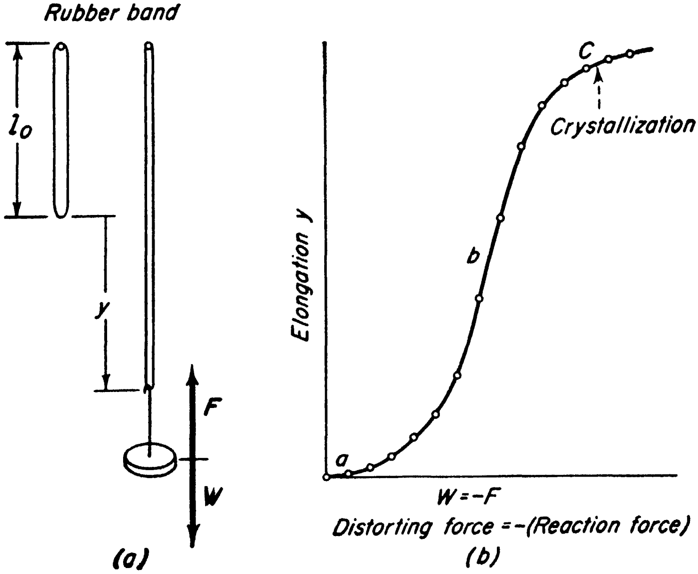

Fig. 5. Elastic properties of rubber.

the elastic limit is reached. Hence it is not surprising that these two types of materials have quite different relations between force and elongation. When subjected to large deformations, the structure of rubber undergoes marked changes, and crystallization sets in when point C of the curve in Fig. 5(b) is reached. This deformation determines the elastic limit beyond which a permanent set is acquired.

PROBLEMS

1. When a body weighing 2 lbf is attached to the end of a spring hanging vertically, the spring is stretched 0.5 in. What is the force constant of this spring?
Ans: 48 lbf/ft.

2. When a 2-kg block is attached to the end of a spring hanging vertically, the spring experiences an elongation of 5 cm. Find the force constant of this spring in kgf/m and in nt/m.

3. Assuming that the spring in Prob. 1 obeys Hooke's law, find the elongation produced when a 32-lb body is suspended from the end of the spring. Ans: 8 in.

4. Assuming that the spring in Prob. 2 obeys Hooke's law, find the elongation produced (a) when a 5-kg block is suspended from the end of the spring and (b) when a force of 50 nt is applied to the end of the spring.

5. What is the potential energy of the stretched spring mentioned in Prob. 3 when it supports the 32-lb body? when it supports a 64-lb body?
Ans: 10.7 ft·lbf; 42.7 ft·lbf.

6. What is the potential energy of the stretched spring for each case mentioned in Prob. 4? Express this energy in joules.

7. The end of a long wire hanging vertically and clamped at the upper end is twisted through an angle of 30° when a torque of 3 lbf·ft is applied to the free end. What is the torsion constant for this wire? What torque is exerted by the clamp on the upper end of the wire? What is the potential energy of this twisted wire?
Ans: $18/\pi$ lbf·ft/rad; 3 lbf·ft; 0.785 ft·lbf.

8. The end of a long brass rod is clamped tightly so as to prevent rotation. When a torque of 9 nt·m is applied to the free end, the free end experiences a rotation of 5°. What is the torsion constant of this rod? What is the potential energy of the twisted rod?

2. LONGITUDINAL STRESS AND STRAIN: GENERALIZATION OF HOOKE'S LAW

In our discussion of the original form of Hooke's law, we have described experiments that can easily be performed by stretching or compressing helical springs, bending thin strips of metal, or twisting wires. These simple experiments serve to verify the original Hooke's law for these particular *bodies* but do not add very much to our knowledge of the elastic properties of the *materials* of which the bodies are composed. For example, we were able to write equation (3) for a twisted wire, but the torsion constant c occurring in the equation applies only to the particular piece of wire used in the experiment. If we twisted another wire of the same material with different cross-sectional area and length, we should not know how to predict the deformation that would be produced by a given torque. Similarly, if we used only the simple form of Hooke's law, we should have to determine the elastic constants of each individual structural member before a building, a bridge, or a machine could be properly designed. Fortunately, it is possible to generalize Hooke's law in such a way as to make it possible to determine elastic constants for *materials.* The elastic constants of *bodies* such as structural members can then be calculated in terms of the elastic constants of the *materials* and the geometrical shapes and sizes of the bodies. This generalization of Hooke's law is therefore a much more valuable tool than the original simple law. In the following paragraphs, we shall discuss the concepts of *stress* and *strain,* in terms of which the general form of Hooke's law is stated.

Stress is related to the *force* causing deformation; *strain* is related to the *amount of deformation.* In order to understand the ideas involved, let us consider Fig. 6, which shows a long elastic rod of initial length l_0 and cross-sectional area A. If forces $\boldsymbol{F}$ are applied to the ends of the rod, the rod experiences an elongation e and we can say that the rod has undergone a *longitudinal strain. Elongation per unit length is the measure of longitudinal strain.* Since any part of the rod in Fig. 6 is lengthened in the same proportion as the whole rod, we may express the longitudinal *strain* σ as the ratio of the elongation e to the initial length l_0:

$$\sigma = e/l_0. \qquad \text{(strain)}$$

Since the rod is elastic, the strain is accompanied by *internal forces* between adjacent parts of the rod. If the cross section of the rod at any point such as M in Fig. 6(b) be considered as a dividing plane, the part of the wire on the right side of the dividing plane exerts a certain force on the part of the wire on the left side of the dividing plane, and the latter part of the wire exerts a force equal in magnitude and opposite in direction on the part of the wire to the right. This effect is shown diagrammatically in part (c) of the figure; the material to the left of M exerts a force of magnitude F toward the left on the material to the right of M, whereas the material to the right exerts a force of magnitude F toward the right on the material to the left of M. These forces must have the same magnitude as the external applied forces, since each part of the wire is separately in static equilibrium. When such forces exist in a body, the body is said to be under *stress,* and the magnitude of the stress is measured by the ratio of the internal force to the area across which the force acts. In the case of the uniform rod shown in Fig. 6, the longitudinal *stress* S is the same at all cross sections and equal in magnitude to the external force F divided by the cross-sectional area of the rod:

$$S = F/A. \qquad \text{(stress)}$$

By performing experiments on rods and wires of different sizes but composed of the same material, we find that the longitudinal stress S is proportional to the longitudinal strain σ, the proportionality constant being independent of the size of the wire. Thus we may write

$$S = E_Y \sigma,$$

where the proportionality constant E_Y is called *Young's modulus** or the *modulus of elasticity in tension,* and is a *constant for a given material.*

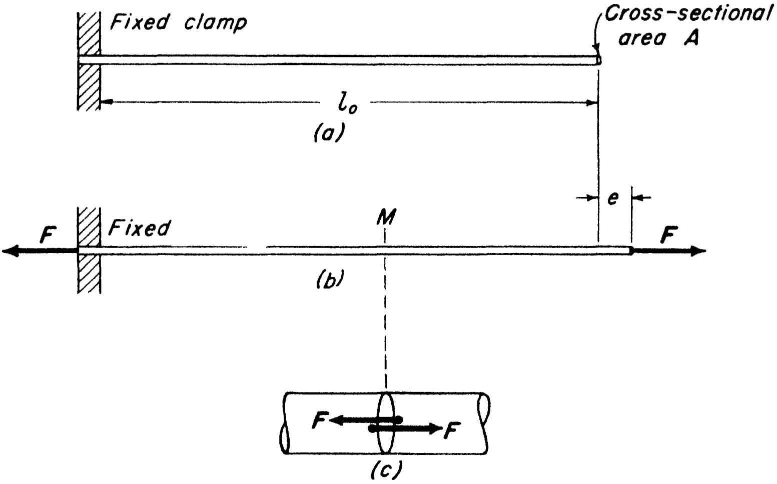

Fig. 6. Longitudinal stress is F/A. Longitudinal strain is e/l_0.

Thus we see that for any size rod or wire,

$$\frac{F}{A} = E_Y \frac{e}{l_0} \tag{5}$$

in the notation of Fig. 6. Since e and l_0 are measured in the same length units, E_Y has dimensions of force/area, such as lbf/ft^2, lbf/in^2, or nt/m^2.

Thus, by introducing the quantities stress and strain we have been able to generalize Hooke's law in the case of stretched rods or wires to the statement:

> *Stress is proportional to strain, the proportionality constant depending only on the material and not on the particular body.*

If we know Young's modulus for a given material and the length l_0 and cross-sectional area A of a given rod, the generalized form of Hooke's law given in equation (5) enables us to determine the elongation or deformation e produced by the application of external forces F. The values of Young's modulus for several materials are given in Table I.

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* Named for Thomas Young, English experimental physicist (1773–1829).

TABLE I

TYPICAL ELASTIC CONSTANTS*

Materials	Young's modulus E_Y			
	lbf/in^2	lbf/ft^2	kgf/cm^2	nt/m^2
Aluminum	10×10^6	14×10^8	7.0×10^5	6.9×10^{10}
Brass	13×10^6	19×10^8	9.1×10^5	9.0×10^{10}
Copper	16×10^6	23×10^8	11×10^5	11×10^{10}
Nickel	30×10^6	43×10^8	21×10^5	21×10^{10}
Steel	29×10^6	42×10^8	20×10^5	20×10^{10}
Tungsten	51×10^6	73×10^8	36×10^5	35×10^{10}
Glass	7.8×10^6	11×10^8	5.5×10^5	5.4×10^{10}

Materials	Shear modulus E_S			
	lbf/in^2	lbf/ft^2	kgf/cm^2	nt/m^2
Aluminum	3.8×10^6	5.5×10^8	2.7×10^5	2.6×10^{10}
Brass	5.1×10^6	7.3×10^8	3.6×10^5	3.5×10^{10}
Copper	6.0×10^6	8.6×10^8	4.2×10^5	4.1×10^{10}
Nickel	11×10^6	16×10^8	7.7×10^5	7.6×10^{10}
Steel	11×10^6	16×10^8	7.7×10^5	7.6×10^{10}
Tungsten	21×10^6	30×10^8	15×10^5	14×10^{10}
Glass	3.3×10^6	4.8×10^8	2.3×10^5	2.3×10^{10}

Materials	Bulk modulus E_B			
	lbf/in^2	lbf/ft^2	kgf/cm^2	nt/m^2
Brass	15×10^6	22×10^8	11×10^5	10×10^{10}
Copper	20×10^6	29×10^8	14×10^5	14×10^{10}
Steel	25×10^6	36×10^8	18×10^5	17×10^{10}
Glass	5.2×10^6	7.5×10^8	3.7×10^5	3.6×10^{10}
Ethyl ether	8.7×10^4	1.3×10^7	6.1×10^3	6.0×10^8
Ethyl alcohol	1.6×10^5	2.3×10^7	1.1×10^4	1.1×10^9
Water	3.1×10^5	4.5×10^7	2.2×10^4	2.1×10^9
Mercury	4.0×10^6	5.8×10^8	2.8×10^5	2.8×10^{10}

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* These are typical values to be used in problem work. The actual values for different samples of the same solid depend upon the previous history of the solid; for example, by heat-treatment and mechanical 'working' one can vary the elastic constants of copper by as much as 12 per cent. The elastic constants for steel, brass, and glass vary widely with the exact composition and treatment of these materials.

In order to gain a better understanding of the way in which the internal forces are produced and to show how Young's modulus can be interpreted in terms of the microscopic properties of a solid, it is desirable to show how the relation $S/\sigma = E_Y$ can be derived in terms of a simple atomic model. We may do this by considering the two-dimensional schematic diagrams shown in Fig. 7, which represent a longitudinal section of a rod. The molecules, atoms, or ions constituting the undeformed solid, represented by dots, occupy equilibrium positions as shown in part (a) of this figure; for convenience we shall call these structural units *atoms*. The equilibrium separation between adjacent atoms in the undeformed solid is denoted by λ_0 in the figure. For simplicity, in Fig. 7 we have chosen a cubic arrangement of atoms in the solid; this is the actual arrangement in some, but by no means all, crystalline solids such as metals.

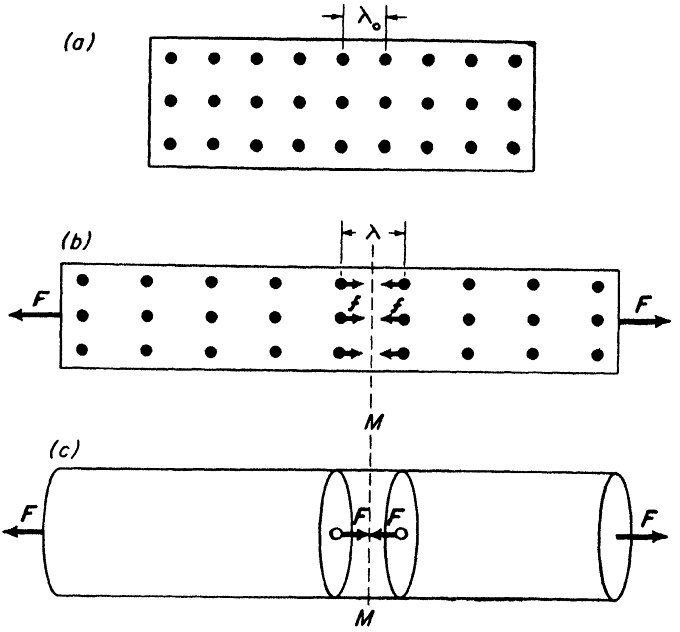

Fig. 7. Model to illustrate longitudinal stress.

When external forces F are applied, the separation between adjacent atoms is increased to λ as shown in part (b) of Fig. 7. This causes interatomic forces f to come into play; these are the forces that tend to restore the atoms to their equilibrium separations.* Considerations of atomic structure show that these interatomic forces obey Hooke's law in the sense that the magnitude of f is proportional to the departure $(\lambda-\lambda_0)$ of the interatomic distance from its equilibrium value. Hence we can write

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* For simplicity in our discussion here, we consider interatomic forces only between adjacent atoms in the deformed body. Actually, there are forces between atoms

$$f=\kappa(\lambda-\lambda_0),$$

where the proportionality constant κ can be regarded as an interatomic force constant. κ of course depends only on the kind of atoms of which the material is composed and not on the shape or size of the body. The material behaves as if each atom were joined to its neighbors by tiny springs of equilibrium length λ_0 and force constant κ.

Now let us consider a cross section of the solid, such as M, that is perpendicular to the line of application of the external forces $\boldsymbol{F}$. The total force acting toward the right on the material to the left of plane M can be expressed as the sum of the interatomic forces acting on the atoms immediately to the left of M; hence, the magnitude of the force acting toward the right on these atoms can be written as

$$F=nAf=nA\kappa(\lambda-\lambda_0),$$

where A is the cross-sectional area and n is the number of atoms per unit area within the section under consideration. Therefore the stress S can be written as

$$S=F/A=n\kappa(\lambda-\lambda_0). \tag{6}$$

Equation (6) enables us to express the stress in terms of quantities involving only atomic spacing and interatomic forces.

Now let us interpret the strain $\sigma=e/l_0$ in terms of our molecular model. The ratio e/l_0 for the whole rod is clearly the same as the ratio $(\lambda-\lambda_0)/\lambda_0$ of the change in distance between atomic layers to the original distance, so that

$$\sigma=e/l=(\lambda-\lambda_0)/\lambda_0. \tag{7}$$

Now let us take the ratio of stress to strain by using the expressions for S and σ given in equations (6) and (7). The resulting expression is

$$\frac{S}{\sigma}=\frac{n\kappa(\lambda-\lambda_0)}{(\lambda-\lambda_0)/\lambda_0}=n\kappa\lambda_0. \tag{8}$$

Thus, we see that the ratio of stress to strain involves only quantities characteristic of the rod *material*, not depending upon the dimensions of the particular rod itself. The quantities n and λ_0 depend upon the

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further apart, but the magnitudes of these forces decrease rapidly as the original positions of the atoms get further apart. Since the magnitudes of *all* the interatomic forces are proportional to $(\lambda-\lambda_0)$, we see that we make no significant error in our attempt to understand the origin of the internal forces by considering only forces between adjacent atoms. The reader can readily elaborate our discussion, to take account of the forces we have neglected, if he so desires.

spacing of the atoms in the solid, and κ is an interatomic force constant. Therefore, the product $n\kappa\lambda_0$ can be replaced by a single constant E_Y which is characteristic of the rod material:

$$S/\sigma = n\kappa\lambda_0 = E_Y. \tag{9}$$

Thus, by using a simple model and applying Hooke's original law to the constituent atoms, we have verified the fact that the constant E_Y in the general form of Hooke's law for a rod in tension, $S/\sigma = E_Y$, involves only the properties of the *material.*

If a stiff rod or cylinder of a material is subjected to a *longitudinal compression,* it will be found that the same value of E_Y will govern the relation between stress and strain, defined as above except for reversed signs. We can consider stress as the *compressive force per unit area* and strain as the *decrease in length per unit length.*

PROBLEMS

1. A weight of 10 kgf is supported by a steel wire 8.0 m in length and 1.0 mm in diameter. What is the resulting elongation? Ans: 5.09 mm.

2. A weight of 8 kgf is supported by a copper wire of length 8 m and diameter 2 mm. What is the resulting elongation?

3. A 2-ton-force load is supported by a steel rod 10 ft long and 1.0 inch in diameter. What is the resulting elongation? Ans: 0.0213 in.

4. An elevator cage weighs 8500 lbf and is supported by two steel cables 60 ft long. If each of the cables has an effective cross-sectional area of 1 in^2, what elongation is produced by the elevator cage?

5. Find the tensional stresses and strains of the supporting members described in Probs. 1 and 3. Ans: 1270 kgf/cm^2, 6.36×10^{-4}; 5090 lbf/in^2, 1.73×10^{-4}.

6. Find the tensional stresses and strains of the supporting members described in Probs. 2 and 4.

7. A brass rod is 1 ft long and has a cross-sectional area of 1.5 in^2. What compressional force must be applied at the ends of this rod in order to produce a decrease of 0.01 inch in the length of the rod? Ans: 16,200 lbf.

8. What compressional stress is involved when the length of a steel rod 10 ft long is decreased by 0.01 in?

9. By considering forces acting on a rod, show that the potential energy per unit volume accompanying elastic distortion is given by the expression $\frac{1}{2}\sigma S$, where σ is the strain and S is the stress. Discuss the units in which S and the potential energy per unit volume should be measured in the MKS and British systems.

10. By considering forces acting on an elastic rod, show that the potential energy per unit volume can be expressed as $\frac{1}{2}E_Y\sigma^2$ or $\frac{1}{2}S^2/E_Y$, where σ is the strain, S is the stress, and E_Y is Young's modulus for the material of which the rod is composed. What units should be used for S, E_Y, and the potential energy per unit volume in the MKS and British systems?

3. VOLUME ELASTICITY: BULK MODULUS

We have seen how the problem of longitudinal extension or compression can be treated satisfactorily by the application of a generalized form of Hooke's law. The generalization that *stress is proportional to strain* can be applied to problems other than those involving longitudinal deformations, although of course different definitions of stress and strain must be used to suit the different types of deformation.

Let us next consider the problem of bulk or volume elasticity. Figure 8(a) shows a body that has volume V_0 when subjected to a uniform external hydrostatic pressure p_0; Fig. 8(b) shows the same body with smaller volume V when the pressure is increased to $p=p_0+\Delta p$. It should be noted that the change in volume takes place without change in the shape of the body; the cube remains a cube. In this process the *stress*, which is *force per unit area*, is simply defined as the pressure increase Δp. The *strain* is defined as *the decrease in volume per unit volume*, or $(V_0-V)/V_0 \equiv \Delta V/V_0$. Substituting these quantities in the general Hooke's law relation, stress $\propto$ strain, and calling the proportionality constant E_B, we have

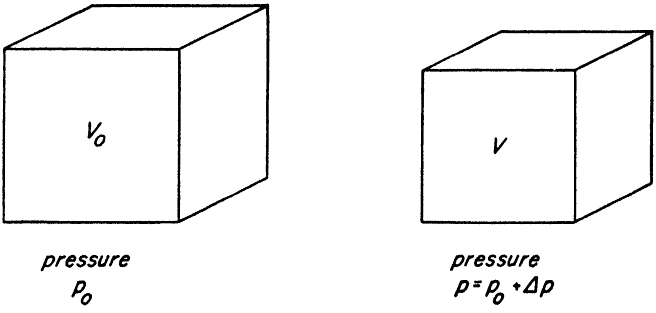


Fig. 8. Volume elasticity: a decrease in volume results from an increase in pressure.

$$\Delta p = E_B \, \Delta V/V_0, \tag{10}$$

where E_B is called the *bulk modulus* of the material of which the body is composed.

The bulk modulus, which has dimensions of pressure or force/area, depends only upon the *kind of material* of which the body is composed. Values of the bulk modulus for several materials are listed in Table I.

Equation (10) enables us to compute the change in the volume of a body when the pressure changes by a given amount Δp, provided we know the initial volume of the body and the bulk modulus of the material of which the body is composed.

The microscopic processes occurring when the volume of a solid is changed by an increase in pressure can be visualized from the two-dimensional atomic-arrangement diagrams shown in Fig. 9. Part (a) of this figure indicates the equilibrium arrangement of the atoms in a solid when the external pressure is very small or zero (vacuum). When the pressure is increased, the external forces F act upon the surfaces of the solid as shown in part (b) of the figure. As a result of these external forces, the atoms are brought closer to one another. The resulting mutual repulsions of neighboring closely spaced atoms accounts for the development of the stress accompanying deformation. If one considers the forces of interaction between the parts of a solid on the two sides of reference planes such as those indicated by the dotted lines M and M' in the figure, it is easy to see how the stress is produced. Material to the left of line M exerts a force toward the right on material to the right of M; the reaction force on the material to the left of M is directed toward the left. Similar forces are exerted across the plane denoted by M'. In an actual solid, the deformation is, of course, three-dimensional; we may construct reference planes corresponding to M and M' with any orientation we wish.

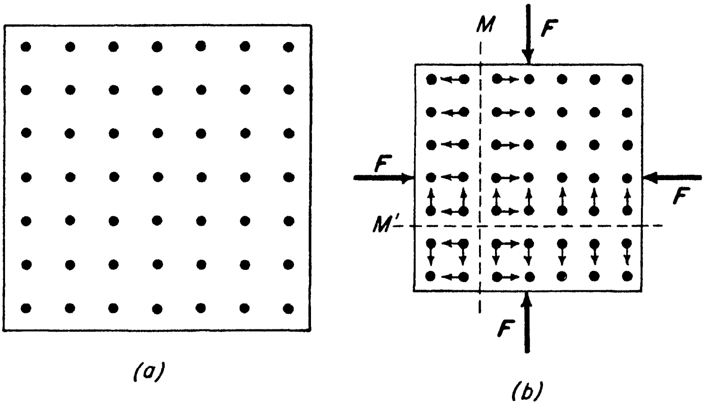


Fig. 9. Model to illustrate volume elasticity.

Since no question of rigidity is involved in volume elasticity, equation (10) applies to liquids as well as to solids. The volume elasticity of gases will be considered in Chap. 17.

PROBLEMS

1. What increase in pressure is required to change the volume of a cubic meter of water by 0.01 per cent? Ans: 2.1×10^5 nt/m^2.

2. What increase in pressure is required to change the volume of a cubic meter of ether by 0.1 per cent?

3. Show that so long as the volume change in (10) is small compared to the original volume, we can write

$$\Delta p = E_B \, \Delta\rho/\rho_0,$$

where ρ_0 is the original density and $\Delta\rho$ the increase in density.

4. At what depth below the surface of a lake is the density of water 0.1 per cent greater than that of surface water at the same temperature?

5. Find the ratio of density of water at a point 500 ft below the surface of a lake to the density of water at the surface. (Neglect temperature differences.)
Ans: 1.000699.

6. What pressure would be required to change the density of brass by 0.1 per cent? the density of glass by 0.1 per cent? the density of ether by 0.1 per cent?

7. What pressure would be required to change the density of steel by 0.1 per cent? the density of copper by 0.1 per cent? the density of water by 0.1 per cent?
Ans: 1.7×10^8 nt/m^2; 1.4×10^8 nt/m^2; 2.1×10^6 nt/m^2.

4. ELASTICITY OF SHAPE: SHEAR MODULUS

The third type of elasticity we shall consider is one in which the *shape* of a body is changed without change in the *volume* of the body. The type of deformation involved is called a *shear*, which can be understood by a consideration of Fig. 10. Part (a) shows a book lying on a table; the book may easily be deformed in the manner indicated in part (b) of the figure. It will be noted that the shape of the book has been altered without change in the volume of the book; consequently, the deformation is a *shear*. A comparison of parts (a) and (b) of Fig. 10 shows that a square element of surface area becomes a rhombus when shear occurs; similarly, a rectangular surface element becomes a parallelogram. The angle ϕ in the figure is called the *angle of shear*.

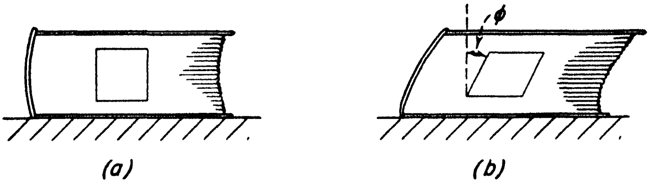


Fig. 10. Shear.

A book is an anisotropic body that is relatively inelastic to deformation of the type indicated in Fig. 10. In order to understand how a shear is produced in an isotropic body, let us consider the volume element shown in Fig. 11; this element of volume has been taken from the interior of a body experiencing a shear. The forces F_1 are exerted on the element in directions tangential to the faces of area A_1, and the forces F_2 are exerted in directions tangential to the faces of area A_2; these forces are exerted by the material outside the volume element on the material inside. The resultant force and the resultant torque on the material in the volume are zero; this must be so, since the material is not experiencing linear or angular acceleration. When the surface forces F_1 and F_2 act in the manner indicated, the face of area A_3, which was a rectangle in the undeformed body, becomes a parallelogram. The *shearing strain* is defined as the *shear angle* ϕ shown in the figure, in radians. Over the elastic range, ϕ is always small enough to be set equal to the ratio aa'/l.

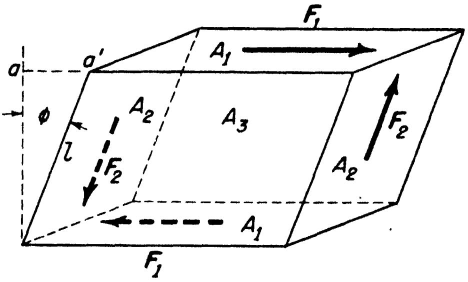


Fig. 11. Shearing stress and strain within a material medium.

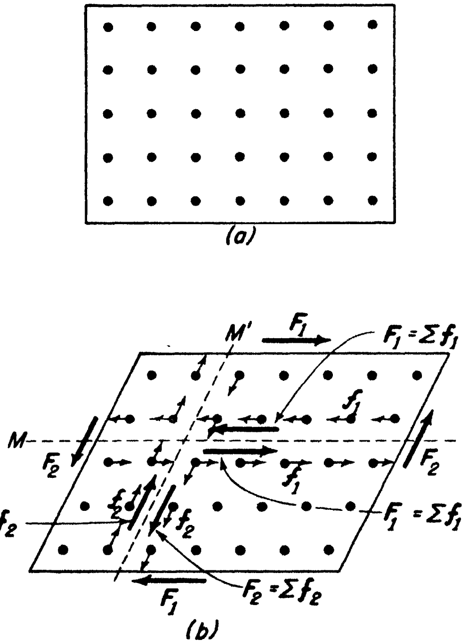
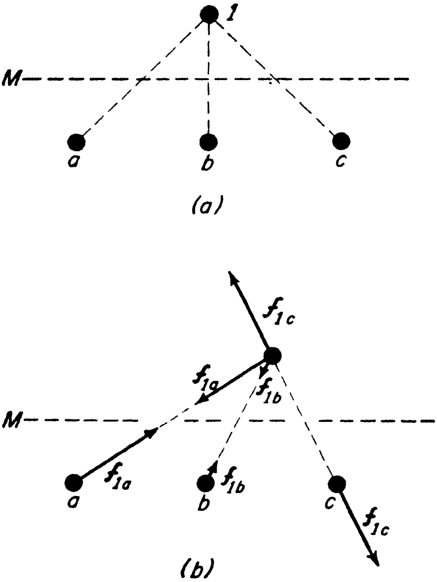


Fig. 12. Model to illustrate shear.

The *shearing stress* S is defined as the *tangential force per unit area* and is therefore given by the ratio F_1/A_1 or F_2/A_2. From the condition for rotational equilibrium, one can readily show that these two ratios are always equal. Applying the general form of Hooke's law, *shearing* *stress* ∝ *shearing strain*, and calling the proportionality constant E_S, we have

$$S = F/A = E_S\phi. \tag{11}$$

The coefficient E_S, which has dimensions of force/area, is called the *shear modulus*, or *modulus of rigidity*. The value of the shear modulus depends only upon the type of material of which the deformed body is composed. Values of the shear modulus for several materials are given in Table I.



(a)

(b)

Fig. 13.

The interatomic mechanism in shearing stresses can be visualized from the two-dimensional atomic-arrangement diagrams shown in Fig. 12. Part (a) of this figure shows the atoms at their normal equilibrium positions before deformation occurs. The sketch in part (b) shows the atomic arrangement after deformation of the body is produced by the tangential surface forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$. As in the case of longitudinal and volume strains, we may set up reference planes indicated by M and M' in our two-dimensional sketch. The material below M exerts a force on the material above M; this force is equal in magnitude to F_1 and is directed toward the left. Similarly, the material above M exerts a force directed toward the right on the material below M. Similar forces are developed on the material on opposite sides of M', but these forces are equal in magnitude to F_2. The forces on the individual atoms adjacent to the reference planes are indicated by the small interatomic forces $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$. If the sums of the interatomic forces are taken along either side of the reference planes, we may write $\Sigma \boldsymbol{f}_1 = \boldsymbol{F}_1$, and $\Sigma \boldsymbol{f}_2 = \boldsymbol{F}_2$.

The reason the interatomic forces $\boldsymbol{f}_1$ act tangentially to the reference plane M can be understood qualitatively by considering the forces exerted on a reference atom above the reference plane by its three nearest neighbors below the plane. Let the reference atom be denoted by 1 and its three nearest neighbors below the reference plane be denoted by *a*, *b*, and *c*. The arrangement of the four atoms before deformation is shown in Fig. 13(a); the arrangement after deformation is shown in Fig. 13(b).

Since the distance between atoms 1 and *a* is increased by the deformation, attractive interatomic forces denoted by f_{1a} act as indicated to return the atoms to their original separation. For similar reasons, forces f_{1b} of attraction act between atoms 1 and *b* after deformation occurs. After deformation, the separation between atoms 1 and *c* is less than the original separation; consequently, repulsive forces f_{1c} act on the two atoms in the manner indicated. Now let us consider the three forces f_{1a}, f_{1b}, and f_{1c} which act on the reference atom 1. Since all these forces have components directed toward the left, it is not difficult to convince ourselves that the resultant f_1 of all forces *exerted on atom* 1 *by all atoms below plane M* is directed toward the left as indicated in Fig. 12; this statement can be proved for the case of Fig. 13 if the interatomic forces are assumed to obey Hooke's law and the deformation is small.

Now let us see how the shear modulus may be determined experimentally by measurements made on deformed macroscopic bodies. Let us consider a thin-walled hollow cylinder that is twisted about its geometrical axis by clamping one end of the cylinder in a fixed position and applying a torque to the free end; this operation results in a uniform shear. A thin-walled cylinder before deformation is shown in Fig. 14(a); a small 'square' surface element is shown in black and a long 'rectangular' surface element is shown by the dashed lines passing along the cylinder surface from end to end. When the cylinder is twisted as shown in part (b) of the figure, the small 'square' becomes a 'rhombus' and the 'rectangle' becomes a 'parallelogram' of the indicated shape; therefore, the material experiences a pure shear. Now let us find expressions for strain and stress from the labeled diagram in Fig. 14(c). The strain can be expressed as the shear angle ϕ. However, the shear angle ϕ is not as easy to measure as θ, the angle of rotation of the twisted end. In the elastic range ϕ is small, and we can write

$$\text{shearing strain} = \phi = r\theta/l, \tag{12}$$

where l is the length of the cylinder. The torque L must appear across any section such as MM, since the left end of the bar is in rotational equilibrium. If the stress is S at section MM, it acts across a total cross-sectional area $2\pi r\,\Delta r$, so that it exerts torque $L = (2\pi r\,\Delta r)S\cdot r$. Hence,

$$\text{shearing stress} = S = L/2\pi r^2\Delta r. \tag{13}$$

The ratio of shearing stress (13) to shearing strain (12) is by (11) the shear modulus. Hence we have

$$E_S = \frac{L/2\pi r^2\Delta r}{r\theta/l}, \quad \text{or} \quad E_S = \frac{Ll}{2\pi r^3\theta\,\Delta r}. \tag{14}$$

Equation (14) enables us to determine the shear modulus E_S for a material from measurements of the rotation θ produced by applying a torque L to a thin-walled cylinder of the material. If the modulus E_S is known, equation (14) can be used to predict the rotation θ produced by the application of a torque L to the end of a thin-walled tube. A similar equation can be derived for a twisted *solid* rod or wire of radius r and length l. This equation is

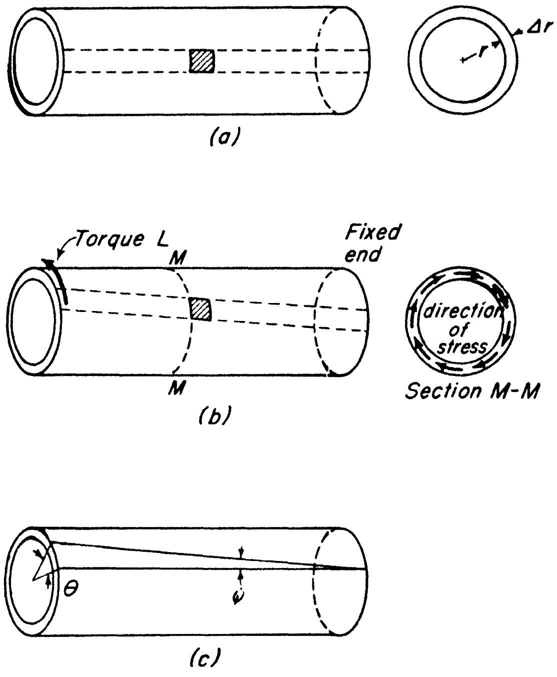


Fig. 14. Shear resulting from twist of a thin-walled cylinder, of length l, radius r, and wall-thickness Δr.

$$E_S = 2Ll/\pi r^4\theta, \tag{15}$$

where θ is the twist angle for torque L.* In making actual measurements, the torque $L = W{\cdot}R$ is frequently applied by hanging weights $\boldsymbol{W}$ on a wheel of radius R clamped to the free end of the section of rod or tube being studied, in the manner indicated in Fig. 15.

The ratio L/θ, the torque required per radian of twist, is called the

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* This relation is derived by dividing the rod up into cylindrical shells of radius ρ and thickness $d\rho$. According to (14), the torque dL required to twist such an element through angle θ is

$$dL = (2\pi\theta E_S/l)\rho^3\,d\rho.$$

Integration of this expression from $\rho = 0$ to r gives (15).

torsion constant of the tube or rod. This constant can be computed from (14) or (15).

The spring constant of a *spring* can be computed once we know the torsion constant of the wire from which it is made. We do not give the details here, but merely point out that when a spring elongates, the wire of which it is formed does not stretch, it merely *twists*. Any small section of the wire behaves exactly like a piece of wire in torsion, and hence the torsion constant governs its elastic behavior.

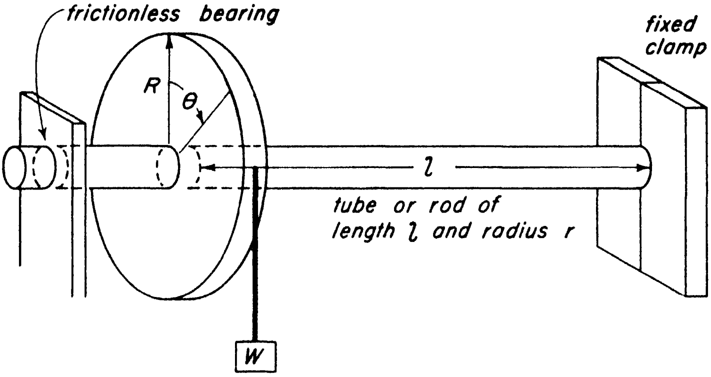

Fig. 15. Measurement of torsion constant.

PROBLEMS

1. A copper tube 2.0 cm in radius is 4.1 m in length and has a wall thickness of 1 mm. One end of this tube is firmly clamped and a torque tending to twist the tube is applied to the other end. What is the torsion constant for this tube? What angular displacement is produced by a torque of 32 nt·m?

Ans: 503 nt·m/rad; 0.0636 rad.

2. Find the torsion constants of steel and brass tubes having the same dimensions as the copper tube described in Prob. 1.

3. Find the torsion constant of a solid steel wire ⅛ inch in diameter and 4 ft long.

Ans: 176 lbf·in/rad.

4. What torque is required to twist the wire of Prob. 3 through ¼ rev?

5. When equal and opposite forces of magnitude F_1 are applied to the top and bottom faces of a parallelopiped as in Fig. 11, show that to maintain rotational equilibrium forces must be applied to the right and left faces of such magnitude F_2 that the stresses F_2/A_2 and F_1/A_1 are equal. Show that this statement is true for arbitrary magnitude of the angle ϕ, although only small angles are involved in elastic shear.

5. RELATIONS BETWEEN ELASTIC CONSTANTS

We have thus far introduced three elastic moduli: Young's modulus, the bulk modulus, and the shear modulus. Now, of all the elastic moduli that can be defined for an isotropic elastic material, only two are independent. All elastic moduli can be expressed in terms of any two of them. For example, if we choose the bulk modulus E_B and shear modulus E_S as the independent ones, it can be shown that Young's modulus E_Y can be expressed as

$$E_Y = 9E_BE_S/(3E_B+E_S). \tag{16}$$

That some such relation should exist seems reasonable if we critically re-examine the way in which we defined Young's modulus. We concentrated our attention on the change in a single dimension, the length of a wire. Actually, the stretching of a wire involves a change in both shape and volume of the wire, and consequently both volume elasticity and shape elasticity are involved. As the length increases, the diameter d decreases; this type of lateral contraction is easily observed when a rubber band is stretched. Hence, if we wish to describe completely the changes that occur when a wire is stretched, we must use two elastic constants—for example, either the bulk modulus E_B and the shear modulus E_S or Young's modulus E_Y and an additional elastic constant. The additional constant frequently used with Young's modulus is called *Poisson's ratio*, which we shall denote by ρ. *Poisson's ratio* ρ *is defined as the ratio of the relative lateral contraction* $\Delta d/d$ *to the relative longitudinal extension* $\Delta l/l$. It is dimensionless. Like Young's modulus, Poisson's ratio ρ can be expressed in terms of the bulk modulus E_B and the shear modulus E_S. The relation is the following:

$$\rho = \frac{3E_B - 2E_S}{2(3E_B+E_S)}. \tag{17}$$

For most metals, Poisson's ratio has a value in the neighborhood of 0.3.

The problem of the experimental determination of the elastic moduli is easiest for Young's modulus and the shear modulus, since the measurements involved consist of determining the elongation produced by a known longitudinal force acting on a wire of known length and cross-sectional area for Young's modulus and determining the rotation produced by a known torque acting on a rod of known dimensions for the shear modulus. The direct determination of the bulk modulus is considerably more difficult. The values for the bulk modulus E_B and for Poisson's ratio ρ may be determined from E_Y and E_S by means of the following relations:

$$E_B = \frac{E_SE_Y}{3(3E_S - E_Y)}; \qquad \rho = \frac{E_Y}{2E_S} - 1 \tag{18}$$

Proofs of these relations, which hold rigorously for isotropic solids, will be found in books on the mathematical theory of elasticity.

These relations do not hold for single crystals of material; single crystals have different elastic properties in different directions and require more than two independent constants to describe their elastic behavior. The relations do, however, hold for crystalline materials that are composed of a large number of small crystals oriented at random, as in the usual case for the solid metals of engineering practice. They also hold for amorphous materials such as glass. They do not hold for a material with a 'grain,' such as wood. Wood is not isotropic but has different elastic properties in different directions.

The only elastic modulus applicable to a liquid or gas is the bulk modulus, since the only type of stress that a fluid in equilibrium will withstand is a hydrostatic pressure.

PROBLEMS

1. From the values given for bulk modulus and shear modulus in Table I, compute Young's modulus in lbf/in^2 for brass, copper, and steel, and compare with the tabular values.

2. From the values given for Young's modulus and shear modulus in Table I, compute the bulk modulus in lbf/in^2 for brass, copper, and steel. Why are the discrepancies with the tabular values much greater than in the results of Prob. 1?

3. Compute Poisson's ratio for aluminum, brass, and tungsten.
Ans: 0.33; 0.27; 0.21.

4. Compute Poisson's ratio for copper, steel, and glass.

5. If a wire has Poisson's ratio 0.3, and is subjected to longitudinal strain (change in length per unit length) σ, where $\sigma \ll 1$, show that the change in cross-sectional area per unit cross-sectional area is $-0.6\,\sigma$, and that the change in volume per unit volume is $-0.3\,\sigma$.

6. ELASTIC LIMIT AND ULTIMATE STRENGTH OF MATERIALS

Now that we have introduced the quantities stress and strain, let us return to the subject of the behavior of bodies that are subjected to very large distorting forces. In Sec. 1 of this chapter we indicated that if a wire is subjected to gradually increasing stretching forces the wire at first experiences elastic elongation proportional to the applied force; when the elastic limit is reached, the wire becomes permanently distorted; if the stretching force is further increased, the wire eventually breaks. Now that we have generalized the distorting force as *stress* and the elongation as *strain*, it is possible for us to interpret elastic limit and breaking strength as properties of *materials* rather than as properties of particular bodies. After determining these properties of a given *material*, we can calculate the particular properties of various bodies composed of this material.

As an example, let us return to the simple case of the elongation e of a wire of cross-sectional area A and length l_0 produced by longitudinal forces F applied to the ends of the wire. A typical plot of data on stress required to produce various strains for a steel wire is shown in the graph of Fig. 16. The curve between A and B is a straight line, showing that stress is proportional to strain for small strains, as would be expected from Hooke's law. The elastic limit is reached at B; after stresses greater than that for point B have been attained, the wire no longer regains its original length when the distorting forces are removed. The value of the stress at point B is called the *elastic limit* for the material.* As the stress is increased beyond the elastic limit, the *yield point* (between C and D on the curve) is reached. At the yield point, a large increase in strain occurs without appreciable increase in stress; in fact, when the yield point is reached, the wire appears to flow like an extremely viscous liquid. As the strain is further increased, the stress changes in the manner indicated in the plot shown in Fig. 16 until rupture occurs. The magnitude of the stress at rupture is called the *breaking strength* of the material. The *ultimate strength* is the highest point of the curve of Fig. 16, the maximum stress that the material can withstand. The ultimate strength is frequently slightly higher than the breaking strength, and may be much above the yield point, as in steel. The yield point is usually close to the elastic limit.

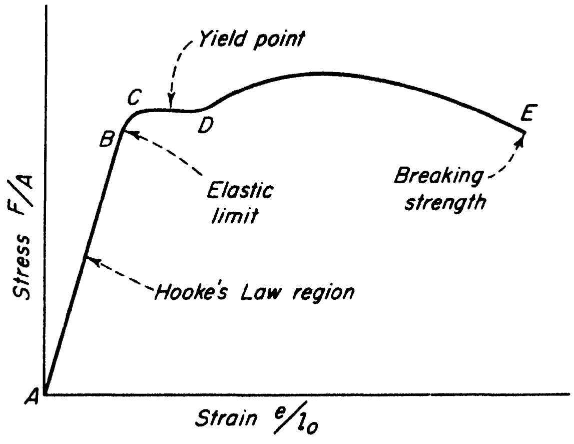

Fig. 16. Typical stress-strain diagram for a material in tension.

Table II shows the elastic limits and the ultimate strengths of several materials. These properties vary widely for a given material. The reason for this variation is that elastic limits and ultimate strengths are determined to a considerable extent by the previous history of a given sample. For example, a work-hardened sample and an annealed sample have very different properties in the case of copper and steel.

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* The point B, at which the stress-strain curve ceases to be linear, is properly called the *proportional limit.* For most metals the proportional limit agrees with the elastic limit, but this is not true for such materials as wood and rubber (cf. Fig. 5).

TABLE II

TYPICAL ELASTIC LIMITS AND ULTIMATE STRENGTHS FOR SEVERAL MATERIALS IN TENSION*

Material	Elastic limit			
	lbf/in²	lbf/ft²	kgf/cm²	nt/m²
Aluminum	19×10^3	27×10^5	13×10^2	13×10^7
Brass	55×10^3	79×10^5	39×10^2	38×10^7
Copper	22×10^3	32×10^5	15×10^2	15×10^7
Steel, medium	36×10^3	52×10^5	25×10^2	25×10^7
Steel, spring	140×10^3	200×10^5	98×10^2	97×10^7

Material	Ultimate tensile strength			
	lbf/in²	lbf/ft²	kgf/cm²	nt/m²
Aluminum	21×10^3	30×10^5	15×10^2	14×10^7
Brass	67×10^3	96×10^5	47×10^2	46×10^7
Copper	49×10^3	71×10^5	34×10^2	34×10^7
Steel, medium	72×10^3	104×10^5	51×10^2	50×10^7
Steel, spring	170×10^3	240×10^5	120×10^2	120×10^7

* These values are to be used in problem work. Exact values of these constants depend upon the *exact* composition and previous history of the test sample.

PROBLEMS

1. What is the largest weight that can be supported by a medium steel wire 2 mm in diameter? by a copper wire 2 mm in diameter? Ans: 160 kgf; 110 kgf.

2. What is the greatest weight that can be supported by a brass wire 2 mm in diameter? by an aluminum rod 2 mm in diameter?

3. What is the maximum tension that can be supported by a medium steel structural member with a cross-sectional area of 4 in²? Ans: 288,000 lbf.

4. What is the maximum tension that can be supported by a brass structural member with a cross-sectional area of 2 in²?

CHAPTER 10

IMPACTS AND CONSERVATION OF MOMENTUM

In this chapter we shall treat briefly certain problems involving collisions between bodies. In problems of this kind it is difficult if not impossible to determine the instantaneous values of the forces involved or even the exact time intervals during which the forces act. Therefore, it is impossible to write numerical equations for Newton's second law in the form we have used thus far. However, even though we cannot give definite values for force or acceleration, it is still possible to apply Newton's laws to these problems; we can do this conveniently by introducing the quantities *momentum* and *impulse.* We shall indicate the methods by which these quantities are used in the solution of impact and collision problems.

The principles of conservation of linear and angular momentum will be discussed and applied to a few simple problems. We shall also state and prove certain important theorems regarding the motion of *systems* of particles or bodies. Some of these theorems have been stated earlier without proof.

Before plunging into formal definitions and details, it might be well to discuss the general ideas involved in the definitions of momentum and impulse in a simple case. Consider a golf ball being driven from a tee. A large force acts on the ball during a short interval of time Δt, accelerating the ball from rest to a final velocity $\boldsymbol{v}$. The magnitude of the force and the time interval Δt are difficult to determine, but the quantity $\bar{\boldsymbol{F}}\,\Delta t$ has a value that is readily computed. From Newton's second law, $\boldsymbol{F}=m\boldsymbol{a}$, we infer that $\bar{\boldsymbol{F}}=m\bar{\boldsymbol{a}}$, and hence that $\bar{\boldsymbol{F}}=m\boldsymbol{v}/\Delta t$, from the definition of average acceleration. Therefore,

$$\bar{\boldsymbol{F}}\,\Delta t=m\boldsymbol{v}.$$

This type of equation is so useful in problems involving impacts and collisions that the quantities occurring are given special names. The product $m\boldsymbol{v}$ is called the *momentum* of the golf ball. The product $\bar{\boldsymbol{F}}\,\Delta t$ is called the *impulse* required to change the momentum of the ball from zero to its final value $m\boldsymbol{v}$. Impulse and momentum, rather than force and time, are the quantities readily determinable in collision processes.

1. LINEAR MOMENTUM

Certain properties of moving bodies depend directly on the product of the mass of the body and its velocity. For example, it is readily shown that the constant force required to bring a moving body to rest within a given time interval is proportional to the product, *mass times velocity.* Galileo and Newton found this product so useful in the study of motion that they give it the special name *momentum* or 'quantity of motion.' We still use the term *momentum* or *linear momentum* for this quantity and find it very useful in solving many types of problems.

> The linear momentum $\boldsymbol{p}$ of a body is defined as the product of its mass m and the linear velocity $\boldsymbol{v}$ of the center of gravity of the body: $\boldsymbol{p}=m\boldsymbol{v}$.

Since mass is a scalar quantity and velocity is a vector quantity, *momentum is a vector quantity.* It is measured in units that are the product of a mass unit by a velocity unit, for example, in kg·m/sec, slug·ft/sec, or lb·ft/sec. From the definition given above, it is evident that the momentum of a body changes whenever either the mass or the velocity of the body changes. As we have already indicated, the mass of a given body may be regarded as constant except for bodies having enormous velocities, comparable with the speed of light.* Hence, in all cases of interest to us at present, we may attribute any change in the momentum of a body to a change in its velocity.

Now let us consider the processes involved in momentum changes. Let us suppose that a particle experiences a change of momentum $\Delta\boldsymbol{p}$ in the time interval Δt. The average time rate of change of momentum is therefore $\Delta\boldsymbol{p}/\Delta t$. In the limit as $\Delta t \rightarrow 0$, we obtain $d\boldsymbol{p}/dt$ as the instantaneous rate of change of momentum; this may be written

$$d\boldsymbol{p}/dt=m\,d\boldsymbol{v}/dt=m\boldsymbol{a},$$

since we can regard the mass as constant. But by Newton's second law, $\boldsymbol{F}=m\boldsymbol{a}$, and therefore we may write

$$\boldsymbol{F}=d\boldsymbol{p}/dt. \tag{1}$$

This equation is actually a restatement of Newton's second law in terms of momentum:

The time rate of change of the linear momentum of a body is equal to the resultant external force acting on the body and is in the direction of the resultant force.

From equation (1), we may write the change in momentum in time dt as

$$d\boldsymbol{p}=\boldsymbol{F}\,dt. \tag{2}$$

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* Speed of light = 186,000 mi/sec = 3×10^8 m/sec.

This equation will be useful in our study of impacts. It is also useful to note that if $\boldsymbol{F}$ is the *average* force acting over a time interval Δt, it will produce change in momentum $\Delta \boldsymbol{p}$ given by

$$\Delta \boldsymbol{p} = \bar{\boldsymbol{F}}\,\Delta t. \tag{3}$$

In equations (1), (2), and (3), the quantities p, F, and t must be measured in kg·m/sec, nt, and sec, or in slug·ft/sec, lbf, and sec.

PROBLEMS

1. An automobile with a mass of 1500 kg has a velocity of 20 m/sec eastward. What is the momentum of the automobile? Ans: 30,000 kg·m/sec eastward.

2. An automobile that weighs 3220 lbf has a velocity of 60 mi/hr northward. What is the momentum of this automobile in slug·ft/sec?

3. A certain 11-kg body is subjected to a set of external forces. If the resultant force acting on the body is 55 nt downward, what is the time rate of change of the momentum of the body? Ans: 55 kg·m/sec² downward.

4. If a constant resultant force of 25 lbf eastward acts on a certain body, what is the time rate of change of the momentum of the body?

5. If the body in Prob. 3 is initially at rest, what is its momentum after the resultant force of 55 nt has acted for 10 sec? What is its velocity at the end of this 10-sec interval? Ans: 550 kg·m/sec; 50 m/sec downward.

6. If the body in Prob. 4 is initially at rest, what is its momentum at the end of 8 sec? What is its velocity after the resultant force has been applied for 8 sec if its mass is 8.05 lb?

7. A billiard ball of mass 200 g is thrown against a steel plate. The ball strikes the plate normally and rebounds. If the magnitude of the velocity before impact was 20 m/sec and the magnitude of the velocity after impact was 18 m/sec, what was the total change in the momentum of the ball?
Ans: 7.6 kg·m/sec, away from the plate.

8. A rubber ball weighing 4 ozf is dropped from rest at the top of a building to the sidewalk. Its downward velocity just as it strikes the sidewalk is 50 ft/sec; the ball rebounds vertically from the sidewalk with a velocity of 30 ft/sec. What is the change in the momentum of the ball during its fall from the top of the building to the sidewalk? What is the change in the momentum of the ball during its collision with the sidewalk?

2. IMPULSE

Let us now consider a process of the type occurring when a golf ball* is struck with a golf club. The club is actually in contact with the ball only for a very short interval of time. During this short time interval a very large force is exerted on the golf ball; this force varies with time in a complex manner which cannot in general be determined. During the time the force acts, the golf ball and club undergo considerable deformation; the details of the process cannot be described completely even from the data obtained by modern high-speed photographic methods. Forces of the kind exerted on the golf ball are called *impulsive forces.*

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* We are concerning ourselves here only with changes in the linear momentum of the golf ball and are therefore neglecting rotational effects. In a sense, we are treating the golf ball as a 'particle.'

In order to see how the actions of such forces can most easily be treated, let us consider the graph shown in Fig. 1(a). Let us assume that this graph shows the magnitude of the actual force exerted on the golf ball as a function of time. By equation (3), we see that the magnitude of the change in momentum in the time interval $\Delta t = t_1 - t_0$ during which the force acts is given by

$$\Delta p = \bar{F}\,\Delta t, \tag{4}$$

where $\bar{F}$ is the average magnitude of the force during this time interval. By definition, the time-average force $\bar{F}$ is such that $\bar{F}\,\Delta t$ is the area under the force curve. The value of $\bar{F}$ is shown in Fig. 1(b), where the rectangle in Fig. 1(b) is drawn so as to have the same area as the cross-hatched area under the curve of Fig. 1(a).

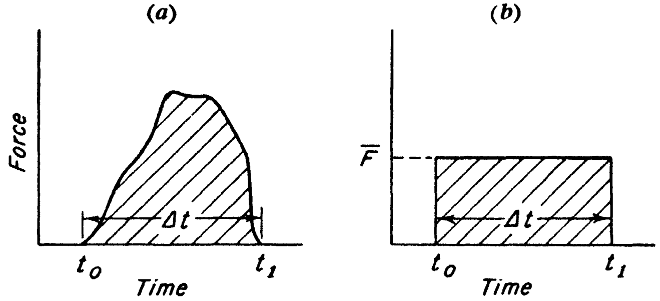

Fig. 1. An impulsive force.

Hence the impulsive force causes a change in momentum of magnitude equal to the area under the force curve of Fig. 1(a). This area has the physical dimensions of force multiplied by time; it is called the *impulse.*

> The impulse associated with an impulsive force is defined as the area under the force curve plotted as a function of time. This is the same as the time average of the force multiplied by the time interval during which the force acts.

Equation (3) states that *the change in momentum of a body is equal to the impulse of the force acting on the body.* The impulse is a vector quantity having the direction of the average force $\bar{\mathbf{F}}$. It is measured in units of force × time. These are seen to be the same as the units in which momentum is measured, in accordance with the identities

$$1\ \text{nt·sec} \equiv 1\ \text{kg·m/sec}, \quad \text{and} \quad 1\ \text{lbf·sec} \equiv 1\ \text{slug·ft/sec},$$

which follow from equations (5), p. 128, and (11), p. 134.

In general, in impact problems the total change in momentum is a quantity that can be measured, whereas data of the type plotted in Fig. 1(a) are generally unknown and cannot be observed. In other words, *experimental measurements can in general give us information on change of momentum of a particle and consequently information on impulse but not on instantaneous values of the force;* as a result, we are likewise unable to determine instantaneous values of acceleration.

PROBLEMS

1. A croquet ball with a mass of 1 kg, initially at rest, is struck with a mallet and acquires a velocity of 4 m/sec. What is the magnitude of the change in the momentum of the ball? What impulse is associated with the force exerted by the mallet? If the mallet is in contact with the ball for 0.1 sec, what is the magnitude of the average force exerted on the ball? Ans: 4 kg·m/sec; 4 nt·sec; 40 nt.

2. A 2-lb ball, initially at rest, acquires a velocity of magnitude 20 ft/sec when it is struck by a club. What is the magnitude of the change in the momentum of the ball? What impulse is associated with the force exerted by the club? If the club makes contact with the ball for 0.02 sec, what is the magnitude of the average force exerted on the ball?

3. A 500-g steel ball is dropped vertically on a concrete sidewalk. It strikes the concrete normally at a velocity of 30 m/sec and rebounds with an initial upward velocity of 20 m/sec. What is the impulse of the forces active on the ball during this collision? If the ball was in contact with the sidewalk for 0.01 sec, what average force did it exert against the walk? Ans: 25 nt·sec upward; 2500 nt downward.

4. A 16.1-lb lead ball is dropped from a height of 81 ft into a mud bank, where it comes to rest in 0.5 sec. What average force was exerted on the ball in bringing it to rest?

3. CONSERVATION OF LINEAR MOMENTUM

We have thus far introduced the concept of linear momentum $\boldsymbol{p} = m\boldsymbol{v}$ and have shown, by stating Newton's second law in terms of momentum, that the change in momentum of a particle is equal to the impulse of the resultant external force acting on the particle. Now let us consider a collision between two particles.

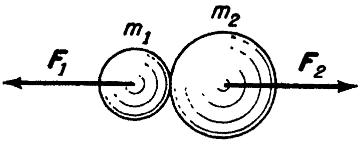

Fig. 2. Interaction between two particles in collision.

Consider, for example, the two particles of masses m_1 and m_2 shown in Fig. 2. Let us assume that during a collision these two particles exert forces on each other but that no other forces from outside this two-body system act on either particle. $\boldsymbol{F}_1$ is the force exerted on particle 1 by particle 2 and $\boldsymbol{F}_2$ is the force exerted on particle 2 by particle 1. We may now write the change in momentum that particle 1 suffers during the collision as

$$\Delta \boldsymbol{p}_1 = \bar{\boldsymbol{F}}_1 \, \Delta t, \tag{5}$$

where Δt is the time F_1 acts on body 1. Similarly, we may write the change in momentum of particle 2 as

$$\Delta \boldsymbol{p}_2 = \bar{\boldsymbol{F}}_2\,\Delta t,$$

where Δt has the same value as in (5), since $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ act simultaneously. Moreover, by Newton's third law, at each instant

$$\boldsymbol{F}_2 = -\boldsymbol{F}_1, \quad \text{hence} \quad \bar{\boldsymbol{F}}_2 = -\bar{\boldsymbol{F}}_1.$$

Therefore

$$\Delta \boldsymbol{p}_2 = -\bar{\boldsymbol{F}}_1\,\Delta t. \tag{6}$$

Now let us consider the two particles as constituting a 'system' and define the *total momentum of the system as*

$$\boldsymbol{P} = \boldsymbol{p}_1 + \boldsymbol{p}_2.$$

The total change $\Delta\boldsymbol{P}$ occurring in the *momentum of the system* as a result of *internal* forces acting within the system, that is, the forces exerted by the two particles on each other, is given by

$$\Delta\boldsymbol{P} = \Delta\boldsymbol{p}_1 + \Delta\boldsymbol{p}_2,$$

or, by (5) and (6),

$$\Delta\boldsymbol{P} = \bar{\boldsymbol{F}}_1\,\Delta t - \bar{\boldsymbol{F}}_1\,\Delta t = 0.$$

This equation expresses the important result that the total linear momentum of the system is not changed by the action of *internal* forces, a result known as the principle of conservation of momentum.

Although we have demonstrated this principle only for a system consisting of two particles, the principle is perfectly general and can be derived for any system by an extension of the method we have already used for two particles. For example, consider the four particles shown in Fig. 3. At a given instant, we assume that the particles are acting on each other with any system of forces whatsoever, arising from collisions or any type of action at a distance, provided only that Newton's third law is satisfied. *No other* forces act on the particles.

At a particular instant, particle 1 experiences force $\boldsymbol{F}_{21}$ from particle 2, $\boldsymbol{F}_{31}$ from particle 3, and $\boldsymbol{F}_{41}$ from particle 4. At this instant the rate of change of momentum of particle 1 is, by (1),

$$d\boldsymbol{p}_1/dt = \boldsymbol{F}_{21} + \boldsymbol{F}_{31} + \boldsymbol{F}_{41}. \tag{7a}$$

Employing a similar notation, we may write the rate of change of momentum of the other particles as

$$d\boldsymbol{p}_2/dt = \boldsymbol{F}_{12} + \boldsymbol{F}_{32} + \boldsymbol{F}_{42}, \tag{7b}$$

$$d\boldsymbol{p}_3/dt = \boldsymbol{F}_{13} + \boldsymbol{F}_{23} + \boldsymbol{F}_{43}, \tag{7c}$$

$$d\boldsymbol{p}_4/dt = \boldsymbol{F}_{14} + \boldsymbol{F}_{24} + \boldsymbol{F}_{34}. \tag{7d}$$

The total momentum of the system is defined as

$$\boldsymbol{P} = \boldsymbol{p}_1 + \boldsymbol{p}_2 + \boldsymbol{p}_3 + \boldsymbol{p}_4.$$

The rate of change of total momentum $d\boldsymbol{P}/dt$ is the vector sum of (7a), (7b), (7c), and (7d). Because of Newton's third law, $\boldsymbol{F}_{12} = -\boldsymbol{F}_{21}$, $\boldsymbol{F}_{13} = -\boldsymbol{F}_{31}$, and so on, and we find that this vector sum vanishes:

$$d\boldsymbol{P}/dt = 0. \tag{8}$$

The total momentum of the dynamical system does not change with time, and hence $\boldsymbol{P}$ is a constant vector, unchanged by any type of interactions between the particles, so long as the system does not experience forces from bodies outside the system.

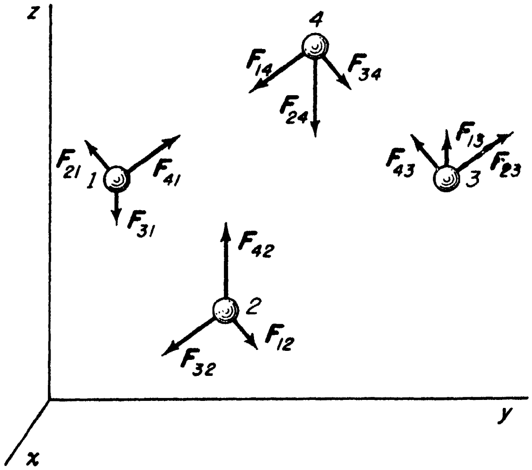

Fig. 3. A four-particle system. To derive the principle of conservation of momentum, we need not assume that the forces between pairs of particles act along the line joining the particles, but only that these forces are equal and opposite in accordance with Newton's third law.

The forces between bodies of the system are called *internal forces;* any forces coming from bodies outside the system are called *external forces.* Our analysis for four particles can be directly generalized to any number of particles and leads to one of the most important general principles of physics, known as the *principle of conservation of momentum.* This principle can be stated as follows:

*The total linear momentum of any system of interacting bodies remains constant unless the system is acted upon by external forces.**

The internal forces between particles within the system may be caused by gravitational attraction, by electrostatic attraction or repulsion, or by elastic or frictional forces resulting from contact or impact. The nature of the forces is immaterial. The important point involved in the proof of this principle is that, at each instant, action and reaction forces are equal in magnitude and opposite in direction.

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* When applied to a single body, this principle is an expression of Newton's first law in terms of momentum.

We shall now prove an important theorem regarding the total momentum of a system of particles:

THEOREM: *The total momentum of a system of particles is the same as the momentum of a single particle, of mass equal to the total mass, located at the center of gravity and moving with the center of gravity.*

If the particles of our system have masses m_1, m_2, $\cdots$, with total mass

$$M = m_1 + m_2 + \cdots,$$

the x-coordinate X of the center of gravity is defined by

$$MgX = m_1 g x_1 + m_2 g x_2 + \cdots,$$

or

$$MX = m_1 x_1 + m_2 x_2 + \cdots. \tag{9a}$$

Similarly,

$$MY = m_1 y_1 + m_2 y_2 + \cdots, \tag{9b}$$

$$MZ = m_1 z_1 + m_2 z_2 + \cdots. \tag{9c}$$

Because of these relations, the *center of gravity* is also called the *center of mass.*

If we differentiate (9a) with respect to t, we obtain the equation

$$MV_x = m_1 v_{1x} + m_2 v_{2x} + \cdots, \tag{10a}$$

where V_x is the x-component of velocity of the center of mass. There are two similar equations for MV_y and MV_z, which together express the vector equation

$$M\mathbf{V} = m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2 + \cdots. \tag{10b}$$

The right side is the total momentum of the system; the left side is the momentum of a particle of mass equal to the total mass moving with the velocity of the center of mass. This result proves the theorem stated.

We can now consider the motion of a system of particles such as those of Fig. 3, interacting with each other in any way but acted on in addition by the external force of gravity. To equation (7a) we must add, on the right, the force $m_1\mathbf{g}$ of gravity. To (7b) we must add $m_2\mathbf{g}$, and so on. When these equations are added we no longer obtain the result (8), but rather

$$d\mathbf{P}/dt = m_1\mathbf{g} + m_2\mathbf{g} + \cdots = M\mathbf{g}.$$

But since, by the theorem we have just proved, $\mathbf{P} = M\mathbf{V}$, where $\mathbf{V}$ is the velocity of the center of mass, $d\mathbf{P}/dt = M\mathbf{A}$, where $\mathbf{A}$ is the acceleration of the center of mass. Hence, $M\mathbf{A} = M\mathbf{g}$, or

$$\mathbf{A} = \mathbf{g}. \tag{11}$$

We have thus proved the following:

THEOREM: *Irrespective of the internal forces which may act between the particles or bodies considered as constituting a dynamical system, the center of gravity moves, under the force of gravity, like a simple particle, in a parabolic trajectory with downward acceleration g.*

One type of system of particles is that constituting a rigid body. The internal forces are the forces that hold the different particles in rigid connection. We have now proved rigorously, for the first time, that the *center of gravity* of such a rigid body, when thrown or projected, moves in a parabolic trajectory like a simple particle. Air resistance, of course, is still neglected, since it is an external force which we have *not* taken into account.

An interesting application of the above theorem concerns the trajectory of an exploding shell—still neglecting air resistance. The first part of the trajectory has the familiar parabolic form as shown in Fig. 4. At point X the shell explodes and shell fragments are blown out in all directions. However, the forces of the explosion are *internal* forces, so the center of gravity of the shell fragments must complete the usual parabolic trajectory. In fact, after a projectile has been launched, it is impossible to modify the trajectory of the projectile's center of gravity by the action of any forces acting between particles *within* the projectile. The path of a projectile moving through air can, of course, be modified by a shift of guiding vanes, but this operation involves *external* forces exerted on the projectile by the surrounding air. The trajectory of a rocket body can also be modified by ejecting gaseous material, but the trajectory of the center of mass of rocket body + ejected gas, all considered as a dynamical system, is not altered by this action—air resistance being entirely neglected. Neglect of air resistance is definitely valid if the rocket is traveling at sufficient height to be above the earth's atmosphere.

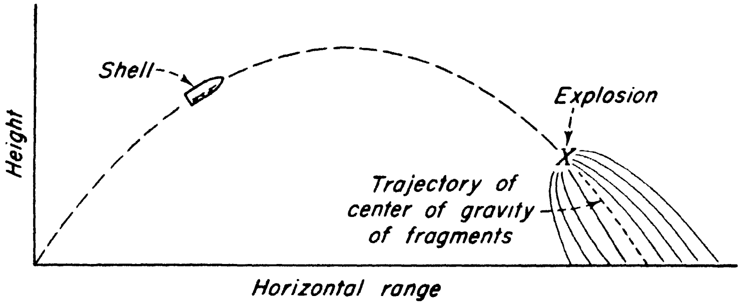

Fig. 4. Trajectory of an explosive shell.

We now state an important theorem which is a generalization of the one proved above about a system of particles acted on by the external force of gravity. The student can readily prove the following theorem by the methods used in the previous proof.

THEOREM: *Irrespective of the internal forces which may act between the particles or bodies considered as constituting a dynamical system, the center of mass of the system moves like a simple particle with acceleration* $\boldsymbol{F}/M$, *where* $\boldsymbol{F}$ *is the resultant of all external forces acting on the system, and* M *is the total mass of the system.*

A special case of this theorem, applied to the motion of the center of mass of a rigid body, was stated without proof in Chap. 8 on p. 205. We shall have occasion to use this theorem in the next chapter.

PROBLEMS

1. A 200-lb man stands in a 300-lb boat at rest on the surface of a lake. If the man dives from the boat with an initial horizontal velocity of 6 ft/sec eastward, what will be the initial horizontal velocity of the boat? Neglect momentum given to the water. Ans: 4 ft/sec westward.

2. A 90-kg man dives from a 150-kg boat at rest on a lake surface. If the initial horizontal velocity of the man was 8 m/sec northward, what was the initial horizontal velocity of the boat? Neglect momentum given to the water.

3. A 10-lb projectile is fired horizontally from a gun of 4000-lb mass. If the muzzle velocity of the projectile is 1200 ft/sec, what is the initial recoil velocity of the gun? Ans: 3 ft/sec.

4. If a 200-lb projectile were fired horizontally from the gun in Prob. 3 with a muzzle velocity of 1200 ft/sec, what would be the recoil velocity of the gun?

5. A 20-ton railroad gondola car moves freely along a horizontal railroad track at a velocity of 30 ft/sec when 15 tons of snow fall vertically into the car from a snow bank overhanging the track. What is the final velocity of the car? Ans: 17.1 ft/sec.

6. If the snow in Prob. 5 melts and leaks from the bottom of the car, what will be the final velocity of the car? (Assume that the water falls downward *vertically with respect to the car.* Ignore frictional effects.)

7. Prove the last theorem stated above.

8. If the initial recoil energy of the gun in Prob. 3 is all transformed into potential energy in a spring, what must be the force constant of the spring if the recoil is to be limited to 2 ft? Ans: 280 lbf/ft.

9. Repeat Prob. 8 for the gun of Prob. 4.

4. ELASTIC COLLISIONS

The problem of the collisions of bodies is a difficult one because the conservation-of-momentum principle is the only rigorous principle applicable to this case, and this principle alone is insufficient to determine the motion after collision from the motion before. The reason for this inadequacy is that an undetermined amount of energy may be converted into heat by deformation of the bodies during the collision, without violating conservation of momentum. *Momentum is always conserved,* but some *energy is always lost* during collision of macroscopic bodies.

There are two extreme cases in which the problem is soluble. One is the case in which we assume that no mechanical energy is dissipated—energy as well as momentum is conserved. Such collisions are said to be *perfectly elastic.* Collisions between ivory or glass balls fall approximately into this category. Under many circumstances, collisions between the fundamental particles (cf. Chap. 44) and collisions between individual atoms fall exactly into this category. The other extreme category is that in which the bodies *stick together* after the collision. Such collisions are called *perfectly inelastic.* Collisions between two balls of putty are likely to be in this category, and a collision between a bullet and a block of wood, in which the bullet remains embedded in the wood, certainly is. It is to be noted that the usage of the terms elastic and inelastic here is not synonymous with their usage in Chap. 9 in considerations of elasticity. There we spoke of elastic materials, here we speak of elastic collisions.

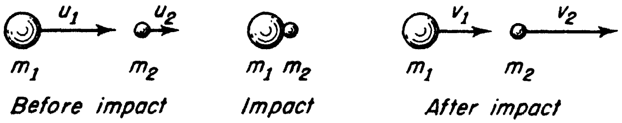

Fig. 5. Direct impact.

In this section we shall consider certain simple cases of perfectly elastic collisions; imperfectly elastic collisions will be considered in Sec. 5, and perfectly inelastic collisions in Sec. 6.

We shall first consider collisions between two 'particles' in which the velocities both before and after collision are along the same straight line. To visualize such a collision, consider two smooth nonrotating spheres colliding exactly 'head-on' when they are moving horizontally through the air. The instant after they separate they will be moving along the same straight line, still without rotation.

Such a collision is represented in Fig. 5, in which the masses of the bodies are m_1 and m_2, the velocities before collision are u_1 and u_2, and those after collision are v_1 and v_2. These 'velocities' are really velocity components along the line of collision; they and the corresponding momenta are taken as positive when directed to the right. Necessarily, in the configuration of Fig. 5, $u_1>u_2$ if the bodies are to collide, and $v_2>v_1$ if the bodies are to separate.

The total linear momentum of the system is unaltered by the impact, hence

$$m_1u_1+m_2u_2=m_1v_1+m_2v_2. \qquad (12)$$

Since we have assumed that the collision is perfectly elastic, we may write another equation for the conservation of energy during the collision process:

$$\text{K.E. before impact} = \text{K.E. after impact}$$

$$\tfrac{1}{2}m_1u_1^2+\tfrac{1}{2}m_2u_2^2=\tfrac{1}{2}m_1v_1^2+\tfrac{1}{2}m_2v_2^2. \tag{13}$$

Rearranging this equation, we may write

$$m_1(u_1^2-v_1^2)=m_2(v_2^2-u_2^2),$$

or

$$m_1(u_1+v_1)(u_1-v_1)=m_2(v_2+u_2)(v_2-u_2). \tag{13'}$$

By rearranging equation (12), we may also write

$$m_1(u_1-v_1)=m_2(v_2-u_2). \tag{12'}$$

By dividing (13′) by (12′), we get

$$u_1+v_1=v_2+u_2,$$

or

$$u_1-u_2=v_2-v_1 \tag{14}$$

This result indicates that, in an elastic collision in which the motion is confined to one dimension, *the relative velocity of approach before collision is equal to the relative velocity of separation after collision.*

Equations (12) and (14) are the simplest pair to use in determining the velocities v_1 and v_2 after collision from the velocities u_1 and u_2 before collision.

Several special cases are of interest. For example, if $m_1=m_2$, equations (12) and (14) take the forms

$$u_1+u_2=v_1+v_2$$

and

$$u_1-u_2=v_2-v_1.$$

By adding and subtracting these two equations we can obtain the relations

$$u_1=v_2 \quad\text{and}\quad u_2=v_1. \tag{15}$$

Equations (15) show that in direct, or 'head-on,' collision of two spheres having equal masses, the spheres *interchange* velocities during impact.

We shall now consider cases in which the body of mass m_2 is originally at rest. Since $u_2=0$, equations (12) and (14) become

$$m_1u_1=m_1v_1+m_2v_2$$

and

$$u_1=v_2-v_1.$$

These two equations are readily solved for v_1 and v_2 in terms of u_1, with the result

$$v_1=\frac{m_1-m_2}{m_1+m_2}u_1, \qquad v_2=\frac{2m_1}{m_1+m_2}u_1. \tag{16}$$

First, let us consider the case illustrated in Fig. 6(a), in which $m_1 \gg m_2$. Equations (16) show that in this case $v_1 \approx u_1$, the velocity of the heavy body being almost unchanged; whereas $v_2 \approx 2u_1$, the final velocity of the body of small mass being equal to approximately *twice* the velocity of the heavy body.

Second, let us consider the case in which the bodies have equal mass. Equations (16) then give $v_2 = u_1$, and $v_1 = 0$. In other words, when the two masses are equal, momentum and kinetic energy are completely transferred to the body that was originally at rest. This type of collision is shown in Fig. 6(b).

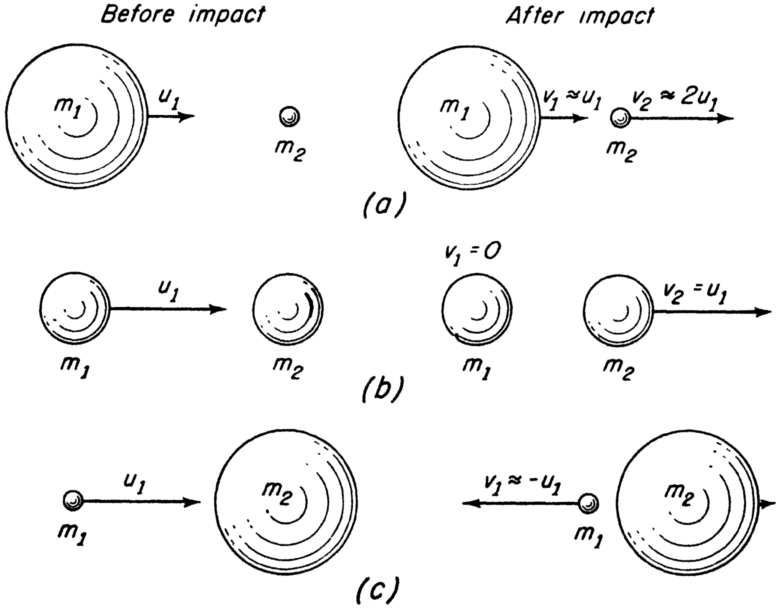

Fig. 6. Direct elastic impact between a moving body and a body initially at rest.

Third, consider the case in which $m_2 \gg m_1$. Equations (16) show that in this case m_2 is given only an extremely small velocity in the positive direction and the velocity of m_1 is practically reversed during impact. This type of collision is shown in Fig. 6(c).

Consideration of these three cases shows that a moving particle loses only a small fraction of its kinetic energy to a stationary particle if it is *either* much heavier or much lighter than the stationary particle. The only case in which *all* the kinetic energy of the 'bombarding' particle m_1 is lost to the 'bombarded' particle m_2 is that in which m_1 and m_2 are equal.

Use of this fact is made in nuclear physics when it is desired to reduce the kinetic energies of energetic neutrons. Since neutrons and protons* have nearly equal masses, high neutron energies can be quickly reduced by sending high-energy neutrons through some substance like water or paraffin that contains many protons. The initial energy of a neutron may be enormous, but after a few collisions with protons, its mean kinetic energy becomes equal to the small mean 'thermal energy' of the protons at room temperature. The neutron usually does not lose all of its energy at a single collision because the probability of its making an exactly 'head-on' collision with a proton is rather small.

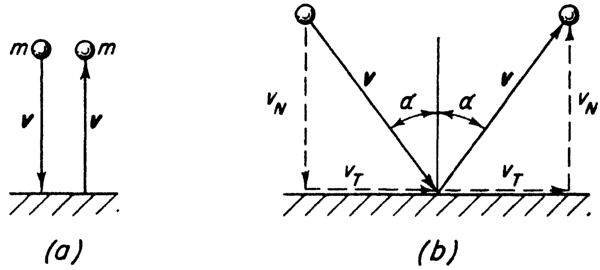

Fig. 7. Elastic impacts between a particle and a smooth fixed plane.

Let us now proceed to the problem of an elastic collision between a sphere and a smooth 'fixed' plane, by which we mean a plane rigidly attached to the earth. The mass factor associated with the plane is the mass of the earth, and, therefore, the velocity imparted to the plane during the impact is negligible. Hence, the problem is essentially similar to the case shown in Fig. 6(c), and the component of the sphere's velocity perpendicular to the plane is reversed in direction and unchanged in magnitude after impact. This situation is shown in Fig. 7(a) for a sphere striking the plane perpendicularly. If the particle strikes the plane at an angle α measured from the normal to the plane, the perpendicular component of the velocity v_N is reversed; and since the plane is 'smooth' and therefore no tangential forces can act, the velocity component v_T tangent to the plane is unchanged. The result is that the particle rebounds from the plane at an angle with the normal that is equal to α, the angle of incidence, as shown in Fig. 7(b).

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* See Chap. 44 for a description of the properties of these fundamental particles. Protons are hydrogen nuclei. Neutrons are electrically neutral particles that play an important role in nuclear reactions, particularly in the nuclear-fission processes that take place in the manufacture of plutonium in nuclear piles and in the explosion of an 'atomic' bomb.

PROBLEMS

NOTE: Assume that all collisions are head-on, that energy is conserved, and that there is no rotational motion.

1. A 20-lb ball *A* moving at a velocity of 20 ft/sec in the $+x$-direction strikes a 1-lb ball *B* initially at rest. Find the velocity, momentum, and kinetic energy of each ball after collision. Ans: *A*: 18.1 ft/sec, 11.2 slug ft/sec, 102 ft·lbf; *B*: 38.1 ft/sec, 1.18 slug·ft/sec, 22.6 ft·lbf.

2. A 14-kg ball *A* traveling at a velocity of 10 m/sec in the $+x$-direction strikes a 1-kg ball *B* initially at rest. Find the velocity, momentum, and kinetic energy of each ball after collision.

3. A 2-kg ball *A* moving with a velocity of 12 m/sec eastward strikes ball *B* of mass 2 kg initially at rest. Find the velocity, momentum, and kinetic energy of each ball after the collision.
Ans: *A*: 0, 0, 0; *B*: 12 m/sec eastward, 24 kg·m/sec eastward, 144 joules.

4. A 2-kg ball *A* moving with a velocity of 12 m/sec eastward makes a head-on collision with another 2-kg ball *B* moving with a velocity of 12 m/sec westward. Find the velocity of each ball after impact.

5. A 2-kg ball *A* moving with a velocity of 10 m/sec westward strikes a 20-kg ball *B* that is moving at 5 m/sec eastward. Find the velocity of each ball after the collision. Ans: *A*: 17.3 m/sec eastward; *B*: 2.27 m/sec eastward.

6. A 1-lb ball *A* moving with a velocity of 40 ft/sec eastward strikes a 40-lb ball *B* that is moving at 2 ft/sec westward. Find the velocity of each ball after the collision.

7. A 2-lb ball *A* moving with a velocity of 30 ft/sec eastward strikes a 2-lb ball *B* moving with a velocity of 70 ft/sec westward. Find the final velocities of the two balls. Ans: *A*: 70 ft/sec westward; *B*: 30 ft/sec eastward.

8. What percentage of its kinetic energy is transferred to the stationary particle when a moving particle strikes a stationary particle of 10 times its mass? of equal mass? of $\frac{1}{10}$ its mass?

9. What percentage of its kinetic energy is transferred to the stationary particle when a moving particle strikes a stationary particle of 100 times its mass? of equal mass? of $\frac{1}{100}$ its mass? Ans: 3.92%; 100%; 3.92%.

10. Show that when a moving particle collides with a stationary particle that is either k times as massive, or $1/k$ times as massive, it transfers to the stationary particle the fraction $4k/(1+k)^2$ of its kinetic energy.

11. *During* the collision depicted in Fig. 5, the spheres exert forces on each other. These are elastic forces (in the sense of the theory of elasticity) arising from the deformation of the spheres. There is first a period of 'compression' during which the spheres sink into each other and are brought to the same velocity, say U; then a period of 'expansion' during which the spheres push each other apart and are brought to their final velocities. Show that in an elastic collision the impulse that sphere 1 exerts on sphere 2 during the period of 'compression' is equal to the impulse that sphere 1 exerts on sphere 2 during the period of 'expansion.'

5. IMPERFECTLY ELASTIC COLLISIONS*

During the collision depicted in Fig. 8, the spheres exert forces on each other. These are elastic forces (in the sense of the theory of elasticity) arising from the deformation of the spheres. There is first a period of

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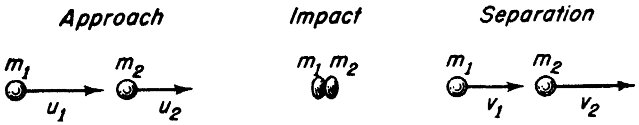
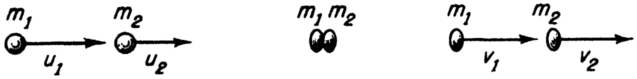
* This section may be omitted without loss of continuity.

'compression' during which the spheres sink into each other and are brought to the same velocity. During this period, some of the original kinetic energy is changed into elastic potential energy and some heat may be generated. Then there is a period of 'expansion' during which the spheres push each other apart. During this period elastic potential energy is changed back into kinetic energy and possibly also into heat. If there is no heat generated, all of the original kinetic energy is recovered and the collision is perfectly elastic. If there is heat generated, some of the original kinetic energy is dissipated and the collision is *imperfectly elastic.*

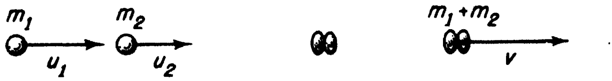
In a perfectly elastic collision, the forces between the spheres are the same at each stage of the 'compression' process as they are at the corresponding stage of the 'expansion' process. In an imperfectly elastic collision, the forces are less during the 'expansion' process so that the work done in deforming the sphere during 'compression' is incompletely recovered during 'expansion.' The spheres may or may not be left with a 'permanent set' as defined on p. 216; when forces are applied with extreme rapidity in collisions, elastic behavior may be quite different from that given by the quasi-static theory of Chap. 9, and even materials like glass, which cannot be given a permanent set because it ruptures before reaching its elastic limit, are imperfectly elastic in collisions.



(a) Elastic collision, $v_2 - v_1 = u_1 - u_2$



(b) Imperfectly elastic collision, $v_2 - v_1 = e(u_1 - u_2)$



(c) Inelastic collision, $v_2 - v_1 = 0$

Fig. 8. Direct impacts.

In an imperfectly elastic collision, the kinetic energy after the collision is less than that before the collision. Some of the kinetic energy is converted into heat. In such collisions, equation (13) cannot be used and we must introduce some other condition, in addition to the conservation-of-momentum relation (12), in order to solve the problem of finding the velocities after impact.

In an elastic collision it can be shown (cf. Prob. 11, p. 254) that the impulse of the force acting on one of the bodies during the period of 'compression' equals the impulse of the force acting during the period of 'expansion.' In an imperfectly elastic collision the latter impulse is smaller than the former. This is reasonable since the force acting during the period of expansion is smaller than that acting during compression.

From experiment, Newton concluded that the ratio of the impulse during expansion to that during compression is approximately constant for collisions between bodies composed of a given material, and he called this constant ratio the *coefficient of restitution.* We can show (see Prob. 7 at the end of this section) that the ratio of impulses mentioned in the preceding sentence is equal to the ratio of the velocity of approach of the two bodies before impact to the velocity of separation after impact. We shall use this velocity ratio as our definition of e, the coefficient of restitution, since velocities can be readily determined by direct experiment.

> The ratio of the relative velocity of separation after direct impact to the relative velocity of approach before impact is defined as the coefficient of restitution
>
> $$e=\frac{v_2-v_1}{u_1-u_2}.$$

Typical coefficients of restitution for several materials are:

glass	0.95	wood	0.50	mild steel	0.55
ivory	0.90	lead	0.20	clay, putty	0

The processes occurring during imperfectly elastic direct impact of two spheres are shown diagrammatically in Fig. 8(b). Let us now see how the coefficient of restitution e can be used in solving the problem of such a collision. Since no forces other than interaction forces between the spheres are involved, we can still use the conservation-of-momentum principle:

total momentum before impact = total momentum after impact

$$m_1u_1+m_2u_2=m_1v_1+m_2v_2. \tag{17}$$

We can also use the relation defining the coefficient of restitution; this relation gives

$$eu_1-eu_2=v_2-v_1. \tag{18}$$

Equations (17) and (18) are sufficient to determine v_1 and v_2 if u_1 and

u_2 are given. As an illustration, let us consider the case of Fig. 6, in which $u_2=0$. In this case (17) and (18) become

$$m_1u_1=m_1v_1+m_2v_2,$$

and

$$eu_1=v_2-v_1,$$

whose solution is

$$v_1=\frac{m_1-em_2}{m_1+m_2}u_1,\qquad v_2=\frac{(1+e)m_1}{m_1+m_2}u_1. \tag{19}$$

In the case $m_1 \gg m_2$, these give $v_1 \approx u_1$; $v_2 \approx (1+e)u_1$.

In the case $m_1=m_2$, they give $v_1=\frac{1}{2}(1-e)u_1$; $v_2=\frac{1}{2}(1+e)u_1$.

In the case $m_2 \gg m_1$, they give $v_1 \approx -eu_1$; $v_2 \ll u_1$.

In the case $m_1=m_2$, we readily verify that these results satisfy conservation of momentum, and we can compute the loss in kinetic energy as follows. The kinetic energy before collision is $\frac{1}{2}m_1u_1^2$. After collision it is $\frac{1}{2}m_1\cdot\frac{1}{4}(1-e)^2u_1^2+\frac{1}{2}m_1\cdot\frac{1}{4}(1+e)^2u_1^2=\frac{1}{4}m_1u_1^2(1+e^2)$. Hence, the energy loss in the collision is $\frac{1}{4}(1-e^2)m_1u_1^2$. This is zero for a purely elastic collision ($e=1$), increases as e decreases, and is half the initial kinetic energy for a purely inelastic collision ($e=0$).

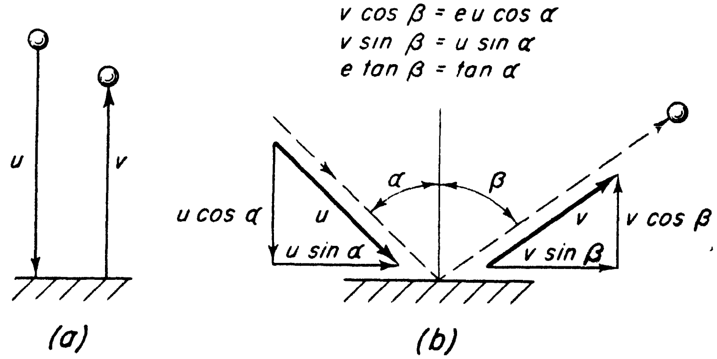


Fig. 9. Imperfectly elastic collisions between a particle and a fixed smooth plane.

Let us now consider the case of the imperfectly elastic collision of a sphere with a *smooth* fixed plane. In this collision the normal component of the sphere's initial velocity is reversed in direction and altered in magnitude, but since a smooth plane is incapable of exerting tangential forces, the tangential component is unchanged. If the sphere strikes the plane perpendicularly with initial velocity u, the final velocity is oppositely directed and has the magnitude $v=eu$, as shown in Fig. 9(a). This corresponds to the case of a collision with an extremely heavy body which we have discussed previously.

If the sphere approaches the plane at an angle α to the normal with velocity $\boldsymbol{u}$ as shown in Fig. 9(b), the initial tangential component is $u \sin\alpha$, and since this component is unchanged, we may set it equal to the tangential component of the velocity of separation $\boldsymbol{v}$. Hence, if the sphere rebounds at angle β with the normal, we may write

$$u \sin\alpha = v \sin\beta. \tag{20}$$

The normal component of the velocity of approach is $u \cos\alpha$ and the normal component of the velocity of separation is $v \cos\beta$. The relation between these is

$$eu \cos\alpha = v \cos\beta. \tag{21}$$

By dividing (20) by (21), we can determine the relation between α and β:

$$e \tan\beta = \tan\alpha,$$

and by squaring both sides of equations (20) and (21) and adding, we may express the magnitude of the final velocity $\boldsymbol{v}$ as

$$v = u \sqrt{\sin^2\alpha + e^2 \cos^2\alpha}.$$

This discussion assumes that the sphere and the surface are made of the same material, so that the value of e for this material can be used.

From the above results, it can be shown that if a sphere is dropped from height h onto a surface of the same material, it rebounds to a height e^2h. Proof of this statement is left as a problem. This experiment furnishes a convenient method of measuring the coefficient of restitution.

PROBLEMS

1. A casehardened steel ball is dropped from a height of 16.1 ft above a casehardened steel plate. If the coefficient of restitution is 0.90, how high will the ball rise on the first bounce after striking the surface? Ans: 13.0 ft.

2. To what vertical height will the ball in Prob. 1 rise on the second bounce? the third bounce? the nth bounce?

3. Solve Prob. 1 for a lead ball striking a lead surface. Ans: 0.64 ft.

4. Solve Probs. 1 and 2 for a glass marble striking a glass plate.

5. The steel ball in Prob. 1 moves at a speed of 40 ft/sec and strikes the steel plate at angle $\alpha = 45°$ as shown in Fig. 9. What will be the value of angle β?
Ans: 48°0.

6. Solve Prob. 5 for a lead ball striking a lead plate.

7. *During* the collision of Fig. 8, the spheres exert forces on each other. As in the text, a period of 'compression' and a period of 'expansion' are defined. Consider an imperfectly elastic collision in which the velocity of separation is $v_2 - v_1$ and the velocity of approach is $u_1 - u_2$. Show that the impulse exerted by sphere 1 on sphere 2 during 'expansion' is smaller than the impulse exerted by 1 on 2 during 'compression' in the ratio $(v_2 - v_1)/(u_1 - u_2)$, as stated in the text.

8. Show that if a sphere is dropped from height h onto a horizontal surface of the same material, it rebounds to a height e^2h.

6. INELASTIC COLLISIONS

Let us now consider inelastic collisions, in which the two bodies have the same velocity after the collision. Collisions of this type are ordinarily accompanied by pronounced deformation because one or both of the colliding bodies has been subjected to forces greatly exceeding its elastic limit. An inelastic collision may be visualized by imagining direct impact between a billiard ball and a ball of putty as shown diagrammatically in Fig. 8(c). As indicated in the figure, the two spheres remain in contact after collision and therefore have a common final velocity v. This final velocity can be determined directly from the relation for conservation of momentum:

$$m_1u_1+m_2u_2=(m_1+m_2)v. \qquad (22)$$

If an inelastic sphere strikes a smooth fixed plane at an angle of α with the normal, the normal component of the velocity $u\cos\alpha$ disappears when the impact occurs, and the deformed sphere slides along the plane with velocity $u\sin\alpha$ as shown in Fig. 10(a). (Imagine a ball of putty thrown down on a surface of ice.)

When a stream of water impinges on a fixed plate, the impact practically follows the laws of inelastic collision. The stream spreads out during impact and the water flows along the surface of the plate as indicated in Fig. 10(b). We can find the force F exerted by such a jet of water on the plate. In time Δt, let mass Δm of water strike the plate. Before collision this water had y-component of momentum $-\Delta m\, u\cos\alpha$. After collision its y-component of momentum is zero. Therefore, in the collision its y-component of momentum is *increased* by $\Delta p=\Delta m\, u\cos\alpha$. According to (3), this means that the plate must exert an average upward force

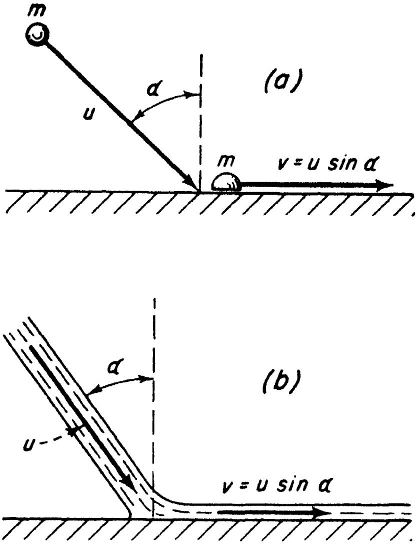


Fig. 10. Inelastic collisions at a smooth fixed plane.

$$F=\Delta p/\Delta t=u\cos\alpha\,\Delta m/\Delta t.$$

But $\Delta m/\Delta t$ is the *mass of water striking the plate per unit time;* call this μ. Then

$$F=\mu u\cos\alpha.$$

The force is constant and hence equal to its average value in time interval

Δt. Hence F is the upward force exerted by the plate on the water, or, by Newton's third law, F is *the magnitude of the downward force exerted by the water jet on the plate.*

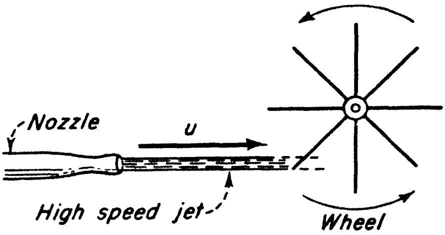
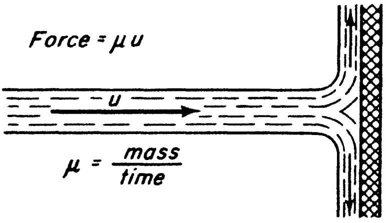
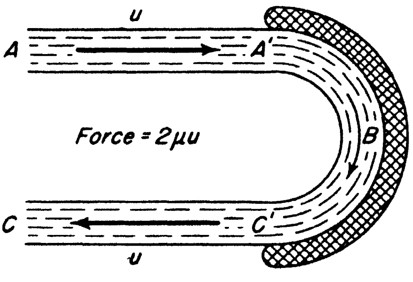


Fig. 11. Principle of the impulse turbine.

Engineers sometimes call this force an *impulse force.* The force applied in this way is employed in the *impulse turbine,* a water wheel operated by means of a high-speed water jet applied in the manner indicated in Fig. 11. The blades of the wheel are flat and the impinging stream of water exerts an impulse force on the wheel. The efficiency of such a turbine can be increased somewhat, as in the Pelton wheel, by 'dishing' the blades. The principle involved can be understood by considering Fig. 12, which shows a dished section of a fixed body. A high-speed water jet is incident in direction AA' toward the dish. The



Stationary flat blade



Stationary 'dished' blade

Fig. 12. Stream of water striking turbine blades. In the discussion, the blades are considered as fixed in position.

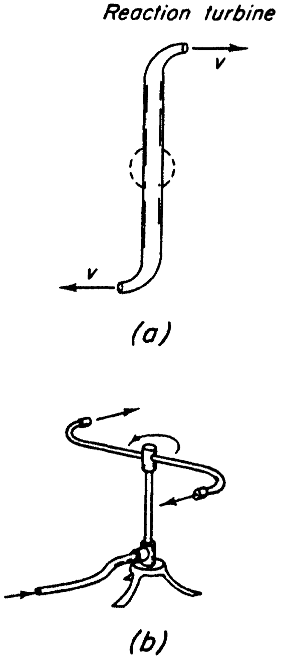


Fig. 13. Simple reaction turbine.

direction of the water stream is changed when it strikes the curved surface of the dish, and if there is no turbulence it leaves the dish in direction $C'C$ with a velocity equal in magnitude to the velocity of the incident jet. The total force exerted on the dish is therefore given by $2u\mu$, since the change in velocity is $2u$. Engineers sometimes think of the total force as being made up of two parts: (1) the impulse force produced while the initial velocity to the right in Fig. 12 is reduced to zero and (2) the reaction force produced while the water is given a velocity toward the left. Such reaction forces are utilized in reaction turbines, of which the rotating sprinkler for lawns is a familiar example. A turbine of this type is shown in Fig. 13. Water flowing out from the center inside the turbine is ejected in the manner indicated, and the reaction force sets the turbine arms into rotation.

PROBLEMS

1. A 20-g lead bullet moving horizontally at a speed of 400 m/sec strikes a 6-kg block of wood resting on a horizontal sheet of ice. The bullet penetrates the surface of the block, and the two bodies move off together along the surface of the ice. What is the magnitude of their common velocity after the collision? How many joules of energy are changed into heat in the collision? Ans: 1.33 m/sec; 1595 joules.

2. Solve Prob. 1 for a 4-oz bullet moving at a speed of 1200 ft/sec and striking a 20-lb block.

3. A stream of water strikes a wall as shown in Fig. 10(b). If angle α is 30°, find the force exerted on the wall when 10 ft^3 of water strikes the wall each minute and the stream velocity u is 40 ft/sec. Ans: 11.2 lbf.

4. Find the force exerted on the wall in Prob. 3: (a) when the stream strikes the wall at angle $\alpha = 60°$ and (b) when the stream strikes the wall normally.

5. Find the force that would be exerted by the stream of water mentioned in Prob. 3 if the stream were directed against a stationary dished blade in the manner shown in Fig. 12. Ans: 25.8 lbf.

6. If the stream of water mentioned in Prob. 5 were directed against the dished blades of a turbine, approximately what power is developed by the turbine if the speed of the water (with respect to the ground) is 10 ft/sec in the opposite direction from that at impact when it leaves the turbine, and the turbine blades are moving at 12 ft/sec at the point of water impact?

7. ACCELERATION OF PROJECTILES; ROCKETS

The conservation-of-momentum principle is useful in the analysis of problems encountered in the launching of projectiles. For example, let us consider the firing of a shell from a gun. As indicated in Fig. 14, let the mass of the shell be m and the mass of the gun be M. Before the gun is fired, the total momentum of the system consisting of gun and shell is zero. From the conservation-of-momentum principle, we can assert that the total momentum of the system remains zero after the gun is fired. Unless external forces act, this statement means that the momentum of the projectile (toward the right) and the momentum of the gun (toward the left) are equal in magnitude at the instant the projectile leaves the muzzle, or

$$MV = mv,$$

where v is the muzzle velocity of the shell and V is the initial velocity of the gun. The velocity of the gun at this instant is called the *recoil velocity;* a large gun is allowed to move backward for an appreciable distance by the *recoil mechanism.*

In order to find the average force exerted on the shell while it is being accelerated, we may use the relation $\bar{F} = \Delta p/\Delta t$. This gives

$$\bar{F} = mv/\Delta t,$$

where Δt is the time required for the shell to move down the barrel.

Now let us examine some of the factors to be taken into consideration in firing large projectiles over long ranges. If air resistance is neglected, it can be shown (Sec. 6 of Chap. 5) that the maximum range of a projectile is proportional to the square of the muzzle velocity. Consequently, for a projectile of given mass, the range is proportional to the initial kinetic energy. If a large projectile is to be used for long-range bombardment, the kinetic energy must be very large and serious problems are encountered in imparting the required kinetic energy to the projectile. In the first place, if the chemical energy of the propellant is to be delivered efficiently to the projectile rather than to recoil of the gun, the mass M of the gun must be extremely large compared to that of the projectile. Transportation of extremely large guns is inconvenient from a practical viewpoint. Second, the force applied to the projectile must be enormous and hence the gun barrel is subjected to high pressure. If we let F_{av} be the force averaged with respect to distance down the barrel (not with respect to time like $\bar{F}$), the work done on the projectile is equal to the kinetic energy, so that

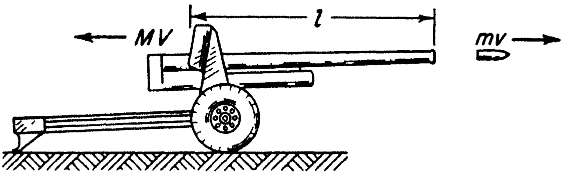


Fig. 14. Acceleration of a projectile from a gun.

$$\text{K.E.} = F_{av}\, l, \quad \text{or} \quad F_{av} = \text{K.E.}/l. \tag{23}$$

Thus, for a given range the distance-average force is inversely proportional to the length l of the gun barrel, which should therefore be made as long as practicable, in order to keep the average pressure inside the gun within allowable limits. It must be remembered that equation (23) gives only the value of the force averaged along the barrel, and that, unless special slow-burning propellants are used, the maximum force and, consequently, the maximum pressure may be much larger. For example, if old-fashioned fast-burning gunpowder were used in modern long-range guns, the guns would be blown to bits. Actually, satisfactory propellants have been developed, but even with these there prove to be definite limitations to the ranges obtainable with projectiles shot from 'conventional' guns of the type we have discussed. The maximum range yet attained in actual combat in this way is 75 miles during the German bombardment of Paris in 1918, but ranges of more than 100 miles have been obtained in more recent ordnance field tests.

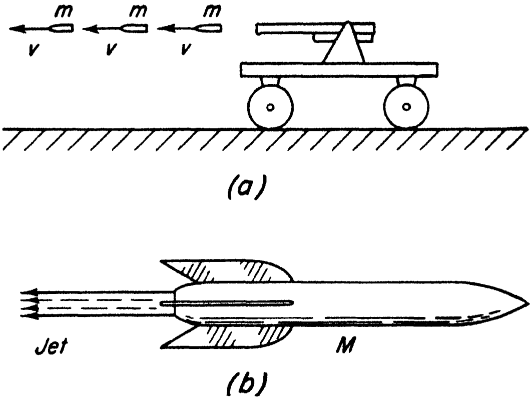


Fig. 15. Acceleration of bodies by reaction forces.

For extremely long-range projectiles, rockets have proved more practicable than missiles propelled from guns. A rocket is propelled by reaction forces supplied from a high-speed jet ejected toward the rear in much the same manner that reaction forces are used to set a reaction turbine in motion. In order to understand how the reaction is used, consider the sketch shown in Fig. 15(a), which shows a machine gun mounted on a car and firing a stream of bullets of mass m with a muzzle velocity v. Each bullet receives momentum mv as it leaves the gun. Hence, if n bullets are fired per second from the gun, they receive momentum $\Delta p = n\,mv$. From (3), with $\Delta t = 1$ sec, we see that the average force exerted by the gun on these bullets is $F = n\,mv$. The magnitude of the reaction force acting on the gun is equal and opposite. Hence,

$$\text{reaction} = \text{change of momentum of projectiles per sec} = n\,mv.$$

This reaction force accelerates the gun and the car on which the gun is mounted.

An accelerating force is applied to the rocket in a similar manner, as indicated in the sketch in Fig. 15(b). In this case, we shall denote the jet velocity *relative to the rocket* by v and shall denote the constant rate of discharge (mass/sec) of hot gases by μ. Hence, the change of momentum per second of the material passing from the rocket into the jet is given by μv, and the magnitude of the reaction force which accelerates the rocket is given by the equation (derived exactly as in the case of the machine gun)

$$F=\mu v. \tag{24}$$

This reaction force is called the *thrust.*

The true rocket is a reaction-propelled missile that carries all the materials from which the propulsion jet is formed, whereas the jet engine ordinarily used on manned aircraft carries only fuel and depends upon atmospheric oxygen to support combustion. The basic principles of propulsion are essentially the same for jet planes and for rockets. A true rocket operates best in the absence of an atmosphere.

The force (24) will give a rocket of mass M an acceleration

$$a=v\mu/M.$$

If we neglect the change in mass of the rocket as the propellant is expelled, this is a constant acceleration which gives the rocket a velocity

$$V=aT=v\mu T/M$$

in time T. Now if the total mass of fuel is m, the fuel will burn for a time given by $\mu T=m$, and therefore the rocket will acquire a velocity

$$V=vm/M, \tag{25}$$

directly proportional to the velocity with which the propelling gases are expelled.

In many rockets, the fuel is an appreciable part of the initial weight. In this case the acceleration increases as the mass of the rocket decreases and the highest attainable velocity is greater than (25) if M represents the initial mass of the rocket. In this case it can be shown that (25) should be replaced by

$$V=v\log_e[M/(M-m)],$$

which has the series expansion

$$V=v\, m/M+\tfrac{1}{2}v\,(m/M)^2+\tfrac{1}{3}v\,(m/M)^3+\cdots,$$

of which the first term is our result (25).

Because of the occurrence of the factor v, the jet velocity, in these expressions, it is important to eject the jet at as high a speed as possible. In the German V-2 rocket, which burned alcohol in liquid oxygen to furnish the propelling gas, the jet velocity was of the order of 6000 ft/sec.

PROBLEMS

1. A 32.2-lb shell is fired from a gun. If the length of the barrel is 16 ft and the muzzle velocity of the shell is 1200 ft/sec, what is the average force (averaged with respect to distance) exerted on the projectile during its passage down the gun barrel?
Ans: 45,000 lbf.

2. What average force would be required to give the projectile in Prob. 1 a muzzle velocity of 3000 ft/sec? 4000 ft/sec?

3. The mass of a certain rocket is 800 lb exclusive of the mass of the propellant. If the mass of the propellant is 200 lb and the velocity of the exhaust gases is 4000 ft/sec, what is the *approximate* value of the maximum velocity of the projectile as given by (25)? Give a more accurate value for the maximum velocity.
Ans: 800 ft/sec; 892 ft/sec.

4. If the mass of propellant in Prob. 3 were increased to 400 lb, what would be the maximum velocity of the projectile?

5. The V-2 rocket had about 250,000 lbf thrust, and a jet velocity of about 6000 ft/sec. How many lb/sec of gas must be exhausted in the jet to achieve this thrust?
Ans: 1340.

8. CONSERVATION OF ANGULAR MOMENTUM

In Chap. 8 we saw that many of the properties of rotational motion are analogous to those of rectilinear translational motion. For rotation about a fixed axis, we derived the expression

$$L = I\alpha, \tag{26}$$

which, because of its similarity to the second law of motion $F = ma$, is sometimes called the 'second law for rotational motion.' The rotational quantity that is analogous to linear momentum $p = mv$ is called *angular momentum* and is denoted by

$$H = I\omega. \tag{27}$$

It is readily seen that the 'second law' given in (26) can be restated in terms of angular momentum, since for a rigid body $dH = I d\omega$, and we may therefore write

$$L = I\alpha = I\ d\omega/dt,$$

or

$$L = dH/dt. \tag{28}$$

Actually, equation (28) is more general than implied by this derivation and can be shown to apply to a body with a variable moment of inertia. The equation states that:

The time rate of change of the angular momentum of a body rotating about a fixed axis is equal to the resultant externally applied torque.

If in (28), $L = 0$, then $dH/dt = 0$, and consequently $H =$ constant. This result is a statement of the principle of conservation of angular momentum:

If no resultant external torque acts, the angular momentum of a body rotating about a fixed axis is constant.

This statement is the rotational analogue of Newton's first law. However, there is one important difference between this principle for angular momentum and the corresponding principle for the linear momentum of a body. This difference is due to the fact that the mass m of a body can in most practical problems be regarded as constant, whereas the moment of inertia I of a nonrigid body can be changed by a redistribution of mass about the axis of rotation. Consequently, although we may say that the *linear velocity* v of the center of mass of any body is constant unless a resultant external force acts on the body, only in case the body is rigid can we state that the angular velocity ω is constant unless a resultant external torque acts. In the more general case of a nonrigid body acted on by no external torque and rotating about a fixed axis, we use the equation $H=\text{constant}$. If the moment of inertia changes from I to I', then ω must change to the value ω' given by

$$I\omega = I'\omega'. \tag{29}$$

This equation states that, in the absence of external torques, the angular velocity of a rotating body is inversely proportional to the moment of inertia of the body.

The relationship given in equation (29) can be strikingly demonstrated by placing a man on a rotating platform that has little friction in its bearings. After he has been set in rotation, the man can regulate his angular velocity by varying the position of his arms. If he extends his arms to the side, thus increasing his moment of inertia, his angular velocity will be decreased; but if he draws his arms in close to his body, thus decreasing his moment of inertia, his angular velocity will increase. A dancer or skater makes use of this same effect to acquire a high velocity of spin; an acrobat or diver to regulate his rate of 'tumble.'

In some problems it may be desirable to make use of the quantity *angular impulse*, denoted by J. This quantity, analogous to impulse, is defined by $J=\bar{L}\,\Delta t$ and from (28) we may write

$$\Delta H = \bar{L}\,\Delta t = J. \tag{30}$$

In other words, the change in the angular momentum of a body is equal to the angular impulse associated with the externally applied torque.

PROBLEMS

1. A man stands on a frictionless rotating platform that is rotating at a speed of 1 rev/sec; his arms are outstretched and he holds weights in each hand. With his hands in this position, the total moment of inertia of the man and the platform is 6 kg·m^2. If by drawing in the weights, the man decreases the moment of inertia to 2 kg·m^2, what is the magnitude of the resulting rotational velocity of the platform?

Ans: 3 rev/sec.

2. What would be the rotational speed of the man in Prob. 1 if he held the weights in an intermediate position at which $I=4$ kg·m^2?

CHAPTER 11

VIBRATORY MOTION

In earlier chapters we considered in some detail the motion of a particle subjected to a *constant* resultant force. In this chapter we shall consider some types of motion in which the resultant force is not constant but varies periodically. These types include the motion of a particle moving at constant speed in a circular path and the vibratory motion of a particle about an equilibrium position when the restoring force is proportional to the displacement. We have already discussed uniform circular motion in connection with rotating wheels and cylinders but shall wish to reconsider this type of motion briefly. Vibratory motion results when a body hanging from a spring is pulled downward and then released; other common examples of this type of motion are the vibrations of strings and air columns of musical instruments, the vibrations of bridges and buildings, and the oscillation of the balance wheel of a watch or of the pendulum of a clock.

Uniform circular motion and vibratory motion are both *periodic* motions; a periodic motion is one in which a certain path is successively traversed with exactly the same velocities in successive time intervals of length T. For example, a particle moving at constant speed in a circle traverses the same path over and over again with the same velocities; a pendulum repeats its to-and-fro motion again and again. One complete execution of the motion in time T is called a *cycle* of the periodic motion. *The time T required for the completion of a cycle is called the period of the motion. The number of cycles completed per unit time is called the frequency f of the motion.* From their definitions, it is readily seen that the period T of a periodic motion is the reciprocal of the frequency f, or

$$T = 1/f.$$

For example, if the rotational *frequency* of a motor shaft is 30 cycles/sec (30 rev/sec), the *period* of the rotational motion is $\frac{1}{30}$ sec.

The principles we shall learn in studying periodic motions in this chapter will have applications not only in mechanics but also in the study of sound, light, and electricity.

1. UNIFORM CIRCULAR MOTION

We have already shown that a particle moving at constant speed v in a circular path must be acted upon by a resultant force that is *centripetal.*

The *magnitude* of this force is constant, but the *direction* of the force changes continuously and is always at right angles to the motion of the particle.

Consider the particle shown in Fig. 1. The angular displacement θ of the particle from its initial position $\theta_0=0$ at $t=0$ is given by the relation

$$\theta=\omega t \tag{1}$$

in radians, where $\omega=v/R$ in rad/sec. We may also write θ in terms of the frequency f in rev/sec. Since $\omega=2\pi f$,

$$\theta=2\pi ft. \tag{1'}$$

Knowing θ and R, we may write the rectangular coordinates of the particle as a function of time as $x=R\cos\theta$, $y=R\sin\theta$, or

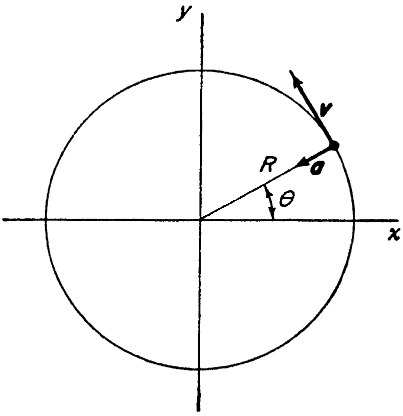


Fig. 1. Uniform circular motion.

$$\left.\begin{aligned} x&=R\cos 2\pi ft,\\ y&=R\sin 2\pi ft.\end{aligned}\right\} \tag{2}$$

If we wish to obtain the rectangular components of the velocity of the particle, we may differentiate x and y in (2) with respect to time t. This differentiation gives

$$\left.\begin{aligned} v_x&=dx/dt=-2\pi fR\sin 2\pi ft,\\ v_y&=dy/dt=2\pi fR\cos 2\pi ft.\end{aligned}\right\} \tag{3}$$

This result is reasonable, since the magnitude of the maximum value of either v_x or v_y is just the constant tangential speed $v=\omega R=2\pi fR$, and we verify that $v_x^2+v_y^2=(2\pi fR)^2=v^2$. The correctness of the signs in (3) can be verified by computing the instantaneous velocity vectors for the particle in Fig. 1 for various values of $\theta=2\pi ft$, such as 0, $\frac{1}{4}\pi$, $\frac{1}{2}\pi$, $\cdots$.

In order to determine the rectangular components of the acceleration, we may differentiate v_x and v_y in (3) with respect to time t. This gives

$$a_x=dv_x/dt=-4\pi^2f^2R\cos 2\pi ft,\quad a_y=dv_y/dt=-4\pi^2f^2R\sin 2\pi ft. \tag{4}$$

These are the rectangular components of the *centripetal acceleration*, whose magnitude is

$$a=v^2/R=\omega^2R=4\pi^2f^2R.$$

The correctness of the signs in equation (4) can be verified by computing the centripetal acceleration vectors in Fig. 1 for various values of θ.

Equations (1), (2), (3), and (4) give a complete *kinematical* description of the motion of a particle at constant speed v in a circular path of radius

R. The magnitude F of the centripetal force required to produce the centripetal acceleration is given by

$$F = ma = m(4\pi^2 f^2 R),$$

where m is the mass of the particle. Solving this equation for the frequency f of the rotational motion of the particle, we obtain

$$f = \frac{1}{2\pi}\sqrt{\frac{F}{Rm}}. \tag{5}$$

Equation (5) gives the rotational frequency f of a particle of mass m moving in a circular path of radius R when subjected to centripetal force of magnitude F. By substituting this value for f in equations (2), (3), and (4), we obtain a complete description of the motion of the particle in terms of the constants F, R, and m and the time variable t.

PROBLEMS

1. An automobile moving at a speed of 60 ft/sec traverses a circular path of radius 400 ft. What is the angular velocity of this automobile in its motion around the center of the circular path? What is its frequency of revolution? What is its centripetal acceleration? What is the car's maximum northward velocity?

Ans: 0.15 rad/sec; 0.0239 rev/sec; 9 ft/sec^2; 60 ft/sec.

2. An automobile moving at a speed of 20 m/sec traverses a circular path of radius 120 m. What is the angular velocity of this automobile in its motion around the center of the circular path? What is its frequency of revolution? What is its centripetal acceleration? What is its maximum southward velocity?

3. A small sphere supported by a metal rod is set into circular motion in a horizontal plane. The motion of the sphere is counterclockwise when viewed from above and the radius of the circular path is 2 ft. If the sphere completes 3 revolutions every 2 sec, what is its tangential velocity? its centripetal acceleration? What is the direction of its centripetal acceleration when the tangential velocity is directed eastward? when the tangential velocity is northward?

Ans: 18.9 ft/sec; 178 ft/sec^2; northward; westward.

4. A metal sphere at the end of a slender metal rod moves at constant speed in a circular path 50 cm in radius in a horizontal plane. The sphere makes one complete revolution every 2 sec. What is the tangential velocity of the sphere? the centripetal acceleration? If the motion is counterclockwise when viewed from above, what is the direction of the centripetal acceleration when the sphere is moving directly westward? when the sphere is moving directly southward?

5. If the sphere in Prob. 3 is directly *east* of the center of its path at time $t = 0$, write expressions for the northward components of (a) its displacement from the center, (b) its tangential velocity, and (c) its centripetal acceleration, as functions of time in seconds. (Use a coordinate system like that in Fig. 1.)

Ans: $y = 2\sin 3\pi t = 2\sin 9.42t$ ft; $v_y = 18.8\cos 9.42t$ ft/sec; $a_y = -178\sin 9.42t$ ft/sec^2.

6. If the sphere in Prob. 4 is due south of the center of its path at time $t = 0$, write expressions for the northward components of (a) its displacement from the center, (b) its tangential velocity, and (c) its centripetal acceleration, as functions of time.

7. If the sphere in Prob. 5 has a mass of 1 lb, what is the magnitude of the centripetal force acting on the sphere? Write an expression for the northward component of the centripetal force as a function of time.

Ans: 5.53 lbf; $F_y = -5.53\sin 9.42t$ lbf.

8. If the sphere in Prob. 6 has a mass of 500 g, what is the magnitude of the centripetal force acting on the sphere? Write an expression for the northward component of the centripetal force as a function of time.

9. A centripetal force of 20 nt acts on a 500-g body moving in a circular path 80 cm in radius. Find the rotational frequency of the body in its motion about the center of the circle. (Treat the body as a point mass.) Ans: 1.12 rev/sec.

10. Find the rotational frequency the body in Prob. 9 would have if the radius of the circle were reduced (a) to 40 cm and (b) to 20 cm.

2. SIMPLE HARMONIC MOTION

Now let us consider the projection of uniform circular motion on a diameter of the circle. In particular, let us consider the projection of the circular motion of the particle in Fig. 1 on the y-axis of the rectangular coordinate system with origin at the center of the circle.

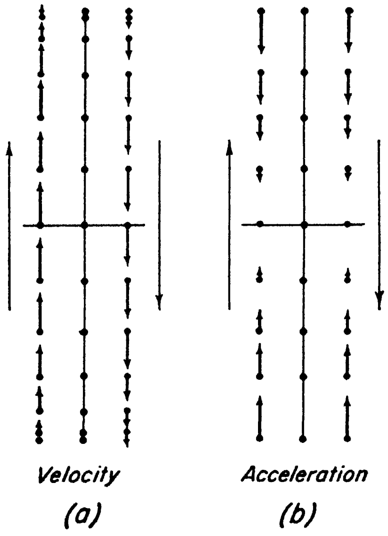


Fig. 2. Schematic representation of the projections of the velocities and accelerations on the y-axis for a particle executing uniform circular motion. Also, representation of the velocities and accelerations of a particle executing simple harmonic motion along the y-axis.

The projected coordinate y is given as a function of time by (2); the velocity and acceleration are given by (3) and (4). These projections have the values

$$\left.\begin{aligned} y &= R\ \sin 2\pi f t, \\ v_y &= 2\pi f R\ \cos 2\pi f t, \\ a_y &= -4\pi^2 f^2 R\ \sin 2\pi f t. \end{aligned}\right\} \qquad (6)$$

The projected motion can be represented by the sketch in Fig. 2, which shows the velocity v_y in part (a) and the acceleration a_y in part (b), for one complete cycle of the motion from $y=-R$ to $+R$ and back to $-R$.

Now let us consider a particle that actually *moves along the y-axis* in the manner of (6) and Fig. 2. We can think of y as giving the *displacement* of the particle from its initial position $y=0$. From (6), we note that

$$a_y = -(4\pi^2 f^2)\ y. \qquad (7)$$

Since $4\pi^2 f^2$ is a constant, the acceleration is proportional to the displacement y from the origin but opposite in direction, alternating in sign as y alternates in sign. Linear motion of the type given by (6) and (7) is called *simple harmonic motion.*

> Simple harmonic motion is a linear motion in which the magnitude of the acceleration is proportional to the magnitude of the displacement of the particle from an origin, and the direction of the acceleration is always opposite to the displacement.

In order for a particle to execute simple harmonic motion, it must experience a restoring force that is proportional to the displacement but opposite in direction, so that the restoring force always gives the particle an acceleration toward the origin. Since the origin is the position of zero force, it is called the *equilibrium position.* According to (6), the particle makes equal excursions on the two sides of its equilibrium position.

Elastic forces obeying Hooke's law have the character that gives rise to simple harmonic motion. For example, suppose that we have a steel ball of mass m supported by a Hooke's-law spring as shown in Fig. 3. At the equilibrium position of the ball, the downward pull of gravity and the upward pull exerted by the spring are equal and opposite. If we pull the ball downward a distance A *below* its equilibrium position, we do work on the ball and increase its potential energy. If we then release the ball, the spring pulls it back to its equilibrium position, and the potential energy is changed into kinetic energy. The ball continues to move upward until it is at a distance A *above* its equilibrium position and then starts downward again. In the absence of frictional effects, the ball continues to move up and down about its equilibrium position; the motion is *oscillatory* or *vibratory.* Energy relations will be considered in the next section. Here we shall consider the forces involved.

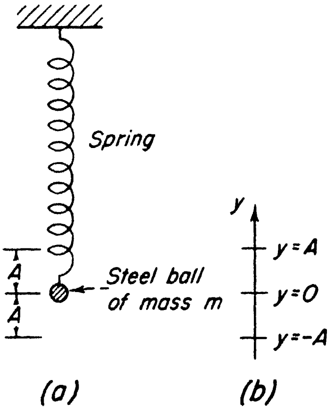


Fig. 3. A steel ball supported by a spring will execute simple harmonic motion if moved away from its equilibrium position and then released.

At the equilibrium position, the spring will be stretched sufficiently so that the upward force of the spring on the ball will be just equal and opposite to the downward force of gravity on the ball. In this position, which we call $y=0$, the net force on the ball will be zero and the ball will be in equilibrium. If now the ball is pulled *down* an arbitrary distance b (to $y=-b$), the upward force of the spring will be *increased* by kb, where k is the spring constant. Since the force of gravity is unchanged, the net force on the ball will be upward and of magnitude kb. Since this net force has a direction toward the equilibrium position, it is called the *restoring force*. If we take the restoring force F as positive upward (that is, in the $+y$-direction), we can write $F=kb$ when the ball is at $y=-b$, or, in terms of y,

$$F=-ky. \tag{8}$$

Let us see that this same formula gives the restoring force for positive values of y. Suppose that the ball is *above* its equilibrium position at $y=c$. Then the upward force of the spring on the ball will be *less* than the force at the equilibrium position by an amount kc, and the force of gravity will be unchanged; therefore, the net force will be downward, toward the equilibrium position. Since a downward force is taken as negative, we must write the restoring force in this case as $F=-kc=-ky$, as in (8).

Hence, the system of Fig. 3 has a *restoring* force proportional to the displacement from equilibrium, and gives rise to simple harmonic motion.

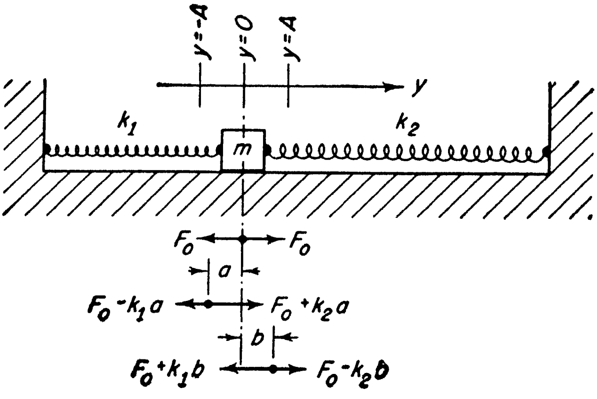


Fig. 4. A mass sliding on a frictionless horizontal plane between two stretched springs. (If one does not like the idea of a frictionless plane, he can visualize a cart with small wheels of negligible rotational inertia.)

As another example of an elastic system that gives rise to simple harmonic motion, consider the mass of Fig. 4, which slides on a frictionless horizontal plane and is attached to two *stretched* springs of spring constants k_1 and k_2. We call the equilibrium position $y=0$. At the equilibrium position, the force exerted to the right by the right-hand spring will be just equal and opposite to the force exerted to the left by the left-hand spring. The magnitudes of these forces are denoted by F_0 in Fig. 4.

If this body is moved to the left of the equilibrium position, to $y=-a$, the force of the right-hand spring to the right is *increased* by

k_2a; that of the left-hand spring is *decreased* by k_1a. The result (see diagram below Fig. 4) is a net force to the right (*restoring force*) $F = (k_1+k_2)a$. If we let $k=k_1+k_2$, the restoring force is

$$F = ka = -ky. \tag{9}$$

Again, if the body is to the right of the equilibrium position, at $y=b$, we see by a similar argument that (9) gives the correct restoring force. k is called the *effective force constant* in this case.

In general, when a body is acted on by *elastic* forces in such a way that it has a definite equilibrium position, and restoring forces come into play when it is displaced from the equilibrium position, the restoring force will be given by a relation such as (8) or (9), with an effective force constant k related to the elastic constants of the system. If the body has mass m, the restoring force will give the body an acceleration toward the equilibrium position when it is displaced from it. By Newton's second law,

$$F = m\, d^2y/dt^2 = -ky,$$

or

$$a_y = d^2y/dt^2 = -(k/m)y. \tag{10}$$

Now let us determine the frequency f of the oscillation of the particle in terms of the effective force constant k and the mass m of the particle. In order to determine f, we must solve equation (10) for y as a function of time. We note that y must be a function whose second derivative with respect to time is a negative constant times the function itself. We have already found such a function in (6). As we see from (6) and (7), the function

$$y = R_1 \sin 2\pi ft \tag{11}$$

has

$$d^2y/dt^2 = -(4\pi^2f^2)y \tag{12}$$

for any value of the constant R_1. The function

$$y = R_2 \cos 2\pi ft \tag{13}$$

also satisfies equation (12) for any value of R_2, as is seen by differentiating. The motion of our body is given by either (11) or (13) or by some linear combination of them, depending on the particular configuration specified at $t=0$. In any case the frequency f must have a value that makes (12) agree with (10). The frequency must satisfy the equation

$$4\pi^2f^2 = k/m,$$

or

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}. \tag{14}$$

This equation gives us the desired expression for f in terms of k, the force constant of the system, and m, the mass of the body.

If the ball of Fig. 3 is pulled down to $y = -A$ and released from rest at

$t=0$, or if the weight of Fig. 4 is pushed to $y=-A$ and released from rest at $t=0$, formula (11) will not satisfy the initial conditions because it gives $y=0$ at $t=0$; but formula (13) will if we take $R_2=-A$. Hence, this motion would be given by

$$\left.\begin{aligned} y &= -A\ \cos 2\pi ft, \\ v_y &= +2\pi fA\ \sin 2\pi ft, \\ a_y &= +4\pi^2 f^2 A\ \cos 2\pi ft, \end{aligned}\right\} \quad (15)$$

with $2\pi f=\sqrt{k/m}$.

We can satisfy any initial conditions $y=y_0$ and $v_y=v_0$ at $t=0$ by using the equations

$$y=R_1\ \sin 2\pi ft+R_2\ \cos 2\pi ft \quad (16a)$$

$$v_y=2\pi f\ (R_1\ \cos 2\pi ft-R_2\ \sin 2\pi ft) \quad (16b)$$

$$a_y=4\pi^2 f^2\ (-R_1\ \sin 2\pi ft-R_2\ \cos 2\pi ft)=-4\pi^2 f^2\ y, \quad (16c)$$

with $4\pi^2 f^2=k/m$. The undetermined constants R_1 and R_2 are fixed by setting $y=y_0$ and $t=0$ in (16a) to get

$$y_0=R_2, \quad (17a)$$

and by setting $v_y=v_0$ and $t=0$ in (16b) to get

$$v_0=2\pi fR_1. \quad (17b)$$

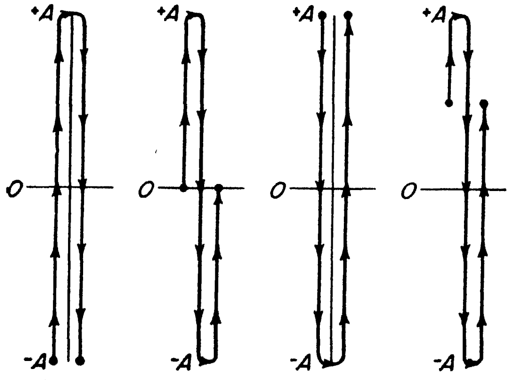


Fig. 5. A complete *oscillation*, or *cycle*, represents the execution of *all* phases of the motion, that is, a complete 'round trip.'

It can be shown that (16a) is itself a sinusoidal curve of the form

$$y=R_1\ \sin 2\pi ft+R_2\ \cos 2\pi ft=A\ \cos(2\pi ft+\phi) \quad (18a)$$

with *amplitude* $$A=\sqrt{R_1^2+R_2^2}, \quad (18b)$$

and *phase angle* $$\phi=\arctan(R_1/R_2). \quad (18c)$$

We shall consider these relations in great detail in Chap. 39 in connection with alternating electric currents, where they will play a fundamental role in our study.

The *frequency* f of a simple harmonic motion that takes place under the action of a resultant force of type (8) is given by (14). The *period* is given by

$$T = 1/f = 2\pi\sqrt{m/k}. \tag{19}$$

The frequency is the number of complete oscillations or cycles per unit time and the period is the time required for a complete oscillation or cycle. It should be noted that by a complete oscillation or complete cycle of a particle executing simple harmonic motion we mean a complete 'round trip' of the particle; thus, if the particle in Fig. 5(a) has an initial displacement $y = -A$ as shown, a complete oscillation includes the upward motion through O to $+A$ and back through O to the original displacement $-A$. Other complete oscillations from other initial positions are shown schematically in parts (b), (c), and (d) of the figure. The *amplitude* A of a simple harmonic motion is the magnitude of the *maximum displacement of the particle from its equilibrium position.*

PROBLEMS

NOTE: In the following problems, consider the mass of the springs as negligible.

1. A force of 10 lbf produces an elongation of 0.25 inch when applied to a certain coil spring. What is the force constant of this spring? If a block weighing 8.05 lbf supported by this spring is displaced 3 inches downward from its equilibrium position and then released, the block executes simple harmonic motion. What is the frequency of this motion? the period? the amplitude?
Ans: 480 lbf/ft; 6.98 cycles/sec; 0.143 sec; 0.25 ft.

2. If a 2-slug body were suspended from the spring in Prob. 1 and were displaced 1 inch upward from its equilibrium position and then released, what would be the frequency, period, and amplitude of the resulting simple harmonic motion?

3. When a body of 2-kg mass is suspended from a spring, the spring experiences an elongation of 4 cm. What is the force constant of this spring? If the body is displaced 2 cm upward from its equilibrium position and then released, the body executes simple harmonic motion. Find the amplitude, frequency, and period of the motion. Ans: 490 nt/m; 0.02 m; 2.49 cycles/sec; 0.401 sec.

4. If a 4-kg block were suspended from the spring described in Prob. 3 and were displaced 1 cm downward from its equilibrium position and then released, what would be the amplitude, the frequency, and the period of the resulting simple harmonic motion?

5. Write equations for the upward displacement y, the velocity v_y, and the acceleration a_y as a function of time for the body described in Prob. 1.
Ans: $y = -0.25\cos(43.8t)$ ft; $v_y = 11.0\sin(43.8t)$ ft/sec; $a_y = 480\cos(43.8t)$ ft/sec^2.

6. Write equations for the displacement, velocity, and acceleration as functions of time for the body described in Prob. 3.

7. The masses of bodies can be compared by observing their frequencies of oscillation when supported by a spring. A body of mass 1.000 kg is supported by a spring; when this body is set into motion about its equilibrium position, the frequency is found to be 0.543 cycles/sec. When the first body is removed and a second body of unknown mass is supported by the same spring and set into motion, the frequency is found to be 1.214 cycles/sec. What is the mass of the second body? Ans: 0.200 kg.

8. If a third body of unknown mass supported from the spring mentioned in Prob. 7 has an oscillation frequency of 0.531 cycles/sec, what is the mass of the body? What is the force constant of the spring?

9. In Fig. 4, if the mass is 50 lb and the springs have force constants of 6 lbf/in and 9 lbf/in, determine the period of oscillation. Ans: 0.584 sec.

10. In Fig. 4, if the mass is 50 kg and the springs have force constants of 6 kgf/cm and 9 kgf/cm, determine the period of oscillation.

11. Show that the expression on the right of (14) has the dimensions of sec^{-1} and that that on the right of (19) has dimensions of sec. Do this for the case in which k and m are expressed in MKS units and also for the case in which they are expressed in British engineering units.

3. ENERGY RELATIONSHIPS IN SIMPLE HARMONIC MOTION

Now let us consider the energy relationships involved in the simple harmonic motion of the particle in Fig. 3 or Fig. 4. To give the body its initial displacement, we must do *work* against the restoring force. As a result of the performance of this work, the system acquires potential energy. When the body is released, this initial potential energy is converted into kinetic energy as the body moves back to its equilibrium position; as a result of its momentum at the equilibrium position, the body moves past the equilibrium position and its kinetic energy is converted into potential energy as work is done against the restoring force. Hence, the energy of the system* is alternately all in the form of potential energy when the displacement is a maximum and all in the form of kinetic energy as the body passes through its equilibrium position. However, if there are no frictional effects, *the mechanical energy* (kinetic plus potential) *is at all times constant and is equal to the initial work done on the system.* Let us now see how this fact can be used in determining the velocity of the particle at various points in the path traversed.

First, we shall write down an expression for the total external work we should have to do against the restoring force to give the body a displacement b. The external force F_{ex} we must exert is at all times equal in magnitude and opposite in direction to the restoring force exerted by the spring, or $F_{\text{ex}}=ky$, where k is the force constant and y is the displacement at any point in the process. Thus, the external force varies from zero when the body at its equilibrium position to a value kb when the body is at $y=b$. The average value (averaged with respect to distance) of the external force F_{ex} during this displacement is therefore

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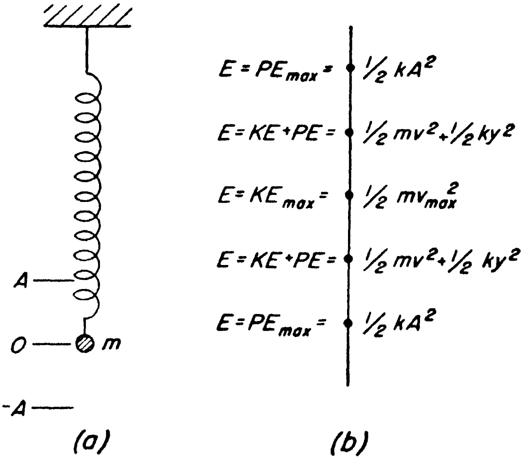
* We ignore the mass of the springs and hence assume that all kinetic energy is associated with the *body*. The potential energy in Fig. 4 is all in the form of elastic energy of the springs. The potential energy in Fig. 3 is partly elastic energy of the spring and partly gravitational potential energy of the weight, the gravitational potential energy of the spring being neglected in our treatment. A detailed analysis of the potential energy in this case is instructive and is called for in Prob. 8 at the end of this section. The treatment of these problems when the mass of the springs is not negligible compared to the mass of the body is more complex than our treatment but does not introduce any new fundamental principles.

$$(F_{ex})_{av}=\tfrac{1}{2}(0+kb)=\tfrac{1}{2}kb$$

and the work W done during the displacement is this average value times the distance, or


A
O
m
-A
$E = PE_{max} =$ $\frac{1}{2}kA^2$
$E = KE + PE =$ $\frac{1}{2}mv^2 + \frac{1}{2}ky^2$
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$E = PE_{max} =$ $\frac{1}{2}kA^2$
(a)
(b)

Fig. 6. The mechanical energy associated with simple harmonic motion is constant in magnitude, but is alternately kinetic and potential. Part (*b*) shows the range from $-A$ to $+A$ on an enlarged scale, and the expressions for K.E. and P.E. at various points in the motion.

$$W=\tfrac{1}{2}kb^2.$$

Therefore, at any position y, we may write

$$\text{P.E.}=\tfrac{1}{2}ky^2. \tag{20}$$

If the body is given an initial displacement A and is released from rest,

$$\left.\begin{aligned} &\text{initial potential energy}=\tfrac{1}{2}kA^2\\ \text{and}\quad &\text{initial kinetic energy}=0.\end{aligned}\right\} \tag{21}$$

This initial potential energy is equal to the total energy E of the system, and at all times after the body has been released the sum of the potential energy and kinetic energy of the system is equal to this total energy E:

$$\text{P.E.}+\text{K.E.}=E=\tfrac{1}{2}KA^2. \tag{22}$$

The kinetic energy of the system is $\frac{1}{2}mv_y^2$, provided the mass of the spring is negligible. Hence, when the ball has displacement y, (22) becomes

$$\tfrac{1}{2}ky^2+\tfrac{1}{2}mv_y^2=\tfrac{1}{2}kA^2. \tag{23}$$

In order to obtain the magnitude of the velocity, we may solve equation (23) for v_y; thus,

$$v_y^2=(k/m)(A^2-y^2),$$

and

$$v_y=\pm\sqrt{k/m}\,\sqrt{A^2-y^2}. \tag{24}$$

This equation gives the velocity component v_y as a function of the position y. When $y=0$, the velocity has its maximum value $\pm\sqrt{k/m}\,A$, in agreement with the value obtained from (15).

One other point should be mentioned concerning mechanical energy associated with a particle of mass m executing simple harmonic motion. Since the total energy

$$E=\tfrac{1}{2}kA^2, \tag{25}$$

we see that *the energy is directly proportional to the square of the amplitude.* Remembering that $4\pi^2f^2=k/m$, we may write

$$E=2\pi^2mf^2A^2. \tag{26}$$

Equation (26) shows that *for a given mass the mechanical energy is also proportional to the square of the frequency of oscillation.* Relation (25) is useful chiefly when we are considering a single oscillator; if we double the amplitude, we quadruple the energy. Relation (26) is useful when we compare different oscillators in which the mass m is the same; if the amplitude of two such oscillating particles is the same but the frequency of the first is twice the frequency of the second, the mechanical energy of the first is four times the mechanical energy of the second.

PROBLEMS

1. A body of mass 0.5 slug is supported by a spring with force constant 120 lbf/ft. If the body is executing simple harmonic motion with an amplitude of 1 inch, what is the mechanical energy associated with the motion? When is the potential energy a maximum? When is the kinetic energy a maximum? Ans: 0.417 ft·lbf.

2. What is the frequency of the harmonic motion executed by the body described in Prob. 1? What is the maximum velocity of the body? What is the kinetic energy of the body when it is 0.5 inch from its equilibrium position? What is the potential energy of the system when the body has a displacement of 0.5 inch?

3. A 2-kg body supported by a spring executes simple harmonic motion at a frequency of 3 cycles/sec with an amplitude of 4 cm. What is the maximum kinetic energy of the body? What is the maximum potential energy of the system? What is the force constant of the spring? Ans: 0.568 joule; 0.568 joule; 710 nt/m.

4. How much additional energy would have to be imparted to the system described in Prob. 3 in order to increase the amplitude of the oscillation to 8 cm? What would be the maximum velocity of the suspended body oscillating with an amplitude of

8 cm? What would be the displacement of the body when the potential energy and the kinetic energy of the system are equal?

5. A 500-g body is supported by a spring having a force constant of 150 nt/m. What maximum external force must be applied to the body in order to give it a displacement of 6 cm? How much work must be done in order to produce this displacement? If the body were released, what would be the frequency of the resulting motion? What is the velocity of the body when it is 3 cm from its equilibrium position? Ans: 9.00 nt; 0.270 joule; 2.76 cycles/sec; ±0.900 m/sec.

6. A body weighing 16.1 lbf is suspended from a spring with force constant 100 lbf/ft. By pushing this body, a man gives it a velocity of 4 ft/sec when it passes through its equilibrium position. What is the kinetic energy of the body at its equilibrium position? What will be the amplitude of the body's oscillatory motion about its equilibrium position? At what position will its velocity be 2 ft/sec?

7. Substitute y and v_y from (15) in (23), and, using (14), show that an identity is obtained.

8. If a body of weight w is hung from a vertical spring, the spring is lengthened by $l=w/k$ when the body is in its equilibrium position. The equilibrium position is not the position where the spring exerts no force, but is the position where it exerts upward force w.

(a) Show that if the body is pulled a distance b below its equilibrium position, the elastic potential energy increases by $wb+\frac{1}{2}kb^2$; the gravitational potential energy decreases by wb; so the total potential energy increases by $\frac{1}{2}kb^2$.

(b) Show that if the body is pushed a distance c above its equilibrium position, the elastic potential energy decreases by $wc-\frac{1}{2}kc^2$; the gravitational potential energy increases by wc; so the total potential energy increases by $\frac{1}{2}kc^2$.

(c) Draw typical curves showing elastic potential energy, gravitational potential energy, and total potential energy, as functions of the position of the body. Show that the total potential energy has a minimum value at the equilibrium position.

4. TORSIONAL OSCILLATION; ANGULAR SIMPLE HARMONIC MOTION

As might be expected, there is a rotational analogue to the simple harmonic motion of a particle. Consider a body such as the disk shown in Fig. 7. The body is suspended by a vertical wire that obeys Hooke's law for torsional deformations. If the body is rotated through an angle θ from its equilibrium position, the wire exerts a *restoring* torque proportional to θ which we can write as

$$L=-c\theta. \tag{27}$$

The restoring torque produces an angular acceleration $\alpha=d^2\theta/dt^2$ of the body, given by the relation

$$L=I\alpha=-c\theta,$$

where I is the moment of inertia of the body. Hence,

$$\alpha=d^2\theta/dt^2=-(c/I)\theta. \tag{28}$$

The mathematical form of (28) is identical with that of equation (10). Hence, we may immediately write down a solution to the problem of the motion when the body is released from rest at angle Θ from equilibrium.

In analogy with (15),

$$\left.\begin{aligned}\theta&=\Theta\cos 2\pi ft,\\ \omega=d\theta/dt&=-2\pi f\Theta\sin 2\pi ft,\\ \alpha=d\omega/dt&=-4\pi^2f^2\Theta\cos 2\pi ft=-(4\pi^2f^2)\ \theta.\end{aligned}\right\}\quad(29)$$

The frequency f is given by the relation $4\pi^2f^2=c/I$, or

$$f=\frac{1}{2\pi}\sqrt{\frac{c}{I}}.\quad(30)$$

These equations are like those for linear simple harmonic motion, except that all quantities involved in translational motion have been replaced by their rotational analogues; the motion is called *rotational* or *angular simple harmonic motion.* The maximum angle Θ is called the *amplitude* of the angular simple harmonic motion.

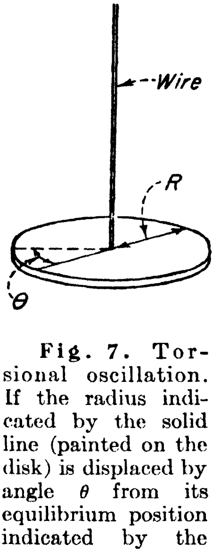

Fig. 7. Torsional oscillation. If the radius indicated by the solid line (painted on the disk) is displaced by angle θ from its equilibrium position indicated by the broken line, the wire exerts the restoring torque $L=-c\theta$, tending to return the disk to its equilibrium position.

Let us compare the expressions for frequency given in equations (14) and (30). It will be noted that the quantities k and c in the numerator under the radical are both elastic constants or *stiffness factors,* whereas the quantities m and I appearing in the denominator serve to measure inertia and hence may be called *inertia factors.* The essential relationship given by (14) and (30) is quite general. For harmonic motion of any simple type, it is possible to express the frequency by a relation of the type

$$\text{frequency}=\frac{1}{2\pi}\sqrt{\frac{\text{stiffness factor}}{\text{inertia factor}}},$$

where a stiffness factor and an inertia factor appropriate to the motion in question must be used.

We can readily write down the energy relationships for a body executing angular simple harmonic motion. The external work done on the system in giving the body its initial angular displacement Θ is equal to $\frac{1}{2}c\Theta^2$; this is the value of the initial potential energy of the system and therefore the value of the total mechanical energy E of the system. If the body is released and permitted to oscillate about its equilibrium position, the mechanical energy E of the system remains constant, provided there are no frictional effects; therefore, we may write

$$\text{P.E.}+\text{K.E.}=E,$$

or

$$\tfrac{1}{2}c\theta^2+\tfrac{1}{2}I\omega^2=\tfrac{1}{2}c\Theta^2.$$

From this equation, we may obtain the value

$$\omega = \pm\sqrt{c/I}\,\sqrt{\Theta^2-\theta^2} \tag{31}$$

for the angular velocity of the disk when its angular displacement has the value θ.

The system of Fig. 7 is frequently called a *torsion pendulum* and is said to execute *torsional oscillations.*

PROBLEMS

1. A solid cylinder of mass 0.5 slug and radius 6 inches is attached to the free lower end of a vertical wire. When an external torque of $L=2$ lbf·ft is applied, the disk turns through an angle $\theta=30°$. What is the torsion constant of the wire? If the external torque is removed, the disk oscillates about its equilibrium position. What is the frequency of the oscillation? What is the total mechanical energy associated with the oscillatory motion?

Ans: 3.82 lbf·ft/rad; 1.25 cycles/sec; 0.524 ft·lbf.

2. A slender uniform rod of mass 0.5 slug and length 4 ft is attached to the wire described in Prob. 1. The point of attachment of the wire is at the mid-point of the rod. What is the frequency of oscillation of the rod? If the amplitude of the oscillation is 30°, what is the maximum angular velocity of the rod?

3. A 1-kg solid sphere of radius 10 cm is attached to the end of a wire. If the sphere is twisted from its equilibrium position and then released, torsional oscillations result. If the frequency of the oscillatory motion is 1.5 cycles/sec, what is the torsion constant of the wire? Ans: 0.355 nt·m/rad.

4. A large wheel of mass 4 kg is attached to the end of the wire mentioned in Prob. 3, the wire coinciding with the axis of the wheel. When the wheel is twisted from its equilibrium orientation and released, it oscillates with a period of 5 sec. What is the moment of inertia of the wheel? its radius of gyration?

5. A wire with a torsion constant of 0.20 nt·m/rad supports a solid disk of mass 4 kg and radius 50 cm as shown in Fig. 7. One joule of work is done in twisting the disk from its equilibrium orientation; it is then released and permitted to oscillate. Find the amplitude of the oscillation and the maximum angular velocity of the disk.

Ans: 3.16 rad; 2.00 rad/sec.

6. Find the values of the kinetic and potential energies and the angular velocity of the system described in Prob. 5 when the angular displacement is (a) 1 rad, and (b) 2 rad.

7. The balance wheel of a watch is supposed to have a period of 1 sec when it oscillates as a torsion pendulum. If the balance wheel has mass 1.5 g and radius of gyration 3 mm, what must be the torque constant of the hairspring?

Ans: 5.33×10^{-7} nt·m/rad.

5. THE MOTION OF A PENDULUM

The rotational motion of a pendulum about a horizontal axis through its point of support approximates simple harmonic motion very closely, provided the amplitude of oscillation about the equilibrium position is sufficiently small. As the first example of pendulum motion, let us consider the motion of a simple pendulum consisting of a point mass m supported by a wire of length l and negligible mass. Such a pendulum can be approximated by the arrangement shown in Fig. 8 and consists of a small sphere of mass m, called the *bob*, supported by a light wire or thread of length l. We can best treat the simple pendulum by considering the motion as the rotational oscillation of a rigid body about a horizontal axis through O, the fixed point of support of the wire.

Considering the pendulum as a rotational motion of small amplitude Θ about a horizontal axis through O, we may write the moment of inertia I of the pendulum as

$$I = ml^2,$$

where m is the mass of the bob and l is the length of the supporting thread. The equilibrium position of the pendulum is the position in which the center of gravity of the bob is immediately below the point of support O, since when the bob is in this position gravity exerts no torque about the horizontal axis through O. If the pendulum has an angular displacement θ from the equilibrium position as shown in Fig. 8, there is a torque which tends to restore the pendulum to the equilibrium position. As the lever arm associated with the weight mg is $l \sin\theta$, the magnitude of the torque is $mgl \sin\theta$. For small angular displacements, $\sin\theta$ may be replaced by θ itself measured in radians. Hence, for small values of θ, we may write

$$L = -mgl\theta \tag{32}$$

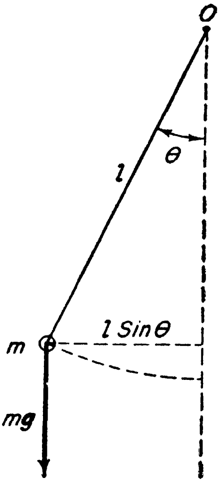

Fig. 8. The simple pendulum.

for the torque, where the negative sign is used because the torque has a direction opposite to that of the displacement θ. This torque gives the pendulum an angular acceleration α given by the relation

$$I\alpha = -mgl\theta$$

or

$$\alpha = d^2\theta/dt^2 = -(mgl/I)\theta. \tag{33}$$

Equation (32) is identical in form with equation (28). Therefore, we conclude that the angular motion of the pendulum is similar to that of the body in Fig. 7. The frequency f of the simple pendulum is therefore given by (30), with c replaced by mgl, as

$$f = \frac{1}{2\pi}\sqrt{\frac{mgl}{ml^2}} = \frac{1}{2\pi}\sqrt{\frac{g}{l}}\,. \tag{34}$$

Using this value for the frequency, we may use equations (29) to determine the angular displacement θ, angular velocity ω, and the angular acceleration α of the pendulum. By multiplying these quantities by l, we may determine the tangential displacement, velocity, and acceleration of the bob in its path about the equilibrium position.

With the same restriction to small amplitudes as above, we may also consider the motion of the pendulum as a simple harmonic motion of the x-coordinate of the *bob* about the equilibrium value $x=0$. From Fig. 9 we see that $x=l\sin\theta$, which for small angles can be written $x=l\theta$. To this approximation, $d^2x/dt^2=l\,d^2\theta/dt^2$, and substitution in (33) gives

$$d^2x/dt^2=-(mgl/I)x=-(g/l)x. \qquad (35)$$

Equation (35) is identical with (10) if we identify g/l with k/m, and hence we conclude that *for small oscillations the x-coordinate of the pendulum bob executes simple harmonic motion about its equilibrium value* with the frequency given by (34).

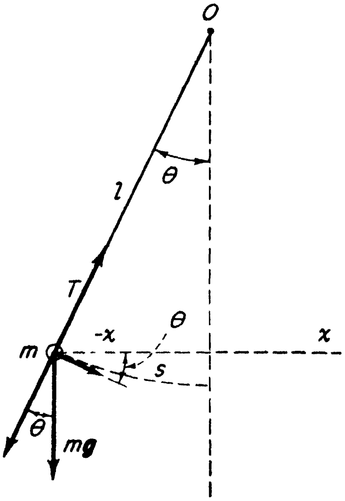

Fig. 9. Forces acting on the bob of the simple pendulum. The force mg of gravity is shown resolved into components tangential and normal to the path.

The results we have obtained apply only to a simple pendulum executing 'small oscillations.' We immediately wonder just how small the oscillations must be in order for the equations we have developed to be applicable. In order to answer this question, we may write down the results for the period T obtained by a more exact analysis. From our approximate analysis, the value of the period $T=1/f$ is given by

$$T=2\pi\sqrt{l/g};$$

an exact treatment gives the series

$$T=2\pi\sqrt{l/g}\,[1+\tfrac{1}{4}\sin^2(\tfrac{1}{2}\Theta)+\tfrac{9}{64}\sin^4(\tfrac{1}{2}\Theta)+\cdots], \qquad (36)$$

where Θ is the amplitude. For $\Theta=20°$, the value of T obtained by our approximate analysis differs from the more exact value by less than one per cent.

The simple pendulum consisting of a point mass at the end of a weightless supporting wire is an abstraction which can be approximated by a small heavy bob supported by a fine thread but can never be completely realized. A real pendulum, such as the familiar clock pendulum, is called a *physical pendulum.* A physical pendulum executes rotational simple harmonic motion like a simple pendulum, provided the amplitude of the motion is small, and we can readily derive a general expression for its frequency of oscillation.

The drawing in Fig. 10 represents a body pivoted about a horizontal axis perpendicular to the drawing and passing through point O located at a distance r from the center of gravity C of the body. The equilibrium position of the body is the position in which C is directly below the axis through O. If the body has an angular displacement θ from this equilibrium position, the restoring torque has the value

$$L = -mgr\sin\theta,$$

or, for small displacements,

$$L = -mgr\theta.$$

The body therefore has angular acceleration $d^2\theta/dt^2$ given by

$$I\,d^2\theta/dt^2 = -mgr\theta \quad \text{or} \quad d^2\theta/dt^2 = -(mgr/I)\theta,$$

where I is the moment of inertia of the body about the axis through O. This equation is similar to equation (33), and hence the frequency of oscillation is given by

$$f = \frac{1}{2\pi}\sqrt{\frac{mgr}{I}}. \qquad (37)$$

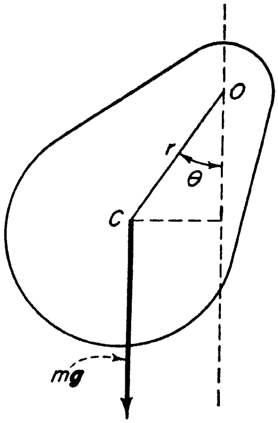

Fig. 10. The physical pendulum.

As an example of a pendulum of this type, let us find the frequency of oscillation of a meter stick pivoted at one end as in Fig. 11. Let l be the total length of the stick; then $r = \frac{1}{2}l$ and $I = \frac{1}{3}ml^2$. Hence, the oscillation frequency is given by

$$f = \frac{1}{2\pi}\sqrt{\frac{\frac{1}{2}mgl}{\frac{1}{3}ml^2}} = \frac{1}{2\pi}\sqrt{\frac{3g}{2l}}.$$

Taking $g = 9.81\ \text{m/sec}^2$, $l = 1$ m, we obtain

$$f = \frac{1}{2\pi}\sqrt{\frac{3\times 9.81\ \text{m/sec}^2}{2\times 1\ \text{m}}} = 0.610\ \text{cycle/sec}.$$

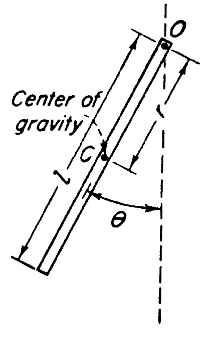

Fig. 11. A slender stick pivoted about a transverse axis through point O, which is supposed to be at the very end of the stick.

It might be noted that if g is known, the moment of inertia I of a body of mass m and any shape whatever can be determined by suspending the body and allowing the body to swing as a pendulum about an axis through some point O. By noting the frequency of oscillation, the moment of inertia I about the axis can be obtained from (37). The moment of inertia I_C about a parallel axis through the center of gravity C can be determined by the parallel-axis theorem

$$I_C = I - mr^2,$$

where r is the perpendicular distance between the axes.

The pendulum offers a precise and convenient method of measuring the acceleration of gravity without resorting to free-fall experiments. The geometry of the pendulum and the period T can be accurately measured. Properly conducted pendulum measurements give absolute values of g with high precision; the accurate values given on p. 110 are based on pendulum data. The pendulum has been used extensively in geophysical methods of exploration for petroleum and for mineral deposits.

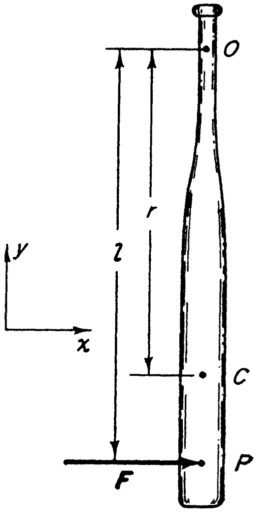

Fig. 12. A baseball bat. A force $\boldsymbol{F}$ acting along a line through P produces no translational acceleration at O, if P is the center of oscillation of the bat when it is arranged to oscillate like a pendulum about an axis through O.

It is always possible to find an *equivalent simple pendulum* whose frequency is equal to that of any given physical pendulum. Thus, if we use l to denote the length of the equivalent simple pendulum, we may set

$$(1/2\pi)\sqrt{g/l} = (1/2\pi)\sqrt{mgr/I}$$

which leads to the expression

$$l = I/mr \qquad (38)$$

for the length of the equivalent simple pendulum. Thus, so far as the frequency of oscillation of a pendulum is concerned, the entire mass can be considered to be concentrated at a point whose distance from the axis of rotation is given by $l = I/mr$. This point is sometimes called the 'center of oscillation' of the pendulum. The center of oscillation of the meter stick of Fig. 11 is seen to be $\frac{2}{3}$ m from the upper end. The meter stick has the same period as a simple pendulum $\frac{2}{3}$ m long.

The center of oscillation (also called 'center of percussion') has an interesting property that is illustrated in Fig. 12, which shows a baseball bat pivoted so that it is free to move in a horizontal plane. If a horizontal impulse $\boldsymbol{F}\,\Delta t$ is exerted through its center of oscillation P, we shall show that the bat will start to rotate freely about O, and no horizontal force would be exerted on a pivot if the bat were pivoted at this point. By the theorem of p. 249 of Chap. 10, the center of gravity of the bat will acquire velocity $\boldsymbol{V}$ given by

$$\boldsymbol{V} = \bar{\boldsymbol{A}}\,\Delta t = \bar{\boldsymbol{F}}\,\Delta t/m.$$

The bat will acquire angular velocity ω that can be obtained by applying the equation $L_C = I_C\,\alpha$ for torque about the center of gravity:

$$\omega = \bar{\alpha}\,\Delta t = \bar{L}_C\,\Delta t/I_C = \bar{F}(l-r)\Delta t/I_C = \bar{F}\,\Delta t(l-r)/I_C.$$

Now the center of gravity C does not serve as the center of rotation because this point has a linear velocity in the x-direction. The motion of the bat can be considered as the superposition of a translation with velocity V to the right and a counterclockwise rotation about the center of gravity, considered as a fixed point, with angular velocity ω. The point that remains at rest and hence acts as the center of rotation is up the handle a distance R above C such that the velocities resulting from these two motions cancel: $V = R\omega$. Hence

$$R = \frac{V}{\omega} = \frac{\bar{F}\,\Delta t/m}{\bar{F}\,\Delta t(l-r)/I_C} = \frac{I_C}{m(l-r)} = \frac{I_C}{ml-mr}.$$

In this relation we substitute $I_C = I - mr^2$, and $l = I/mr$, to obtain

$$R = \frac{I-mr^2}{(I/r)-mr} = \frac{I-mr^2}{(I-mr^2)/r} = r,$$

which proves that the distance up to the center of rotation is r, as we stated.

The property of the center of percussion just described is familiar to baseball players. If, in striking a pitched ball, the batter hits the ball with a part of the bat too near the handle or too near the end, the bat 'stings' his hands; the sting results from forces exerted at the pivot position—the batter's hands. The batter may, however, hit the ball 'solidly' with the bat and feel no reaction at all; this is the case when the force exerted on the bat by the ball acts through the center of percussion.

PROBLEMS

1. A simple pendulum consists of a small bob of mass 100 g at the end of a thread 120 cm long. What is the moment of inertia of this pendulum about an axis passing through the point of suspension of the thread? What is the frequency of oscillation of the pendulum? Ans: 0.144 kg·m^2; 0.499 cycle/sec.

2. If the bob of the pendulum described in Prob. 1 were replaced by a bob of mass 1 kg, what would be the answers to the questions asked in Prob. 1?

3. A 'seconds pendulum,' which beats seconds, has a period of 2 sec. What should be the length of such a simple pendulum at a place where the gravitational acceleration is 9.81 m/sec^2? Ans: 0.994 m.

4. What would be the precise length of a seconds pendulum at a sea-level station at the equator? at 40° N. lat.? at the North Pole? (See Table III, p. 110.)

5. A flat circular disk of 2 ft diameter is pivoted for rotation about a horizontal axis perpendicular to the face of the disk and passing through the periphery of the disk. Find the frequency of oscillation of the resulting pendulum. What is the length of the equivalent simple pendulum? Ans: 0.738 cycle/sec; 1.5 ft.

6. The disk mentioned in Prob. 9 is pivoted for rotation about a certain horizontal axis perpendicular to the face of the disk. If the period of oscillation of the disk about this axis is 4 sec, what is the distance of this axis from the center of the disk?

7. What is the period of a pendulum formed by pivoting a meter stick so that it is free to rotate about a horizontal axis passing through the 100-cm mark? through the 75-cm mark? through the 60-cm mark? Ans: 1.64 sec; 1.53 sec; 1.94 sec.

8. Show that the three periods in Problem 7 are in the ratio $\sqrt{40} : \sqrt{35} : \sqrt{56}$ exactly.

9. If a stick of length L is mounted so as to rotate about a horizontal axis perpendicular to the stick and a distance r from the center, show that the period has a *minimum* value when $r = L/\sqrt{12} = 0.289L$.

10. A bicycle wheel has diameter 26 inches and mass 7 lb. When hung by its rim across a horizontal knife-edge 12.5 inches from the center, it executes small vibrations with a period of 1.53 sec. What is its radius of gyration (with respect to the center of the wheel)?

11. To determine the moment of inertia of a 100-lb flywheel, it is hung with the inside of its rim over a horizontal knife-edge, and found to oscillate at 0.6 cycle/sec. If the knife-edge is 11 inches from the center of the flywheel, determine the moment of inertia about the axis through the center of the wheel. Ans: 124 lb·ft^2.

12. Show that if a thin circular hoop is hung over a knife-edge, it oscillates with the same frequency as a simple pendulum of length equal to the diameter of the hoop.

6. DAMPED HARMONIC MOTION*

In our discussion thus far we have assumed that there are no frictional forces; thus, the particle executing the vibratory motion discussed in Sec. 2 will continue indefinitely the motion described by equation (18a):

$$y = A\cos(2\pi ft + \phi).$$

Actually, a pendulum or a weight on a spring does not oscillate indefinitely; rather, as a result of friction, the amplitude of the oscillation gradually decreases to zero. The motion is said to be *damped* by friction and is called *damped harmonic motion.*

In most cases of practical interest, the frictional forces are due to air resistance and to internal friction in the spring. Frictional forces of this type are not constant but are approximately proportional to the velocity of the moving body, with direction opposite to that of the velocity. We can write this frictional force (called the *damping force*) as $-\mu v_y$, where μ is an appropriate coefficient. This force must be considered along with the elastic restoring force $-ky$.

Thus, for the body supported by the spring described in Sec. 2, the equation for the second law becomes

$$m\, d^2y/dt^2 = -ky - \mu v_y.$$

Since $v_y = dy/dt$, we may write this equation as

$$m\, d^2y/dt^2 + \mu\, dy/dt + ky = 0. \tag{39}$$

Provided the damping is not too great, the solution of this equation is

$$y = Ae^{-\mu t/2m}\cos(2\pi f't + \phi), \tag{40}$$

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* **Sections 6 and 7 may be omitted without loss of continuity.**

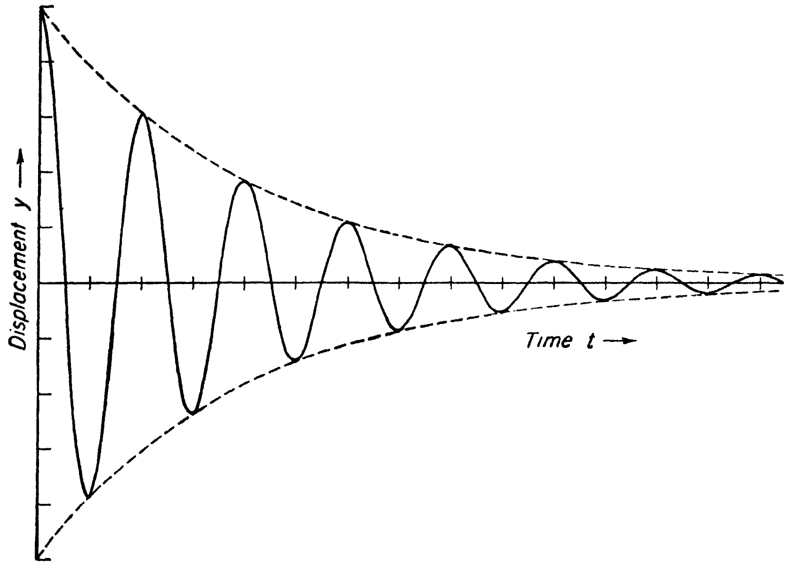

Fig. 13. A typical plot of equation (40). The dashed curve gives a plot of the decreasing amplitude factor $e^{-\mu t/2m}$.

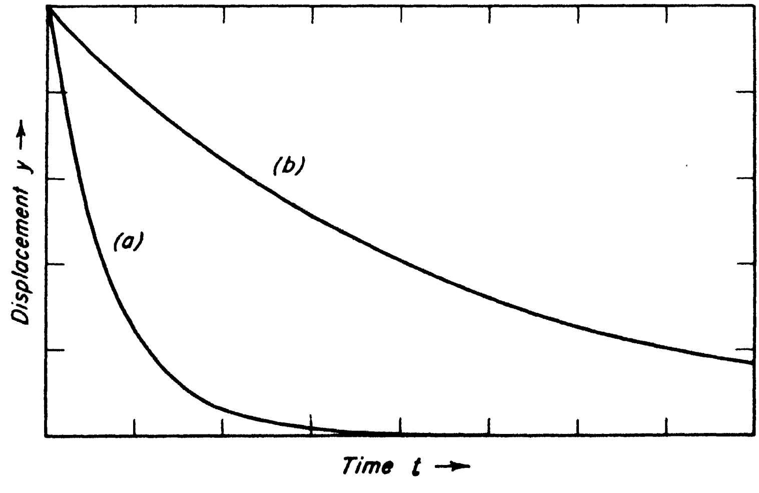

Fig. 14. Non-oscillatory motion results when $\mu^2/4m^2 \geq k/m$. Curve (a) shows displacement as a function of time for $\mu^2/4m^2 = k/m$; the oscillator is said to be *critically damped.* Curve (b) shows the motion for $\mu^2/4m^2 > k/m$; the oscillator is said to be *overdamped.* Critical damping gives the most rapid return to $y = 0$ that can be achieved without y 'overshooting' and becoming negative.

where
$$f' = \frac{1}{2\pi}\sqrt{\frac{k}{m} - \frac{\mu^2}{4m^2}}, \tag{41}$$

and ϕ is a phase angle determined by the initial conditions.

A plot of this motion is shown in Fig. 13. The amplitude of the motion decreases exponentially with time. If the coefficient μ is zero, equations (40) and (41) reduce to equations (18a) and (14), respectively. If the frictional coefficient μ is so large that

$$\mu^2/4m^2 \geqq k/m,$$

the expression for the frequency f' becomes zero or imaginary, and (40) is no longer a satisfactory solution. When this is the case, the body supported by the spring does not oscillate but merely returns to its equilibrium position when released from its initial displacement A. Plots of this motion are shown in Fig. 14.

PROBLEM

1. By direct substitution, verify that (40) is an exact solution of the differential equation (39).

7. FORCED VIBRATIONS; RESONANCE

The type of motion discussed in the preceding sections results when a pendulum or a weight supported by a spring is displaced from its equilibrium position and then left to itself. Another way in which the weight or pendulum can be set in motion is by a series of periodic impulses; a succession of small impulses, if properly timed, will produce a vibration of large amplitude. A child using a swing learns this very quickly by experiment; by 'pumping' at proper time intervals the child can set the swing into motion of relatively large amplitude.

In order to show how problems of this type may be treated, let us set up the equations of motion of a body supported by a spring as in Fig. 3 or 4 when it is subjected to an external force that is alternately in the $+y$- and the $-y$-direction. Let the y-component of the external force vary sinusoidally with time according to the expression $F_0 \cos 2\pi f''t$, where F_0 is the maximum value of the force and f'' is its frequency. The y-component of the force on the vibrating body is the resultant of this external force or *driving force*, the elastic restoring force $-ky$, and the frictional force $-\mu v_y$. Hence, the expression for Newton's second law is

$$F_0 \cos 2\pi f''t - ky - \mu v_y = ma_y$$

or
$$m\, d^2y/dt^2 + \mu\, dy/dt + ky = F_0 \cos 2\pi f''t. \tag{42}$$

The complete solution of this equation is rather involved and will not be given, but we can give a qualitative description of the resulting motion. When the driving force is applied, the amplitude of the resulting motion is small at first but gradually increases if the frequency of the driving force is close to the natural frequency of vibration of the system. After the driving force has been applied for a sufficiently long time, the system settles down to a state of *undamped harmonic motion, not at its own natural frequency but at the frequency f'' of the driving force;* a vibration of this kind is called a *forced vibration.*

The amplitude of the forced vibration of a given system depends upon the frequency f'' of the driving force and also on the frictional force; the greater the friction, the smaller the amplitude of the forced vibration. Figure 15 shows four curves giving the amplitude of the forced vibration as a function of the ratio of the frequency f'' of the driving force to the natural frequency f the system would have if there were no damping. Each of the four curves corresponds to a different value of the damping coefficient μ. Curve (a) shows the amplitude when there is no damping; in this case, the amplitude becomes infinite for $f''=f$ because energy is imparted to the system continuously by the driving force and none of this energy is dissipated. Curves (b), (c), and (d) give the amplitude of the forced vibration for systems with increasing amounts of damping; the small vertical line on each curve indicates the natural frequency f' for each system when damping effects are included.

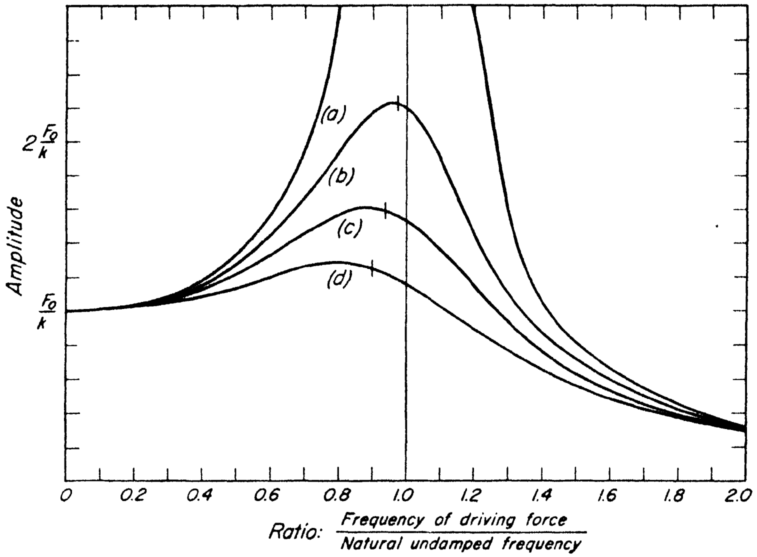

Fig. 15. Amplitude of forced vibration as a function of f''/f. (a): no damping; (b), (c), (d): increasing values of $\mu/\sqrt{km}$. Position of bar shows f'/f. The maxima of the curves give f''/f for *resonance.*

The particular frequency f'' of the driving force that gives the maximum amplitude for the forced vibration of the system is known as the *resonant frequency*. If there is no damping, the resonant frequency is the same as the natural frequency f of the system. If there is damping, the natural frequency f' of the system as given by (41) is less than the value f it would have in the absence of damping. Curves (b), (c), and (d) show that the resonant frequency is still smaller than the natural frequency f'. However, except for systems in which damping effects are large, the resonant frequency and the natural frequency f' are *nearly* the same.

The fact that forced vibrations of large amplitude (large compared to the static displacement F_0/k which the force F_0 would cause) can be produced in a system by the application of a relatively small driving force at the resonant frequency can in some cases lead to effects of great practical importance. For example, a column of marching soldiers can set a bridge into destructively large vibration if the frequency of their marching step happens to be the resonant frequency of some mode of vibration of the bridge structure. For this reason, it is customary for marching columns of soldiers to 'break step' when crossing a bridge. Similarly, a small dog can set the floor of a room into noticeable vibration by trotting across the floor at the proper gait. Another example of an undesirable effect of forced oscillation is the case of the nervous student who sets the floor of the library reading room into objectionable oscillation by tapping his foot on the floor at the proper (or improper) frequency.* The effects of forced vibration are not always undesirable, as we shall see later in our discussion of sound.

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* NOTE: Intentional repetitions of this 'experiment' are not encouraged by librarians.

CHAPTER 12

FLUID DYNAMICS

In Chap. 4 we discussed the laws of fluids at rest—*fluid statics.* In the present chapter we shall give a brief introduction to the behavior of fluids in motion—*fluid dynamics.* This is a subject of considerable importance, since it furnishes the foundation material for *hydrodynamics* and engineering *hydraulics,* which are concerned with the motion of water, and for *aerodynamics,* which is concerned with the motion of air and with the motion of bodies such as airplanes, rockets, and shells through the air.

The discipline of fluid dynamics is mathematically and physically very complex. Certain basic theorems, such as Bernoulli's theorem, are satisfied by real fluids only when velocities are small and velocity gradients low. During the years, more or less satisfactory empirical corrections to the results given by these theorems have been developed in order to reach agreement with the behavior of real fluids. Only since the development of interest in powered flight in the twentieth century has an intensive effort been made to develop the theory to account satisfactorily for the behavior of *real* fluids. An example of a subject for which *no* satisfactory theory yet exists and on which intensive experimental and theoretical work is presently under way is *turbulence* in the motion of fluids. Another subject under intensive development is that of *supersonic flow*—motion at speeds greater than that of sound. Supersonic flow is, both physically and mathematically, very different from motion at subsonic speeds.

1. BERNOULLI'S THEOREM

The most useful theorem in fluid dynamics is Bernoulli's theorem, which is essentially a formulation of the law that work equals change in mechanical energy in the case where mechanical energy is conserved, that is, in the case where the heat generation by viscous friction is negligible. (We shall discuss the fundamentals of viscous friction in Sec. 2.)

We shall derive this theorem for the important special case where the flow is *steady,* and where the change in density of the fluid can be neglected. The latter assumption, usually referred to as the assumption of *incompressibility,* is entirely valid for the flow of liquids such as water, and gives results that are correct to an excellent approximation for flow of a gas such as air up to speeds of about 350 mi/hr.

By *steady flow*, we mean that the velocity and other characteristic properties of the fluid at a given point of space are constant, independent of time. Thus in steady flow through the pipe of Fig. 2, the velocity at a given point in the pipe is constant in time, so that a definite constant volume of fluid passes a given section in the pipe in each unit of time. In order that the fluid should not 'pile up' anywhere in the pipe, or leak out, the same volume per unit time must pass every section in the pipe. The same volume per unit time enters one end of the pipe as leaves the other.

In more general steady flow, such as the flow of air around a model in a wind tunnel, we can represent the velocity directions by means of *streamlines*, as in Fig. 1. At each point x, y, z of space the velocity has a definite magnitude and direction which do not vary with time. The streamlines of Fig. 1 are drawn in such a way that at each point the velocity vector is tangent to the streamline. A tubular region whose generators are streamlines, as indicated near the top of Fig. 1, is called a *stream tube*. Because of the way in which streamlines were defined, the fluid contained within a particular stream tube remains within this tube, never crossing the boundary, because at no point does the fluid have a velocity component across the boundary of a stream tube. Hence the fluid in a particular stream tube behaves in all respects like the fluid flowing down the pipe of Fig. 2, and the same volume of flow crosses each section of the stream tube in each unit of time (if the density of the fluid does not change).

Now let us consider the forces that act on a definite quantity of fluid located within a stream tube, such as the quantity marked out by heavy solid lines in Figs. 1 and 2. On the 'ends' act forces $p_1\,dA_1$ and $p_2\,dA_2$ arising from the pressure of the adjoining fluid. Similarly, on the 'sides' act normal forces arising from the pressure of the adjoining fluid in Fig. 1, or from the walls of the pipe in Fig. 2. *If the viscosity of the fluid is neglected*, there will be no tangential forces on these surfaces. In addition, there will be the force of gravity.

We shall now compute the work done on this quantity of fluid in the time interval dt and equate this work to the increase in potential energy in the field of gravity plus the increase in kinetic energy. In time dt, one end of our portion of fluid moves distance dl_1, the other end dl_2, related in such a way that $dl_1\,dA_1 = dl_2\,dA_2 = dV$, where dV is the volume of fluid passing any point in the stream tube in time dt. The work done by the pressure p_1 is $p_1\,dA_1\,dl_1 = p_1\,dV$; that done by p_2 is $-p_2\,dA_2\,dl_2 = -p_2\,dV$, negative because the force is opposite to the direction of the motion as the fluid moves from the position shown by the solid outline to that shown by the broken outline. The pressures on the sides of the tube do no work because they act perpendicularly to the motion. Hence,

$$\text{work done} = (p_1 - p_2)\, dV \qquad (1)$$

in the time *dt*.

Now we compute the change in energy in the same time *dt*. Here the assumption of steady flow greatly simplifies the discussion, since in steady flow the energy of the fluid that happens to occupy any given region of space at one time is the same as the energy of the fluid that occupies this same region of space at a later time. Hence, the fluid that occupies a given region of space, such as the region denoted by ①, ②, or ③ in Figs. 1 and 2, has an energy that is characteristic of that particular region. Initially, the fluid bounded by solid lines has energy characteristic of regions ①+③. After time *dt*, the same fluid, which has moved into the space outlined by broken lines, has energy characteristic of regions ②+③. The change in energy is the energy characteristic of region ② minus the energy characteristic of region ①. Let us denote the density (mass per unit volume) of the fluid by ρ, the speeds in regions ① and ② by v_1 and v_2, and the vertical heights of regions ① and ② by z_1 and z_2. (In the final analysis, the tube of Fig. 1 is supposed to have infinitesimal area, so that the assignment of a single speed and height to these regions is justified. The speeds and heights assigned in Fig. 2 are some kind of average over the cross section.) The fluid in region ② has potential energy $\rho\, dV\, gz_2$ and kinetic energy $\frac{1}{2}\rho\, dV\, v_2^2$. The fluid in region ① has potential energy $\rho\, dV\, gz_1$ and kinetic energy $\frac{1}{2}\rho\, dV\, v_1^2$. Hence,

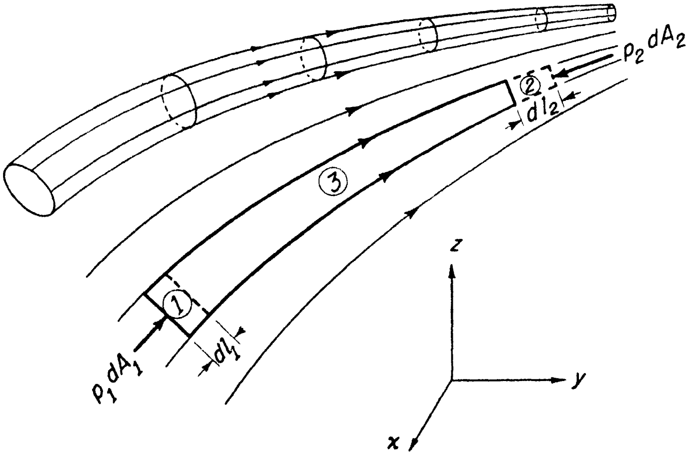

Fig. 1. Streamlines and stream tubes in fluid flow.

$$\text{change in energy} = \rho\, dV\, gz_2 + \tfrac{1}{2}\rho\, dV\, v_2^2 - \rho\, dV\, gz_1 - \tfrac{1}{2}\rho\, dV\, v_1^2 \qquad (2)$$

in time dt. Equating work done (1) to change in energy (2) and dividing by dV gives

$$p_1 - p_2 = \rho g z_2 + \tfrac{1}{2}\rho v_2^2 - \rho g z_1 - \tfrac{1}{2}\rho v_1^2,$$

or

$$p_1 + \tfrac{1}{2}\rho v_1^2 + \rho g z_1 = p_2 + \tfrac{1}{2}\rho v_2^2 + \rho g z_2. \tag{3}$$

The equality in (3) expresses the theorem first stated by Daniel Bernoulli in 1738:

Bernoulli's Theorem: *At any two points along the same streamline in a nonviscous, incompressible fluid in steady flow, the sum of the pressure, the kinetic energy per unit volume, and the potential energy per unit volume has the same value.*

In applying (3) we must use a consistent system of units, such as lbf, ft, slug, and sec; or nt, m, kg, and sec; for force, length, mass, and time. A fluid at rest is a special case of steady flow, and in this case (3) is seen to embody the law of pressure change with height.

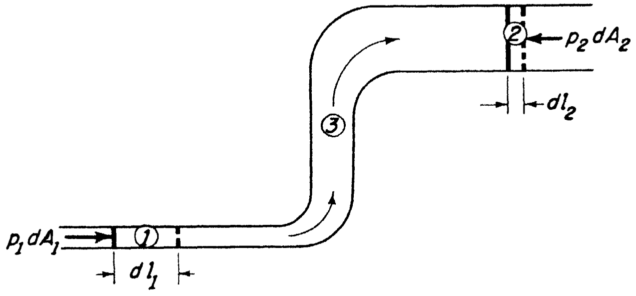

Fig. 2. Flow in a pipe of variable cross section.

In the case of a fluid that has frictional forces acting, particularly in the case of flow through pipes as in Fig. 2, some of the work done goes into heat rather than into increase in mechanical energy. In this case the equality (3) will not hold, but we see that we can definitely say that *the value of the left side of* (3) *is greater than the value of the right side, which is evaluated further downstream.*

As a first application of Bernoulli's theorem, we shall derive Torricelli's theorem for the velocity of efflux of a liquid from an orifice in a tank. Figure 3 shows three types of orifices discharging at height h below the water level in a *large* tank. We can apply Bernoulli's theorem to points ①, ②, and ③, since some streamline will connect ① and ③, also ② and ③. We shall take $z_2 = z_3 = 0$; $z_1 = h$. Since the pressure appears on both sides of Bernoulli's theorem, an arbitrary reference level for pressure may be used. Normally, it is most convenient to use *gauge pressure.* Hence we may set $p_1 = 0$. The gauge pressure at ③, in the stream just beyond the opening, is also zero because the sides of the stream are open to the atmosphere. The speeds v_1 and v_2 can be taken as zero if the tank is large. We desire to find the speed v at ③. Bernoulli's theorem gives

$$\rho gh = p_2 = \tfrac{1}{2}\rho v^2.$$

This equation gives

$$v = \sqrt{2gh} \tag{4}$$

for the velocity of efflux from a tank under 'head' h, or

$$v = \sqrt{2p/\rho} \tag{5}$$

for the velocity of efflux of liquid from an orifice when the tank pressure is p at the same elevation as the orifice. Equation (4) is called *Torricelli's theorem,* after Evangelista Torricelli (1608–1647), who first recognized that in Fig. 3(c) the water column should reach the level of the water in the tank. Equation (5) is useful for flow from a closed tank when the liquid is under greater pressure than just the hydrostatic head.

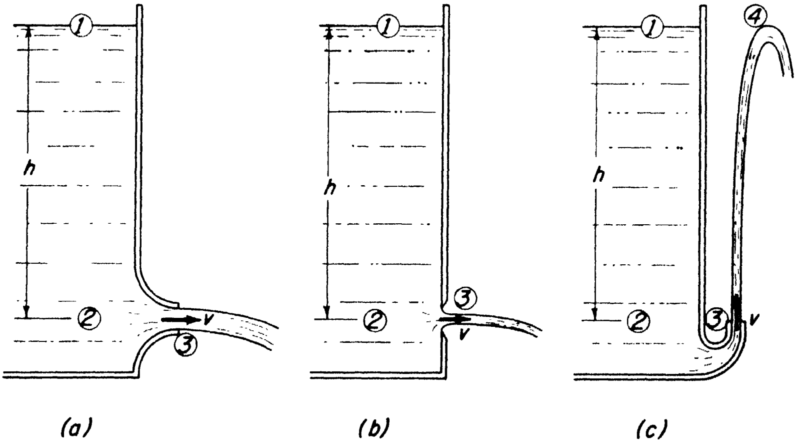

Fig. 3. Efflux from (a) a well-rounded orifice, (b) a sharp-edged orifice, showing *vena contracta,* (c) a vertical nozzle.

In the case of the vertical jet in Fig. 3(c), if we extend the application of Bernoulli's theorem to point ④, where the speed is again zero, we see that the vertical jet should reach just the height h. In practice, the jet falls a little short of h because of frictional losses. This result is in accord with our remark that the expression evaluated in Bernoulli's theorem may in actual cases decrease (but never increase) as one goes *downstream.* It will be noted that the velocity given by Torricelli's theorem is just the velocity of vertical projection required for a body to reach height h in the absence of air resistance.

To find the volume flow out of the tank, one can get a reasonably accurate result by multiplying the velocity v by the area of the opening if the opening is well rounded as in (a). But if the opening is sharp-edged as in (b), the fluid has not completed its acceleration by the time it passes through the opening. It continues to accelerate for a short distance in the jet, and, as the fluid speeds up, the jet of course contracts because the product of velocity by area must remain constant. The jet takes on a characteristic shape called the *vena contracta.* Torricelli's theorem gives the velocity after the contraction is complete. For a sharp circular opening, the contracted area is about 62 per cent of the area of the opening, and correction for this contraction must be applied in finding the volume flow.

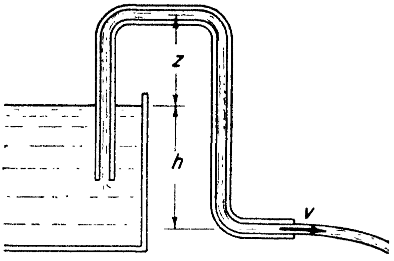

Fig. 4. Siphon.

An interesting application of Torricelli's theorem is to the *siphon* (Fig. 4). We see from arguments similar to the above that the velocity of efflux will be given by the Torricelli formula with h equal to the difference in height between the water level in the tank and the discharging orifice. The depth of the entrance to the siphon is irrelevant. Of course the siphon will not work unless it is filled with liquid. The liquid column will break at the top of the siphon if the absolute pressure there falls to zero (more rigorously, to the vapor pressure of the liquid; see Chap. 18). If we denote the absolute pressure at the top of the siphon by p, absolute atmospheric pressure by p_{at}, and the height of the top above the liquid level in the tank by z, and employ a tube of uniform bore so that the speed at the top is v, Bernoulli's theorem gives, by comparison of the top of the siphon with the top of the tank,

$$p+\tfrac{1}{2}\rho v^2+\rho g z=p_{\mathrm{at}}.$$

The limiting height $z_{\max}$ is obtained by setting $p=0$ to get

$$\rho g z_{\max}=p_{\mathrm{at}}-\tfrac{1}{2}\rho v^2.$$

The greater the flow velocity, that is, the greater the head h, the less the maximum permissible value of z.

In the case of flow through pipes the 'loss of head' because of friction is frequently rather important. However, Bernoulli's theorem is still fundamental and is ordinarily applied with empirical corrections for the loss of mechanical energy. As an important application of Bernoulli's theorem to flow of gas or liquid through a pipe, we shall discuss the measurement of flow by means of the Venturi flowmeter illustrated in Fig. 5. The flow velocities in Fig. 5 will be related to the cross-sectional areas by

$$v_1A_1=v_2A_2. \tag{6}$$

Bernoulli's theorem gives

$$p_1-p_2=\tfrac{1}{2}\rho(v_2^2-v_1^2). \tag{7}$$

The pressure difference p_1-p_2 is read directly on the differential manometer in Fig. 5. If we solve (6) for v_1 and substitute in (7), we obtain

$$p_1-p_2=\tfrac{1}{2}\rho v_2^2(A_1^2-A_2^2)/A_1^2. \tag{8}$$

This equation enables us to obtain the velocity of flow, and hence the volume rate of flow, merely from a measurement of the differential pressure. The Venturi flowmeter is widely applied in the measurement of flows of liquids and gases in pipe lines.

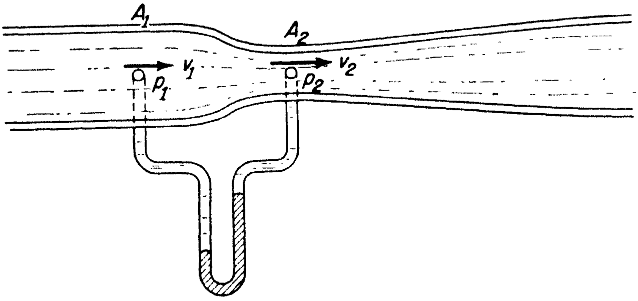

Fig. 5. Venturi flowmeter.

If the flow of liquid through a Venturi tube such as that of Fig. 5 is rapid, it is not at all difficult to get conditions in which Bernoulli's theorem gives a value for absolute pressure in the constricted section that is negative. Since a liquid will not sustain a negative pressure, a phenomenon called *cavitation* ensues, in which cavities or holes are formed in the interior of the liquid. Not only does cavitation introduce a large energy loss, but it produces serious pitting or corrosion on metallic surfaces, apparently owing to sudden collapse of the cavities. Such cavitational corrosion can cause destruction of turbine blades and ship propellers.

The low pressure attained in fluid flow through a constricted section is utilized in the design of suction pumps and atomizers.

PROBLEMS

1. Find the speed of flow of water from a tank through an orifice 15 ft below the water level. Ans: 31.1 ft/sec.

2. Find the speed of flow of water from a closed tank if the gauge pressure at the level of the orifice is 30 lbf/in².

3. Find the volume rate of flow in Prob. 1 if the orifice is well rounded, with an area of 1 in^2. Ans: 0.216 ft^3/sec.

4. Find the volume rate of flow in Prob. 2 if the orifice is a 1-inch-diameter sharp-edged circle.

5. Find the theoretical maximum value of z for a water siphon of uniform bore with $h = 20$ ft in Fig. 4, when the atmospheric pressure is 14 lbf/in^2. Ans: 12.3 ft.

6. Show that the theoretical maximum value of z for a siphon can be written as $z_{\max} = p_{at}/\rho g - h$. Find the theoretical maximum value of z for a water siphon of uniform bore with $h = 40$ ft in Fig. 4, when the atmospheric pressure is 15 lbf/in^2. Might large friction losses in a small pipe make such a siphon possibly operate with a positive z?

7. A Venturi flowmeter in a water line has $A_1 = 4$ ft^2, $A_2 = 3$ ft^2, in the notation of Fig. 5. If the differential pressure is 10 inches of mercury, what is the volume rate of flow of water? Ans: 122 ft^3/sec.

8. If it is desired that a Venturi flowmeter should read a differential pressure of 12 inches of mercury when the flow velocity of water in the full diameter of the pipe is 100 ft/sec, what should be the ratio of the diameter of the constricted section to the full pipe diameter?

9. In Fig. 5, if $p_1 = 20$ lbf/in^2 absolute and $v_1 = 100$ ft/sec, at what area ratio A_1/A_2 will cavitation ensue? Neglect the vapor pressure of the water. Ans: 1.14.

10. In Fig. 5, if the area ratio A_2/A_1 is $\frac{3}{4}$, and $p_1 = 40$ lbf/in^2 absolute, at what speed v_1 will cavitation ensue in water? Neglect the vapor pressure.

11. Neglecting friction, find the gauge pressures and speeds at ①, ②, and ③ in Fig. 6 if $h = 100$ ft, $z = 70$ ft, $A_1 = 1$ ft^2, $A_2 = 1$ ft^2, $A_3 = 0.75$ ft^2, and the water discharges into the free atmosphere at ③.
Ans: ①: 19.0 lbf/in^2, 60.2 ft/sec; ②: -11.4 lbf/in^2, 60.2 ft/sec; ③: 0, 80.2 ft/sec.

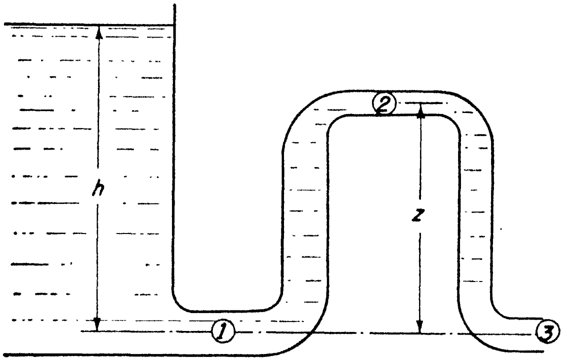

Fig. 6.

12. If water discharges in a stream $\frac{1}{2}$ inch in diameter from a faucet, with a speed of 30 ft/sec, what is the gauge pressure in the 1-inch-diameter pipe which the faucet terminates? Neglect losses.

13. When the units lbf, ft, slug, and sec are used, show that each term in (3) comes out in lbf/ft^2, and each term in (4) and (5) in ft/sec.

14. When the units nt, m, kg, and sec are used, show that each term in (3) comes out in nt/m^2, and each term in (4) and (5) in m/sec.

2. FLUID VISCOSITY*

Whenever the velocity of a fluid changes in magnitude as one moves in a direction perpendicular to the streamlines, viscous forces come into play. To visualize this situation, consider Fig. 7, in which we have two very large horizontal plates, the lower one at rest and the upper one moving in the x-direction at speed V. The space between the plates is filled with a fluid such as air or water. In such an arrangement, it would be found that a pattern of horizontal streamlines would be set up in the fluid, with the fluid velocity entirely in the x-direction, but with magnitude varying linearly from 0 to V as one progressed in the y-direction. There would be no slip of the fluid at either solid surface.

The last statement is a rigorous observational conclusion. *There is no slip between a fluid and the surface of a solid.* The velocity of the fluid immediately adjacent to the solid is always the same as the velocity of the solid surface; that is, this fluid is at rest *relative to* the solid surface. We shall discuss this point again later.

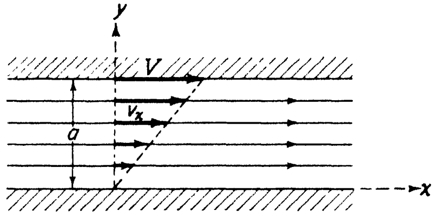

Fig. 7. A layer of fluid between a moving plate and a stationary one.

In the case of Fig. 7, the fluid in any given layer will exert a tangential stress (of the nature of a shearing stress) on the faster-moving fluid above it, tending to retard the motion of this faster-moving fluid. Reciprocally, in accordance with Newton's third law, the faster-moving fluid will exert a tangential stress in the forward direction on the slower-moving fluid below it, tending to accelerate this fluid. The fluid in contact with the upper plate will exert a backward stress on this plate, tending to retard it; whereas the fluid in contact with the lower plate exerts a forward stress on this plate, tending to drag it along with the fluid.

The magnitude of the shearing stress (force/area) we have been discussing is found to be directly proportional to the velocity gradient in a direction perpendicular to the streamlines, that is, to dv_x/dy in the case of Fig. 7. The constant of proportionality, called the *coefficient of viscosity* μ, is characteristic of the fluid. If S denotes the stress,

$$S=\mu(dv_x/dy) \tag{9}$$

for the case of Fig. 7. This law was first stated by Newton. In principle, we could measure the coefficient of viscosity with the apparatus of Fig. 7 by measuring either the force per unit area required to keep the upper plate moving, or the forward drag force per unit area on the lower plate, each of which is S. The velocity gradient in (9) is given by V/a. In practice, we cannot construct the apparatus of Fig. 7, but we can use two cylinders of large diameter with fluid in the thin annular space between. When one of the cylinders rotates, conditions in the annular space approximate the conditions in Fig. 7.

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* Sections 2 and 3 may be omitted without loss of continuity.

The dimensions of μ are seen from (9) to be those of (force·length)/(velocity·area), which, since force has dimensions of mass·acceleration, reduce to mass/(length·time). In MKS units, μ is expressed in kg/m·sec. In British units, μ is expressed in slug/ft·sec.*

Water and air are examples of fluids of very low viscosity. For water at 68° F,

$$\mu_{\text{water}} = 1005 \times 10^{-6}\ \text{kg/m·sec} = 20.9 \times 10^{-6}\ \text{slug/ft·sec}. \qquad (10)$$

The viscosity of water, like that of liquids in general, decreases rapidly with increase of temperature. The viscosity of water changes by a factor of over 6 between the freezing and the boiling points. For comparison, we might note that olive oil has a viscosity about 100 times, and castor oil a viscosity about 1000 times, that of water.

For air at 68° F,

$$\mu_{\text{air}} = 18.1 \times 10^{-6}\ \text{kg/m·sec} = 0.376 \times 10^{-6}\ \text{slug/ft·sec}. \qquad (11)$$

Unlike liquids, gases have viscosities that increase with increase of temperature. The viscosity of gases has the remarkable property of being independent of pressure. Although the density is directly proportional to the pressure as we shall see in Chap. 17, down to extremely low pressures and densities a gas still exerts the same viscous drag at the same temperature for the same velocity gradient. This seemingly startling behavior is adequately explained by the kinetic theory of gases.

Although one might conclude from the values in (10) and (11) that viscous forces would have less effect on the motion of air than on the motion of water under similar flow configurations, actually it is the other way around. The *accelerations* caused by viscous forces are proportional to the ratio μ/ρ, where ρ is the fluid density, and this ratio is 15 times as great for air as for water, so water comes closer to being an ideal nonviscous fluid than does air.

It turns out that in the flow of air around such an object as an airplane, *the viscous forces have an entirely negligible effect on the air motion except in a thin layer immediately adjoining the solid surface;* this layer is called the *boundary layer.* The coefficient of viscosity is so small that the velocity gradients occurring in the bulk of the fluid do not give rise to sufficient viscous force to have any detectable effect on the motion of the fluid. But in the immediate vicinity of the solid surface we must have sufficient viscous force to bring the fluid in contact with the surface to rest relative to the surface, for no slip occurs. Thus the air might be flowing over an airplane wing at 300 ft/sec, relative to the wing. But just at the surface the speed is zero. The layer in which the transition takes place is thin—a fraction of an inch thick in the case just mentioned. In this boundary layer, the gradients are sufficiently large that the viscous forces exert an appreciable, though not large, viscous drag on the solid surface. This is called the *skin-friction drag.*

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* In CGS units, the unit of μ, the g/cm·sec, is called the *poise*, in honor of Poiseuille, who did fundamental work in the flow of viscous fluids.

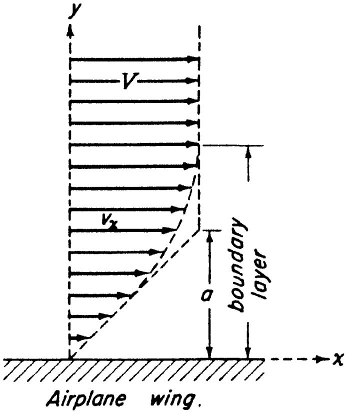

Fig. 8. Velocity distribution in a boundary layer, relative to the material surface.

Fig. 8 shows the manner in which the speed changes from the free-stream speed V to the speed 0 at the surface of the solid. We give all speeds relative to the solid surface. Such speed distributions can be measured by means of the Pitot-static tube we shall discuss in the next section. We are now in a position to compute a typical magnitude for the skin friction shearing stress on a wing or fuselage surface and verify that it is indeed small. The force per unit area is given by (9). A typical value of dv_x/dy is obtained from Fig. 8 if we take $V=300$ ft/sec (about 200 mi/hr), and $a=\frac{1}{10}$ in $=\frac{1}{120}$ ft. This gives the slope of the broken curve in Fig. 8, which is the velocity gradient at the surface of the plate, as (300 ft/sec)/($\frac{1}{120}$ ft) = 36000/sec. When we substitute this in (9) with the value of μ taken from (11), we obtain the stress

$$S=0.38\times10^{-6}\,\frac{\text{slug}}{\text{ft·sec}}\times36000\,\frac{1}{\text{sec}}=0.014\,\frac{\text{slug·ft}}{\text{ft}^2\text{·sec}^2}.$$

Since 1 lbf = 1 slug·ft/sec², this is

$$S=0.014\ \text{lbf/ft}^2=0.224\ \text{ozf/ft}^2. \qquad (12)$$

This is small indeed. Actually, on large surfaces, skin-friction stresses are of somewhat greater order of magnitude than this because the boundary layer becomes turbulent at some distance back from the leading edge and the flow no longer has the simple laminar character of Fig. 8.

The whole drag of a plane is small. If it were not possible to design a plane so this were so, flight would be impossible. The direct action of viscous forces in skin friction contributes only a small part of the total drag. We shall discuss the other contributions to the drag in the next section.

In preparation for our aeronautical discussion of the next section, we need to make one or two further observations.

Bernoulli's theorem is applicable throughout the flow pattern around an airplane except for that part of the fluid that lies in the thin boundary layer where viscous forces have an appreciable effect on the *flow* (even though they do not contribute much drag to the solid surface) and except for that portion of fluid behind the plane which *had been in the boundary layer* as it passed over the plane surface. The latter portion of fluid is called the *wake.* Bernoulli's theorem is not applicable in the wake because the fluid in the wake has a violent eddying motion not at all amenable to the simple steady-flow streamline description, like that in Fig. 1, which is essential to the derivation and application of Bernoulli's theorem. Typical wakes are indicated in Fig. 11 on p. 308.

The pressure (normal stress) *on the solid surface in Fig.* 8 *is the same as the pressure at the outer limit of the boundary layer where the speed is* V. This important fact, which is justified both experimentally and theoretically, shows that if we compute pressures for streamlines that pass just outside the boundary layer and for which Bernoulli's theorem is applicable, these pressures can be transferred to the solid surface to compute the normal forces on it. A misapplication of Bernoulli's theorem to the fluid in the boundary layer would lead to the entirely erroneous conclusion that the pressure at the solid surface, where the speed is 0, should be greater than the pressure in the free stream, where the speed is V. But *the fluid in the boundary layer has been slowed up by viscous forces,* and one cannot at all apply Bernoulli's theorem to a case where viscous forces have played a significant role. The slowing up in the boundary layer is of the nature of a *loss of head.*

As we remarked earlier, the simple form of Bernoulli's theorem given in (3), which was derived on the assumption of incompressibility, is adequately applicable to flow of air up to speeds of about 350 mi/hr or 500 ft/sec. But it is not difficult to remove this assumption and derive a somewhat more complex form of Bernoulli's theorem that takes into account the density and temperature changes of the air with change in pressure as the flow velocity changes. This derivation we are not yet prepared to give because we have not yet studied the properties of gases in detail.

3. AERONAUTICAL APPLICATIONS OF BERNOULLI'S THEOREM

If we set up an airplane model in a wind tunnel and blow air past it at a constant oncoming speed, we get a *steady* flow that is representable by streamlines as in Fig. 1 or Fig. 10 (except in the eddying wake or in the eddying flow behind the plane's propellers), and Bernoulli's theorem is applicable outside of the boundary layer and the wake.

But for a plane in flight, the flow is not at all *steady* (see definition of steady flow in Sec. 1) if we use a coordinate system attached to the ground. However, if the plane is flying at constant velocity and we use *a coordinate system attached to the plane*, then we do get a steady flow pattern which is identical with the pattern observed in a wind tunnel when the plane is standing still and air is blown past it. From the point of view of a coordinate system attached to the plane, the plane *is* standing still and air *is* blowing past it at high speed. Bernoulli's theorem is applicable to this steady flow in this coordinate system. This statement follows from an important theorem known as the

> NEWTONIAN PRINCIPLE OF RELATIVITY: If the laws of mechanics are valid in one coordinate system, they are equally valid in a coordinate system in motion at constant velocity relative to it; hence on the basis of mechanical phenomena there is no way of choosing a fundamental coordinate system, all systems being equally satisfactory provided they are not accelerated.

It is not difficult to see that if Newton's laws are applicable in one coordinate system, they are applicable in a second coordinate system moving at constant velocity relative to the first. Of course, different velocities are assigned to a particle in the two coordinate systems, but if a particle moves at *constant* velocity in one it moves at constant velocity in the other and would be considered as acted on by no forces in either case. Hence, Newton's first law is satisfactory. Although different velocities are assigned to particles in the two coordinate systems, these velocities always differ by a constant which is the relative velocity of the two systems. Hence, a particle has the *same acceleration* in the two coordinate systems, and Newton's second law assigns the *same forces*. Since the forces are the same, Newton's third law is valid in one system if it is valid in the other. This proves the Newtonian principle of relativity. It was by extension of the idea that laws of nature are the same for coordinate systems in uniform relative motion, so that no one coordinate system is more fundamental than any other, to the laws of electrodynamics that Einstein was led to the important conclusions embodied in the Einstein theory of relativity.

A coordinate system for which Newton's laws are valid is called an *inertial coordinate system*. We ordinarily use a coordinate system attached to the earth, which is moving relative to the sun at a speed of 30 km/sec (but with acceleration negligible compared to most accelerations of dynamical interest). A coordinate system attached to the earth departs from an inertial coordinate system principally because the earth is rotating. The discrepancy is small but does give rise to slight observable departures from the results we have predicted in previous chapters. For example, a bullet fired toward the north in the Northern Hemisphere has a slight deviation toward the east, and a simple pendulum suspended from a string does not oscillate in a fixed plane but rather in a plane that rotates with angular velocity of the order of one revolution per day. A pendulum designed to show this rotation is called a *Foucault* pendulum.

By this digression we have really proved rigorously that Bernoulli's theorem holds for a coordinate system attached to an airplane in flight at constant velocity—at least to the same approximation that it holds for a coordinate system attached to the rotating earth. The coordinate system attached to the plane will be employed in the discussions of aeronautical applications in the remainder of this section.

First we shall describe the *Pitot-static tube,* which is the most convenient instrument for measuring air-flow speed. One can see these tubes protruding from the front of the wing or fuselage of airplanes, where they are used to measure the speed of the oncoming air and to actuate the air-speed indicator on the instrument panel.

Figure 9 shows a diagram of a Pitot-static tube pointed in the direction of the oncoming flow. This instrument has two sets of openings to the air stream: one opening that the air strikes head-on leads into a tube called the *Pitot tube* or the *total-head tube;* another set of small openings around the periphery well back from the forward portion leads into a tube called the *static tube.* The two tubes are connected to a differential pressure gauge placed at some convenient location.

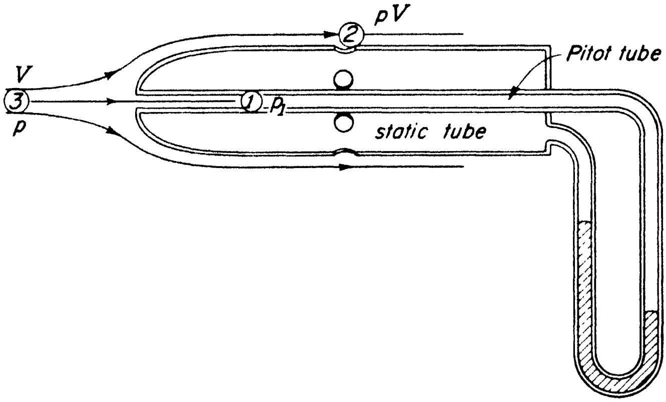

Fig. 9. Cross section of a Pitot-static tube (schematic).

To a good approximation, we can consider that the speed past the openings ②, outside the boundary layer, is the same as the oncoming speed V, and that the pressure here is the same as the free stream pressure p. According to the discussion of the previous section, the free-stream pressure will be transmitted unchanged to the solid surface, and hence to the air inside the static tube through the small holes. Hence, a gauge attached to the static tube alone would read the pressure p of the oncoming stream (called for some reason the '*static*' *pressure*), and can be so used.

In connection with the Pitot tube, apply Bernoulli's theorem to the central streamline, between points ③ and ①. Since the fluid is at rest inside the Pitot tube, this theorem gives

$$p_1 = p + \tfrac{1}{2}\rho V^2. \qquad (13)$$

Hence, the difference $p_1 - p$, as read on the differential pressure gauge, equals $\frac{1}{2}\rho V^2$. From this value one can obtain V, if the density of the air is known. On an airplane, the density may be assumed to be known from the altitude, or may be computed from observations of pressure and temperature. The plane air-speed indicator is calibrated in terms of velocity on the assumption that the air has its sea-level density, and gives what is called *indicated air speed.* Above sea level, the indicated air speed must be multiplied by a correction factor greater than unity to get true air speed. At a high altitude, the density is lower, so the speed is higher, than it would be for the sea-level calibration at the same differential pressure, that is, at the same dial reading on the air-speed indicator.

In (13), the term p is called *static pressure,* the term $\frac{1}{2}\rho V^2$ is called *dynamic pressure,* and the sum p_1 is called *stagnation pressure,* or, if expressed in equivalent feet of water, the *total head.*

Let us now consider the origin of the *lift* of an airplane wing. A wing is called an *airfoil,* and the lift is entirely associated with the characteristics of the flow pattern that is set up around a section having the distinctive characteristic shape of an airfoil, with its sharp trailing edge and greater curvature of the top surface than the bottom surface. This shape results in a greater flow velocity past the top surface of the wing than past the bottom surface, as can be seen from the pattern of streamlines in Fig. 10. As the figure shows, the oncoming air divides in such a way that more than the proper share passes above the wing and less than the proper share below the wing. This inequality results in the flow velocity past the top of the wing being greater than the oncoming velocity

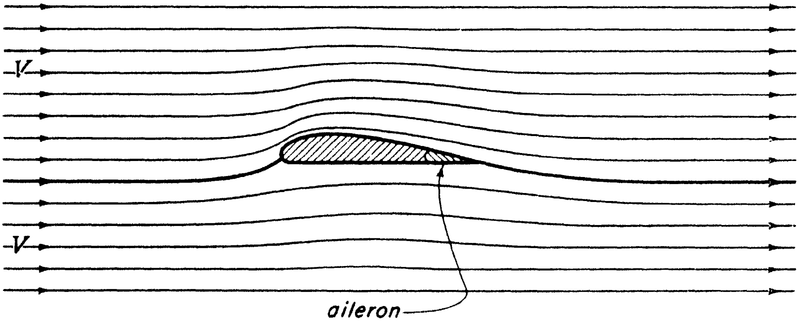

Fig. 10. Flow past an airfoil (schematic). The heavy streamline divides the flow above the airfoil from the flow below.

V, and hence, by Bernoulli's theorem, in the pressure on the top of the wing being *less* than the free stream pressure p. The reverse happens on the bottom of the wing; there the velocity is in general lower than the free stream velocity, and the pressure is greater than the free-stream pressure. The pressure difference between the top and bottom surfaces results in a lift which is proportional to ρV^2 and hence in turn is proportional to the square of the *indicated* air speed, so that an airplane flying at the same indicated air speed at any altitude has the same lift. The lift can be increased by lowering the trailing edge of the wing. Rotation of the trailing section, called the *aileron,* about a transverse axis is used to control the lift. A different amount of rotation of the two ailerons on right and left wings causes the plane to 'roll' about a longitudinal axis.

Finally, we shall discuss the *drag* of airfoils and other bodies moving through fluids. The drag of an airfoil can be considered as made up of three parts. One part is the *skin-friction drag,* which we have already discussed. This is a small part of the drag on an airplane but is the major part of the drag on a blimp or dirigible, which has an enormous surface area.

The second part of the drag is peculiar to an airfoil or to a body having *lift.* It is called the *induced drag* and is intimately associated with the lift, varying in proportion to the square of the lift for a given airplane wing. It arises because the distribution of (normal) pressures that give rise to the lift on a wing in a flow like that of Fig. 10 necessarily integrates to a resultant force that has a component backward, in the direction of V, as well as a component upward. This drag component can be shown to be unavoidable, but on modern airplane wings is a small fraction (about one-fortieth) of the lift component.

The third part of the drag is called *form drag.* It occurs for all bodies and is associated with energy communicated to turbulent eddies in the wake. This type of drag can be minimized by *streamlining,* and for the case of an airfoil, which is a good example of a well streamlined shape, is only a small fraction of the induced drag. Figure 11 shows an example of a well streamlined shape which has a small wake and low form drag, and of a poorly streamlined shape (like a rifle projectile) which has an extensive wake and relatively enormous form drag. One can see readily how this type of drag is transmitted to the body through the system of normal forces on its surface. The pressure on the leading surface is considerably higher than the free-stream pressure because the flow velocities are here lower than the stream velocity. In fact, at the *stagnation point,* marked on Fig. 11, the velocity is zero and the pressure is the stagnation pressure p_1 given by (13). (Note that the velocities we are talking about are those just outside the boundary layer—the velocity exactly at the surface is always exactly zero, but the pressure is determined by Bernoulli's theorem when we employ the velocity just outside the boundary layer.) It follows that the force on the front of the body is higher than that which would arise from free-stream pressure alone. But it turns out that for a well streamlined body the pressure on the trailing surface of the body is also considerably higher than the free-stream pressure, because the flow velocities along these surfaces are also lower than the free-stream velocity. And it can be rigorously proved that *for a nonviscous fluid*, in which there would be no boundary layer and no wake, the excess force on the rear would just balance the excess force on the front, and *there would be no drag.*

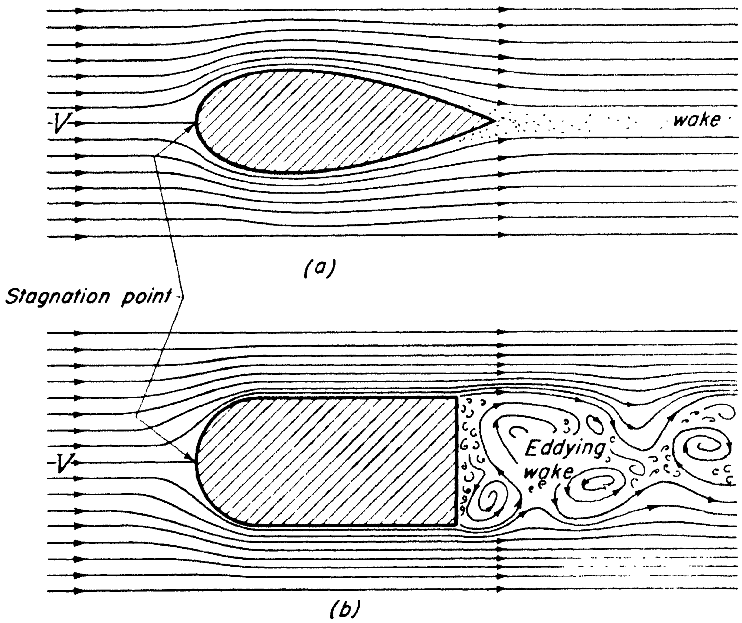

Fig. 11. Flow past (a) a streamlined body, (b) a nonstreamlined body (schematic).

Since the flow of a real fluid *outside* the boundary layer and the wake is practically identical to the flow of a nonviscous fluid, the drag of a body in a real fluid will be small if the body can be so shaped that the wake is kept small. Development of excess pressure (pressure above free-stream pressure) on the rear of a body to compensate for the excess pressure which always exists on the front of a body is called *pressure recovery*, and one can say that pressure recovery is almost complete if the wake is narrow, for then the flow over most of the body will be like the flow of a nonviscous fluid. But in the case of the bluff-tailed body of Fig. 11(b), in which the streamline flow does not continue around the rear, the pressure in the eddying wake at the rear is almost the same as the free-stream pressure. The result is that the pressure on the rear of the body has about the free-stream value whereas the pressure on the front is considerably higher than the free-stream value; the result is that the object of Fig. 11(b) has a drag 20 or more times as great as the drag of the streamlined body of Fig. 11(a) for comparable sizes and flow velocities.

A body designed to minimize form drag is said to be *streamlined.* The term arises from the fact that the attempt is to make the air flow in a streamline pattern over as much of the body surface as possible. A well streamlined body has a rounded nose, no sharp corners, and a long, gradually tapering tail coming to a sharp edge or point at the rear.

It might be noted that in *supersonic* flow, conditions are quite different. In supersonic flow a body has a form drag even for the case of an ideal nonviscous fluid. In the case of a wing section, this 'ideal' form drag is proportional to the maximum thickness of the wing, so that the important requirement is to make the wing sections as thin as possible. In supersonic flow a distinct new type of drag arises, called *wave drag.* This drag is associated with shock waves set up at the leading edge or point of the body. (A shock wave is a *sudden* pressure change propagated outward with a speed related to the speed of sound. The short, sharp shock from an explosion is a good example of a shock wave. The shock wave from a projectile travelling at supersonic speed is exactly analogous to the 'bow wave' of a ship when the ship travels through the water at a speed faster than the speed of water waves.) To minimize the wave drag it is desirable to make a sharp knife-edged type of leading edge or a sharp pointed nose, the sharper the better.

PROBLEMS

1. If an air-speed indicator is calibrated for an air density of 1.40 kg/m^3 and the plane flies at altitude 33,000 ft, where the air density has one-third this value, what is the true air speed when the indicated air speed is 220 mi/hr? Ans: 381 mi/hr.

2. If the air-speed indicator on a Navy plane is calibrated for sea-level air density and the plane flies at altitude of 10,000 ft, where the air density is 74 per cent of the sea-level density, what is the true air speed when the indicated speed is 200 knots?

3. What are the static pressure, the dynamic pressure, and the stagnation pressure, in atmospheres, in an air stream of pressure 1 atm, density 1.29 kg/m^3, and speed 360 km/hr? Ans: 1 atm; 0.0637 atm; 1.0637 atm.

4. What are the static pressure, the dynamic pressure, and the stagnation pressure, in atmospheres, in a water stream of pressure 2 atm and speed 50 ft/sec?

CHAPTER 13

GYROSCOPIC MOTION*

So far we have discussed rotational motion only in the case where there is a fixed axis or where the motion takes place in two dimensions. The general problem of rotational motion in three dimensions is extremely complex and far beyond the mathematical level of this course. However, there is one case for which a comparatively simple discussion can be given; this is the case of a top or gyroscope spinning *very rapidly* about its symmetry axis.

Within recent years, gyroscopes have been developed for the performance of very important functions in connection with the control and guidance of airplanes, ships, weapons, and missiles. For this reason we feel that it will be worth while to give a brief introduction to the behavior of tops and gyroscopes, making no attempt to give rigorous derivations of the formulas used.

1. KINEMATICS

A top or gyroscope can rotate independently about three axes. It is convenient to take these as the nonorthogonal axes shown in Figs. 1 and 2. In the familiar top (Fig. 1), the *spin axis* is the axis of symmetry around which the top spins with *spin angular velocity* ω_s. The *precession axis* is the vertical axis. As the spin axis moves around the vertical axis in a cone of semiangle θ, the top is said to *precess* with a certain *precession angular velocity* ω_p. If the angle θ, called the *nutation angle*, changes so that the spin axis *nods* toward and away from the vertical, the top is said to nod, or to undergo *nutation*. $d\theta/dt$ is called the *nutation angular velocity* ω_n. The *nutation axis* is a horizontal axis perpendicular to the plane containing the spin axis and the

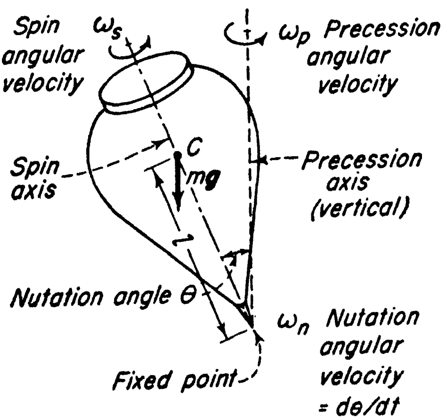
Spin angular velocity
ω_s
ω_p Precession angular velocity
C
Spin axis
mg
Precession axis (vertical)
Nutation angle θ
ω_n Nutation angular velocity = $d\theta/dt$
Fixed point

Fig. 1. Top.

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* This chapter can be omitted without loss of continuity.

precession axis. The point at the end of the top is supposed to be a fixed point in this discussion; with this restriction, any conceivable motion of the top can be considered as compounded of spin, precession, and nutation.

If we now look at the freely mounted gyroscope of Fig. 2, we can understand the system of three gimbals if we note that rotation in the bearings of each of the three gimbals corresponds precisely to one of the three modes of motion of the top discussed above. Thus the gyro wheel rotates in the bearings of the innermost gimbal at the spin angular velocity ω_s; this gimbal rotates in the bearings of the next at the nutational angular velocity ω_n; and this gimbal rotates in the bearings of the outermost fixed gimbal at the precession angular velocity ω_p.

Although the precession axis is drawn vertical in Fig. 2 to correspond to Fig. 1, the gyro of Fig. 2 is supposed to be a *free* gyro, which means that it is perfectly balanced so that gravity can exert no torques whatsoever; consequently, which way is up is immaterial, and there is no significance to taking the precession axis vertical rather than in some other direction. The center of gravity of the gyro wheel in Fig. 2 is supposed to be at its exact center of symmetry, and each of the gimbals is supposed to be perfectly balanced in frictionless bearings.

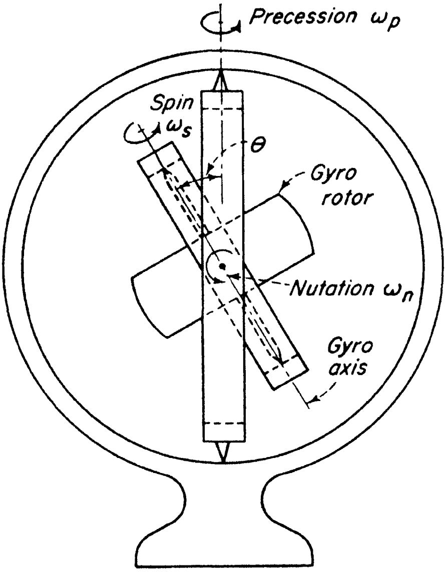

Fig. 2. Gyroscope in three-gimbal mounting.

We now note one important point which a little careful study of Fig. 2 will convince us is true, namely, that for any position of the outer fixed gimbal in Fig. 2, the gyro axis is capable of taking on any direction in space whatsoever by proper rotation of the two inner gimbals in the precessional and nutational bearings.

2. DYNAMICS: THE EQUATION OF MOTION

The top or gyroscope can have, in additior to its spin angular velocity ω_s, angular velocities of precession and nutation ω_p and ω_n. It turns out that these angular velocities can be considered as vectors directed along the respective axes and can be added vectorially. Angular momentum

H is of the nature of moment of inertia times angular velocity and is a vector. But since the moments of inertia associated with the different angular velocities are different, the expression for the angular momentum vector, which we shall not give, is rather complex. However, *in the case where the spin angular velocity is very large compared to the other angular velocities, the angular-momentum vector* $\boldsymbol{H}$ *will to a good approximation be directed along the spin axis and will have magnitude* $I\omega_s$, *where* I *is the moment of inertia about the spin axis.* The direction of the vector $\boldsymbol{H}$ is related to the sense of turning by the right-hand-screw rule which we have already discussed on p. 34 in connection with torque. This direction is illustrated in Fig. 3.

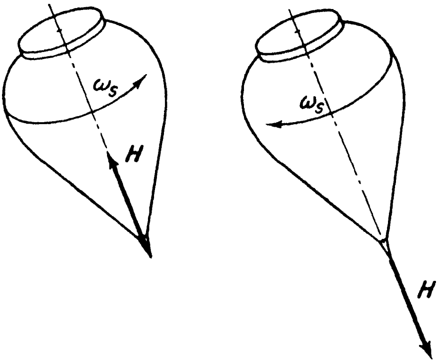

Fig. 3. Relation between direction of angular momentum vector and sense of spin.

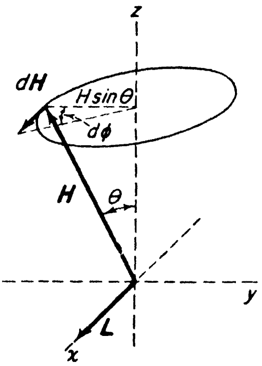

Fig. 4. Rate of precession of the top shown in Fig. 1.

If $\boldsymbol{L}$ is the torque acting on the top (about the fixed point), or on the gyro rotor (about the center of gravity), the equation of motion is the vector equation

$$d\boldsymbol{H}/dt=\boldsymbol{L}, \tag{1}$$

which says that in the time dt, the change in the angular momentum vector $\boldsymbol{H}$ has magnitude $L\,dt$ and direction the same as the direction of the torque.

As an application of relation (1), let us derive the formula for the angular velocity of precession of the familiar rapidly spinning top. Let the top be spinning in the sense of Fig. 1, so that the angular momentum is a vector along the axis in the direction shown in Fig. 4. If at the particular instant for which Fig. 4 is drawn, $\boldsymbol{H}$ lies in the yz-plane, the torque of gravity will be a vector pointing along the x-axis of magnitude $mgl\sin\theta$. Hence $d\boldsymbol{H}$ is parallel to the x-axis, which means that it is perpendicular to $\boldsymbol{H}$ and directed tangent to the circle of radius $H\sin\theta$

shown in Fig. 4. ***H*** changes only in direction, not in magnitude. Furthermore, as the end of ***H*** moves around the circle of Fig. 4, the torque ***L*** changes direction continuously so that this relation between the direction of ***H*** and the direction of $d\boldsymbol{H}$ remains continuously satisfied.

Since $dH = L\,dt = mgl \sin\theta\, dt$, in time dt the end of H turns through angle $d\phi = mgl \sin\theta\, dt/H \sin\theta = mgl\, dt/H = mgl\, dt/I\omega_s$. Since $d\phi/dt$ is the precessional angular velocity ω_p, this gives

$$\omega_p = mgl/I\omega_s \tag{2}$$

for the rate of precession. This formula is only valid provided that ω_p comes out small compared to ω_s so that the assumption that the angular momentum vector is directed along the spin axis is justified. The precession angular velocity (2) increases in proportion to the torque of gravity, and decreases in proportion to the angular momentum; consequently, the faster a given top spins, the less its precession angular velocity. It will be noted that the precession velocity is independent of the angle θ between the axis of the top and the vertical.

The above dynamical law is seen to furnish the explanations of why the torque of gravity does not tend to make a top tip over. So long as the spin angular velocity does not decrease because of friction, the torque of gravity can add increments $d\boldsymbol{H}$ of angular momentum *only* in the direction indicated in Fig. 4 and such increments do not tend to change θ. The situation is analogous to the action of centripetal force on a body moving in a circular path. The centripetal force adds increments of velocity that continuously change the direction of the velocity, but not its magnitude.

PROBLEMS

1. A top of moment of inertia 9 oz·in^2 is spinning at 25 rev/sec at an angle $\theta = 20°$. The top weighs 14 ozf and has its center of gravity 1.5 inches from its point. The spin is clockwise as seen from above. What is the angular velocity of precession of the top axis, and is it clockwise or counterclockwise as viewed from above?

Ans: 0.913 rev/sec; clockwise.

2. A top of moment of inertia 8000 g·cm^2 is spinning at 18 rev/sec at an angle $\theta = 30°$. The top weighs 900 gf and has its center of gravity 5 cm from its point. The spin is counterclockwise as seen from above. What is the angular velocity of precession of the top axis, and is it clockwise or counterclockwise as viewed from above?

3. APPLICATIONS OF GYROSCOPES

The free gyroscope, mounted as in Fig. 2, has the property that it is impossible by means of gravity, or by means of torques applied to the outer gimbal, to exert any torque whatsoever on the gyro wheel. Consequently, the angular-momentum vector remains fixed in magnitude and direction in space. No matter what motion may be given to the frame of the outer gimbal, the direction in space of the gyro axis will not vary.

Such universally mounted gyros are useful for maintaining a fixed direction of reference in a body which may be undergoing changes in direction. The axis of the steering gyro of a torpedo is initially given the desired direction of motion of the torpedo, and the gyro serves to steer the torpedo in this direction. An essentially free gyro, known as the *directional gyro,* is used to maintain constant direction of flight in connection with the automatic pilot of an airplane. Other gyroscopes are used in the rate-of-turn indicator and the artificial horizon in a plane.

Foucault first used a free gyroscope to demonstrate the rotation of the earth, making use of the fact that the axis of a free gyroscope maintains constant direction *in space.*

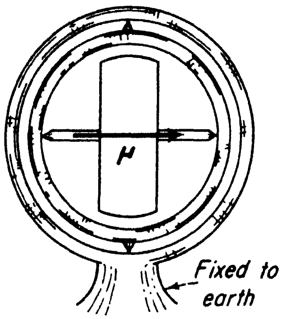

Fig. 5. Principle of the gyrocompass. The angular momentum $\boldsymbol{H}$ is constrained to remain in the horizontal plane. Torques arising from the rotation of the earth cause $\boldsymbol{H}$ to change direction until it points north.

There are many applications of gyroscopes that employ a two-gimbal mounting, so that torques can be applied to the gyro wheel about some axes and not about others. Perhaps the most important application of such a gyro is in connection with the shipboard *gyrocompass.* In the gyrocompass, the gyro axis is constrained to remain horizontal, as indicated in Fig. 5. Analysis of the torques on this gyro wheel, when fixed to the rotating earth, shows that the angular-momentum vector turns and points to the north.

Gyroscopic principles are involved in the explanation of why a rapidly rolling wheel does not tip over; how it is possible to balance a bicycle or motorcycle; how the spin of a rifle bullet prevents it from tumbling under the action of the forces of the air; and why the axis of the earth maintains a practically fixed direction in space, undergoing only a very slow precession because of the torque exerted by the gravitational attraction of the moon and the sun on the equatorial bulge of the earth (the precession of the equinoxes).

PROBLEM

1. A suitcase, with a rapidly turning gyro rotor mounted with its axis fixed horizontally parallel to the long dimension of the suitcase, is carefully handed to a porter.

(a) When the porter, in turning, attempts to turn the suitcase about a vertical axis, how does the suitcase actually move?

(b) When the porter attempts to swing the suitcase about a transverse horizontal axis, how does the suitcase actually move?

In the above show by means of vector diagrams the relation between the directions of the gyro-spin-velocity vector, the torque vector, and the vector that expresses the angular velocity acquired by the suitcase.

PART II

HEAT

CHAPTER 14

TEMPERATURE; THERMAL EXPANSION

With this chapter we begin our study of the branch of physics known as *heat,* which deals with certain phenomena that cannot be described completely in terms of the theories presented earlier under the heading of mechanics.

It is the aim of physics to develop a single consistent body of theory that is sufficiently general to permit a complete description of physical phenomena in terms of *fundamental physical quantities,* the number of such fundamental quantities being kept as small as possible. Thus, the whole of mechanics can be described in terms of *length, mass,* and *time.* For these quantities we chose quite arbitrary units—the kilogram, meter, and second. It is appropriate to take length, mass, and time as fundamental quantities because these quantities cannot be defined in terms of a simpler set of physical quantities, or in terms of each other. (Another set of *three* quantities might have been selected as fundamental; engineers sometimes prefer force, length, and time.) We then proceeded to define other useful quantities, such as *force, momentum, energy, acceleration,* and *density,* in terms of the three fundamental quantities that we had selected. Quantities defined *in terms of* the fundamental quantities are called *derived quantities.*

No difficulty was encountered in defining *all* the quantities needed for the description of mechanical phenomena in terms of just the three fundamental quantities: length, mass, and time. However, in describing *heat* phenomena we find that we need one quantity that cannot be thus defined. This quantity is *temperature.* Until about a hundred years ago still another quantity—the *amount of heat* contained in a body—had to be regarded as fundamental. Then, as we shall discuss in Sec. 1, the work of Rumford and Joule showed that heat is simply a form of mechanical energy associated with random motions of atoms and molecules, so that the amount of heat can be expressed in mechanical energy units. However, as will be pointed out in Sec. 2, *temperature* is a physical quantity of a type distinctly different from *quantity of heat,* and we still have to introduce temperature as a new *fundamental quantity.* Our definition of temperature will consist of a description of the *operations* that must be performed in its measurement.

1. HEAT IS MECHANICAL ENERGY OF RANDOM MOTION ON A MICROSCOPIC SCALE

Little progress was made in the understanding of phenomena involving heat until the truth of the statement contained in the title of this section was realized.

During the eighteenth and early nineteenth centuries, heat was generally regarded as an elastic fluid, which supposedly pervaded all matter, for which the name *caloric* was adopted. It was supposed that the transfer of heat from a hot body to a cool body consisted of a flow of caloric from the hot to the cool body. The hot body lost caloric and became cooler; the cool body gained caloric and became warmer.

The absurdity of this notion of heat as a kind of substance was strikingly pointed out by Count Rumford* in 1789 in connection with the large amount of heat developed by friction when cannon-boring machinery was operated with a blunt boring tool. Little happened except that an enormous amount of heat was created. Rumford concluded that the source of heat generated by friction appeared to be inexhaustible, so that the heat could not possibly be a material substance and must be associated somehow with 'motion.'

Nevertheless, the equivalence of heat with mechanical energy was not firmly established until 1843, when James Prescott Joule showed that there is a definite numerical proportionality between the mechanical work done (in ft·lbf) and the heat developed (in thermal units, BTU) when work is done against friction. From Joule's work it was recognized that the thermal units (kcal, BTU), which we shall define later, were just units of energy like the mechanical units (joule, ft·lbf), and were definite numerical multiples of these mechanical units. Thus, heat was related to dynamics and the study of the relations between the two subjects, called *thermodynamics*, made rapid progress in the last half of the nineteenth century with the work of Clausius, Kelvin, Maxwell, Helmholtz, Gibbs, Boltzmann, and many others.

Now let us try to visualize in more detail the meaning of the title of this section. It will be simplest to think first of a solid body whose atoms are held in a regular array by elastic forces of an electrical origin. These forces behave exactly as if the atoms of the solid were connected to each other by spiral springs, as in Fig. 1. What is your first impression when you look at this figure? That it resembles a bedspring? Well, a bedspring is a good model to visualize. Or does it appear that if you gave the array of Fig. 1 a little push you could set it into vibration like a bowl of jelly? Well, you can. Of course these springs are relatively stiff and there are a tremendous number of them, about 10^{23} per cm^3, so you would have to push pretty hard to get much effect. But a hammer blow on the solid of which Fig. 1 is a piece would definitely set it into vibration. The mechanical energy of the hammer, which apparently disappears when the hammer strikes the solid, does not actually disappear; it all goes into energy of vibration of the atoms of the solid and the atoms of the hammer face. This vibration is on a scale too small and too fast to be seen or photographed. The amplitude of vibration is of the order of 10^{-9} cm, the frequency of the order of 10^{15} per second. But we can *feel* the vibration. If we touch the solid with our finger tips, some of the vibration energy is transmitted to the atoms of our nerve endings, and the body feels *hot.* Yes, this energy of vibration *is* heat.

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* Count Rumford (1753–1814), born Benjamin Thompson in North Woburn, Mass., performed his celebrated experiments in Munich while in charge of the Bavarian arsenal. He was a loyalist at the time of the American Revolution.

Before going further we must take the opportunity to explain two useful scientific adjectives that we shall have frequent occasion to use, namely, *macroscopic* and *microscopic.* The motion of the hammer is called *macroscopic* because it can be seen and photographed. The kinetic energy of motion of the hammer is called *macroscopic mechanical energy,* or just *mechanical energy,* because it can be computed from the measured velocity of a body of *macroscopic size,* containing a tremendous number of atoms all moving together in a manner that can be seen and photographed. Any body that can be seen and photographed, even though it may be necessary to use a microscope, is called *macroscopic.* The word *microscopic* in its scientific sense is reserved for phenomena that take place on a scale much too small to be detected by any mechanical measuring instrument and too small to be seen by the best microscope, a scale in fact of the order of atomic sizes. *Random mechanical motion on a microscopic scale,* such as the vibration of the atoms of Fig. 1, *manifests itself as heat* and needs for its detection types of measuring instruments and sense organs different from those employed for detection of macroscopic motion—so entirely different that the fact that heat and macroscopic motion both involve mechanical energy of the same fundamental type was not generally recognized until 1843.

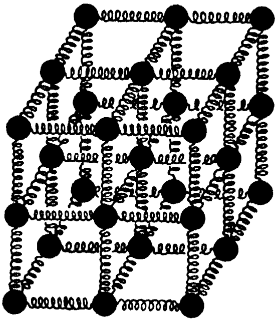

Fig. 1. The forces between atoms in a solid are like those that would be exerted by a set of springs.

Now let us return for a moment to our bedspring analogy. There is one serious discrepancy in the analogy between the vibrations of the bedspring and those of the atoms in Fig. 1. The macroscopic motion of a bedspring gradually subsides, not because the mechanical energy disappears, but because (owing to friction) it is converted from macroscopic into microscopic mechanical energy—that is, into heat. The motion of the atoms of Fig. 1, however, will not subside but will go on indefinitely unless the energy is given to some other body; it cannot be converted into heat because it *is* heat. The 'springs' of Fig. 1 are not ordinary springs with internal friction—they are electrical springs with no property analogous to friction.

The process of adding heat to a solid consists in transferring to it more and more mechanical energy, which appears in the solid as energy of atomic vibration. As a result of this increase in energy, the solid feels hotter to the touch, and we say that the *temperature* of the solid has been raised. If the temperature is raised sufficiently, the vibrations become so violent that the solid *melts* and becomes a *liquid*. In the liquid state the atoms are still held close together by large forces, so that the liquid remains a compact mass of definite volume, but the atoms (or molecules) have sufficient energy of motion to overcome the forces tending to hold them in a definite spatial arrangement and are able to move around rather freely so long as they remain close together. Raise the temperature of the liquid sufficiently and it *vaporizes* and becomes a *gas*. The atoms (or molecules) have acquired sufficient kinetic energy to overcome the forces tending to hold them close to each other. One by one they jump clear of the liquid surface and go wandering off into space at high speed, so that the gas will eventually completely fill a container of any size in which it is confined. The atoms of a gas are, on the average, so far apart that they exert negligible forces on each other. The thermal energy of a gas consists of kinetic energy of random translational motion plus, in the case of diatomic and polyatomic gases, energy of random rotation of the molecules and of vibration of the atoms in the individual molecules.

These matters will be discussed in quantitative fashion in this and the next five chapters. The above discussion and that of the next section are intended to give us a feeling for the basic mechanical explanation of the phenomena involved. The phenomena of heat—manifestations of mechanical energy on a microscopic scale—are so different from macroscopic mechanical phenomena that the connection is not at all obvious to our senses. Yet, as history has demonstrated, little progress can be made in the understanding of heat until this connection is comprehended thoroughly.

So far we have used the word *heat* in its popular non-technical sense. The word is used this way in the title PART II, HEAT of this portion of the book, and in the heading of this section. However, since the nature of thermal phenomena has become clearly understood, the word *heat* has taken on a more restricted technical meaning to which we should adhere. We shall begin our technical definitions by correcting the title of this section and thereby defining *thermal energy:*

> Thermal energy is mechanical potential and kinetic energy of random motion on a microscopic scale.

The term *heat* has the following restricted technical meaning:

> Heat is thermal energy in the process of being added to, or subtracted from, a given material substance, or in the process of being transferred from one material substance to another.

The measurement of *quantities* of heat will be considered in Chapters 15 and 19.

We note that *heat* and *thermal energy* bear the same relation to each other as *work* and *mechanical energy*. *Heat* and *work* represent energy in transition. A review of mechanical principles shows that work is macroscopic mechanical energy in the process of being added to or subtracted from a material substance. This point of view is very helpful in the discussion of thermodynamics.

2. TEMPERATURE

The sensations of 'hotness' and 'coldness' of a given body are determined by what is called its *temperature*. Add heat to a body and its temperature ordinarily rises. Subtract heat (that is, let it give thermal energy to some other body) and its temperature ordinarily goes down. For a *given* body, the quantity of thermal energy determines the temperature, but the relation between thermal energy density and temperature is different for every different substance. *Temperature is not, fundamentally, a measure of thermal energy but, rather, is a measure of ability to transfer heat to other bodies.*

To clarify this idea, consider a block of copper and a beaker of water. When the block of copper is placed in the beaker of water, one of three things will happen:

(a) The copper will cool down and the water will warm up, indicating transfer of heat from copper to water.
(b) Neither will cool down or warm up, indicating no transfer of heat.
(c) The copper will warm up and the water will cool down, indicating transfer of heat from water to copper.

(Until we have discussed *thermometers*, we can assume that we use our crude cutaneous senses, by feeling the copper and the water, to determine whether these have warmed up or cooled down.)

By *definition*, we say that before the block of copper was placed in the water, the following conditions prevailed:

In case (a), *the copper was at a higher temperature than the water.*
In case (b), *the copper and the water were at the same temperature.*
In case (c), *the copper was at a lower temperature than the water.*

In case (b), we shall find that the actual microscopic mechanical-energy contents of the copper and of the water (per unit mass, per unit volume, per molecule, or however we wish to specify it) are quite different; but at the surface of contact, the motions of the water and copper molecules are such that *on the average no energy is transferred from one substance to the other. They are in this case said to have the same temperature.*

It is in this sense that the temperature of different bodies is related to their ability to transfer energy. We shall see that this definition of temperature equality is logically consistent when we compare bodies of more than two different kinds.

Consider Fig. 2(a), showing a block of copper and a block of iron (or any other substance) immersed in a vessel of water. If we wait a long time, we can be sure that the copper and the water are at the same temperature, also that the iron and the water are at the same temperature, by the above definition of temperature equality. But are the copper and the iron then at the same temperature? Suppose that we take them out of the water and place them in contact as in Fig. 2(b). If they are at the same temperature, no energy transfer will take place. If energy transfer takes place, our definition of temperature equality is not a very useful one, because then two bodies at the same temperature relative to a third would be at different temperatures relative to each other. Let us suppose for the moment that energy transfer did take place in Fig. 2(b) and that the copper was warmed and the iron was cooled. Then in Fig. 2(c), we could use the copper to warm part of the same water, since the copper is now at higher temperature than the water, and we could use the iron to cool part of the water. This would be a fine way to heat or cool water, but the possibility of realizing such a situation is completely contrary to all experience.* Hence we must conclude, no experimental evidence to

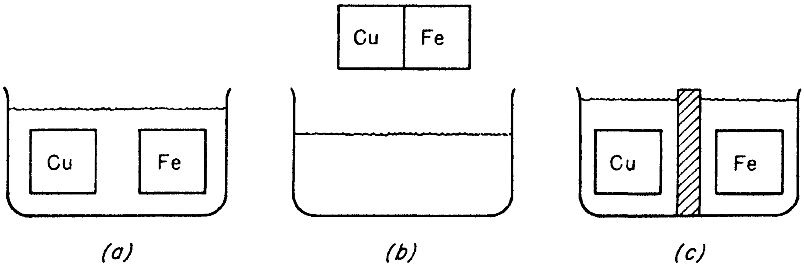

Fig. 2.

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* This body of experience is summarized in what is called the *second law of thermodynamics*, which will be discussed in Chap. 19. One would say that the above scheme for heating and cooling water is in contradiction to the second law of thermodynamics. (The *first* law of thermodynamics is merely a statement that energy is conserved when it is transformed from macroscopic mechanical form to microscopic thermal form, and vice versa.)

the contrary ever having been offered, that no net energy transfer takes place between the copper and the iron in Fig. 2(b). Continuation of this type of logical argument from experience shows that the following definitions of temperature equality and inequality are satisfactory and logically consistent:

> Two bodies have the same temperature if, when placed in contact, no heat flows from one to the other. Body A is at higher temperature than body B if, when they are placed in contact, heat flows from A to B.

A group of objects and substances within a well-insulated enclosure, such as a pot of hot coffee, a bucket of ice, and the furniture, walls, and air in a thermally insulated closed room, will transfer heat in such a way that eventually they will all come to the same temperature. At this stage *thermal equilibrium* is said to be established, and no rearrangement of the objects can result in further transfer of heat.

If the body of Fig. 1 is at room temperature, we do not have to hit it with a hammer to set the atoms into vibration. They must already be in fairly violent vibration, since they are capable of giving up a good deal of heat to a colder body, say a block of ice, with which the body is placed in contact. Hitting the body with the hammer just *adds* thermal energy and sets the atoms into *more violent* vibration. All matter possesses a certain amount of thermal energy. Matter *feels* hot or cold according to whether there is a gain or loss of heat by the skin in contact with the matter, which may be gaseous, liquid, or solid.

As temperature scales are set up, there is no theoretical upper limit to temperature; one can go on increasing the thermal energy, and hence the temperature, of a substance indefinitely. There are practical limits, of course, but it is known that the temperature in the interior of the sun is of the order of some millions of degrees. On the other hand, *there is a definite lower limit to temperature.* Starting with a body at room temperature, one cannot go on taking thermal energy away from it indefinitely. When it has given up all the thermal energy it is capable of giving up,* it has become as cold as it can get; and a logical deduction from our definition of temperature inequality will readily show that no other body can be any colder.

Any body that has given up all the thermal energy it possibly can is said to be at the *absolute zero of temperature.* It has been established experimentally that on the ordinary temperature scales,

$$\text{absolute zero} = -273.16^\circ\text{ C} = -459.69^\circ\text{ F}. \qquad (1)$$

In the laboratory it has been possible to reach temperatures within a few thousandths of a degree of absolute zero. The method of determining the value (1) will be discussed in Chap. 19.

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* One must be cautious in phrasing this statement, since the zero from which mechanical energy (and hence thermal energy) is measured is inherently arbitrary.

PROBLEM

1. (a) Show, from our definition of temperature equality and inequality, that no body can be colder than a body from which all possible thermal energy has been removed.

(b) Show that any body that is capable of transferring heat to any other body is at a higher temperature than a body from which all possible thermal energy has been removed.

(c) Show that all bodies from which all possible thermal energy has been removed are at the same temperature.

3. TEMPERATURE SCALES

Temperature is determined by measurement of some mechanical, electrical, or optical quantity whose value has a one-to-one correlation with temperature. Usually the temperature of a substance is not determined by a measurement made on the substance directly, but by measurement on an instrument called a *thermometer* that is brought into thermal equilibrium with the substance and hence acquires the same temperature.

We must now discuss the way in which numbers are assigned to temperature values, that is, the way in which *temperature scales* are defined. Our definitions will be operational in character.

In defining a temperature scale, two conveniently reproducible temperatures called the *fixed points* are used.

The lower fixed point (or *ice-point*) is defined as the temperature of a mixture of pure ice and water at standard atmospheric pressure. The upper fixed point (or *steam-point*) is defined as the temperature of steam from pure water boiling under a pressure of one standard atmosphere. (A standard atmosphere is the pressure exerted by the weight of a column of mercury exactly 76 cm high at 0° C under the pull of standard gravity, $g = 9.80665\ \mathrm{m/sec^2}$. It is a pressure of 101,325 $\mathrm{newtons/m^2}$.)

Thermometers based on the expansion of a liquid were invented early in the seventeenth century. At first they had completely arbitrary scales, so that each thermometer gave readings peculiar unto itself. The desirability of using two fixed points to standardize the readings was recognized late in the seventeenth century, and our present scales were devised during the first half of the eighteenth century.

The temperature scale used in all scientific work and in common use in many countries is the *Centigrade scale*, in which the fixed points are taken as 0° C and 100° C; hence the name of the scale. It was devised by Swedish physicians, Elvius and Linnaeus.

The common Fahrenheit scale was devised in 1714 by Gabriel Daniel Fahrenheit, scientist and scientific instrument manufacturer in Danzig and Amsterdam. On this scale the fixed points are 32° F and 212° F. Fahrenheit arrived at these figures by taking 0° as the temperature of a freezing mixture of ice and salt in his laboratory (actually a temperature as low as $-9°$ F may be obtained with the proper mixture of ice and salt), and choosing 96°, for some reason, to represent the temperature of the human body (his scale missed this value by 2.6°, but this discrepancy can be attributed to experimental inaccuracy).

A third scale, still in common use in Scandinavian countries and parts of Germany, was devised about 1731 by a Frenchman, René Antoine de Réaumur, who called the fixed points 0° and 80°. We shall not be further concerned with the Réaumur scale.

The original thermometers were of the type still in common use; they make use of the volumetric thermal expansion of liquid mercury from a reservoir into an evacuated glass capillary tube of uniform bore. The position of the liquid at the temperatures of the fixed points is determined and marked. The distance between the marks is divided into 100 equal spaces for the Centigrade scale, or 180 for the Fahrenheit scale, to give the individual degrees. Above and below the fixed points the scale may be extended by marking off degrees of the same size. This calibration is illustrated in Fig. 3.

This method of interpolating and extrapolating the scale relative to the fixed points, depending as it does on the expansion properties of a particular substance, mercury, lacks fundamental significance. Also, it is impossible to extend the scale in this way below the freezing point and above the boiling point of mercury. These difficulties were resolved by Lord Kelvin, who devised a *thermodynamic scale of temperature* that is independent of the physical properties of any particular substance. This is the fundamental scale internationally adopted for thermometer calibrations. Values still have to be assigned to two fixed points, as we have done, to define the scale. But once the fixed points are assigned, the value of any other temperature is determined. This thermodynamic scale is approximated very closely by the constant-volume gas thermometer which we shall describe in Chapter 17. The gas thermometer, employing hydrogen gas, is usable over a very wide temperature range and is employed in national standardizing laboratories for fundamental calibrations. The mercury thermometer, calibrated as

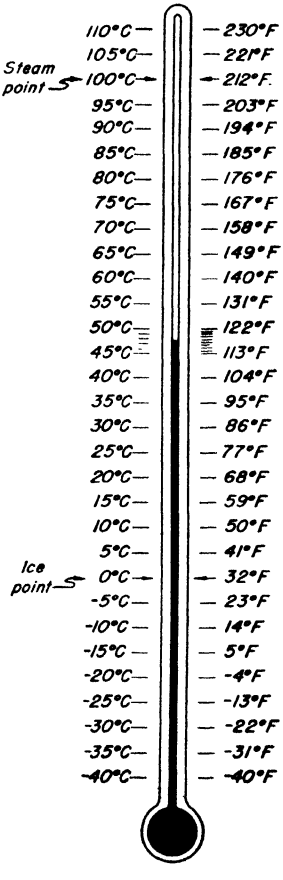

Fig. 3. Centigrade and Fahrenheit temperature scales.

above, gives readings very close to those of the thermodynamic scale, and hence the calibration we have described furnishes a satisfactory definition of temperature in the ordinary range pending a more detailed consideration of the thermodynamic scale in Chap. 19.

Mercury freezes at −39° C. Ethyl alcohol can be employed for temperatures down to −115° C. Because of the low boiling point of alcohol (78° C), the steam-point cannot be determined on an alcohol thermometer, but the alcohol thermometer can be calibrated against a mercury thermometer in the overlapping range and the scale extended uniformly to lower temperatures.

Because we have occasion to use both scales, we must learn to convert temperatures from Centigrade to Fahrenheit, and vice versa. We must also learn to convert temperature intervals or temperature changes. The latter is very simple. Because there are 100 Centigrade degrees and 180 Fahrenheit degrees between the ice-point and the steam-point, we see that

$$100 \text{ C deg} = 180 \text{ F deg},$$

so

$$\left.\begin{aligned} 1 \text{ C deg} &= \tfrac{9}{5} \text{ F deg}, \\ 1 \text{ F deg} &= \tfrac{5}{9} \text{ C deg}. \end{aligned}\right\} \quad (2)$$

and

Note that because of the way temperature scales are set up we must distinguish in notation between a temperature and a temperature difference. We write *temperatures* as °, C, or F; thus,

$$0^\circ \text{ C} = 32^\circ \text{ F},$$

$$5^\circ \text{ C} = 41^\circ \text{ F}.$$

The difference between two temperatures is not a temperature but a temperature interval. It would be nonsense to conclude, by subtracting the first of the equations immediately above from the second, that 5° C = 9° F, when we have already noted that 5° C = 41° F, but we can properly conclude that

$$5 \text{ C deg} = 9 \text{ F deg}.$$

This is an equation that relates the *sizes* of Centigrade and Fahrenheit degrees rather than relating actual temperatures. We write *temperature intervals* as C deg or F deg, and these symbols can be handled like ordinary units by using the relations in (2).

We now return to the problem of converting temperatures from one scale to the other. Rather than trying to remember a formula for this, it is best to remember the fixed points of each scale, and hence the relative size of the degrees, and to use the scheme of the following examples:

(a) ***What is the Centigrade temperature corresponding to* 68° F?** **We start by noting that**

$$68^\circ \text{ F} = 36 \text{ F deg above ice-point},$$

which, from (2), $= 36 \times \frac{5}{9}$ C deg above ice-point

$= 20$ C deg above ice-point

$= 20°$ C.

(b) *What is the Fahrenheit temperature corresponding to* $-273.16°$ C (the most accurately determined value of absolute zero)?

$-273.16°$ C $= 273.16$ C deg below ice-point

$= 273.16 \times \frac{9}{5}$ F deg below ice-point

$= 491.69$ F deg below ice-point,

which, since the ice-point is $+32°$ F,

$= -459.69°$ F.

For much of the theory of heat, it is necessary to express temperatures on an *absolute scale*, in which temperatures are measured from absolute zero. There are two absolute scales in use, one in which temperatures are measured in degrees of Centigrade size, the other in which Fahrenheit-size degrees are used.

The absolute scale in which temperatures are measured from absolute zero in Centigrade-size degrees is always used in scientific work, and is known as the *Kelvin scale*. Since absolute zero is $-273°$ C,* a temperature may be expressed in ° K by adding 273 to the value in ° C. Thus,

absolute zero $= 0°$ K,

$0°$ C $= 273°$ K,

$10°$ C $= 283°$ K,

$-10°$ C $= 263°$ K, etc.

Temperature differences on the Kelvin scale are the same as on the Centigrade scale and are measured in units that can be called either C deg or K deg.

The absolute scale employing Fahrenheit-size degrees is in frequent use in engineering practice and is known as the *Rankine scale*. Since absolute zero is $-460°$ F,* a temperature may be expressed in ° R by adding 460 to the value in ° F. Thus,

absolute zero $= 0°$ R,

$0°$ F $= 460°$ R,

$32°$ F $= 492°$ R,

$-10°$ F $= 450°$ R, etc.

Temperature differences on the Rankine-scale are the same as on the Fahrenheit scale.

Since the Kelvin and Rankine scales employ the same zero, temperatures may be converted from one to the other by using the ratio 5:9. Thus,

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* For ordinary purposes this round number is sufficiently accurate.

$$0° \text{ C} = 32° \text{ F} = 273° \text{ K} = 492° \text{ R},$$

and we verify that

$$273:492 = 5:9.$$

Any property of matter that varies with temperature in a measurable way can be made the basis of a thermometer. The various properties that are used for thermometric purposes are:

(a) Expansion of a liquid.
(b) Expansion of a solid (see next section).
(c) Variation of pressure or volume of a gas (see Chap. 17).
(d) Variation of electrical resistance (see Chap. 33).
(e) Thermoelectricity (see Chap. 34).
(f) Variation of quantity of radiated energy (see Chap. 30).
(g) Variation of color of radiated light (see Chap. 30).
(h) Variation of vapor pressure (see Chap. 18).

In calibrating thermometers of various types, the following accurately reproducible experimental temperature values are, by international agreement, employed in addition to the fundamental ice-point and steam-point. All values are measured under a pressure of one standard atmosphere. These temperature values are determined on the thermodynamic scale. Those marked with a dagger are considered more accurate than the others:

†Oxygen, boiling point of liquid	−182.97° C
Carbon dioxide, sublimation point	−78.5 ° C
Mercury, freezing point	−38.87° C
Sodium sulphate, transition temperature	32.38° C
Naphthalene vapor, condensing temperature	218.0 ° C
Tin, freezing point	231.8 ° C
Benzophenone vapor, condensing temperature	305.9 ° C
Cadmium, freezing point	320.9 ° C
Lead, freezing point	327.3 ° C
Zinc, freezing point	419.4 ° C
†Sulfur, boiling point of liquid	444.60° C
Antimony, freezing point	630.5 ° C
†Silver, melting point	960.5 ° C
†Gold, melting point	1063 ° C
Copper, freezing point	1083 ° C
Palladium, freezing point	1555 ° C
Tungsten, melting point	3400 ° C

A revision of the present international temperature scale has been proposed by scientists at the National Bureau of Standards. The revised scale would replace the present two fixed points (ice-point and steam-point) by two more accurately reproducible fixed points. The new fixed-points would be the absolute zero of temperature, which would be called 0° K or −273.16° C, and the triple-point of water (defined in Chap. 18), which would be called 273.17° K or 0.01° C. The new scale would agree with the present scale within the present experimental error of ±0.01 C deg.

PROBLEMS

NOTE: Use the method of the two examples given in the preceding text in working Probs. 1–8.

1. What is normal body temperature, 98.6° F, on the Centigrade scale?
Ans: 37.0° C.

2. What is the pasteurization temperature, 165° F, on the Centigrade scale?

3. What is the melting point of silver, 960.5° C, on the Fahrenheit scale?
Ans: 1761° F.

4. What is the melting point of gold, 1063° C, on the Fahrenheit scale?

5. On a cold winter day, the temperature is 0° F. What would a Centigrade thermometer read? Ans: −17.8° C.

6. On a very cold winter day, the temperature is −10° F. What would a Centigrade thermometer read?

7. A Fahrenheit thermometer reads f° F. What is the reading c° C of a Centigrade thermometer? Ans: $c = \frac{5}{9}(f - 32)$.

8. A Centigrade thermometer reads x° C. Find the reading y° F of a Fahrenheit thermometer.

9. At what temperature will the reading of a Fahrenheit thermometer be exactly double that of a Centigrade thermometer? Ans: 320° F.

10. At what temperature will the reading of a Centigrade thermometer be exactly double that of a Fahrenheit thermometer?

11. Show that only at −40° do the two temperature scales agree.

12. Express 212° F and −50° C in ° K and ° R.

13. Express 68° F and −100° C in ° K and ° R.
Ans: 68° F = 293° K = 528° R; −100° C = 173° K = 312° R.

14. Express 400° K in ° R. Express 400° R in ° K.

15. Express 500° K in ° R. Express 500° R in ° K. Ans: 900° R; 278° K.

16. What is the difference between 212° F and 0° C in C deg? in F deg?

17. What is the difference between 20° C and 32° F in C deg? in F deg?
Ans: 20 C deg; 36 F deg.

4. THERMAL EXPANSION OF SOLIDS

Let us look again at Fig. 1 and attempt to visualize the atoms in violent thermal agitation. Since the average amplitude of vibration of the molecules increases with temperature, it seems quite reasonable that the average distance between the atoms should increase with temperature, and hence that the over-all size of the solid should increase with temperature; this thermal expansion is observed. The increase in dimensions of a solid with increasing temperature is small. The order of magnitude is easy to remember: *a meter length of solid lengthens by about 1 millimeter for a temperature rise of 100 Centigrade degrees.*

Thermal expansion is of sufficient magnitude, however, to be an important factor in many engineering and practical problems. Thermal expansion makes necessary the provision of expansion joints in buildings, bridges, and steam pipes, and provision for temperature compensation of pendulums of clocks and balance wheels of watches; it makes possible the shrink-fitting of steel tires on wheels and of collars on shafts; and it results in the breakage of ordinary glass when heat is irregularly applied.

The change in any linear dimension of a solid, such as the length, width, height, radius, or distance between two marks, is known as the *linear expansion.* We shall denote the length in question by l; the increase in length that arises from an increase in temperature of ΔT we shall denote by Δl. The quantities ΔT and Δl are considered to be positive for increases and negative for decreases. Thus, $\Delta T = +10$ C deg means a temperature rise of 10 C deg; this might be accompanied by an increase of length represented by $\Delta l = +0.15$ mm.

Experiment shows that the change in length, Δl, is proportional to the change in temperature, ΔT. It is of course also proportional to the length l itself, so we can write

$$\Delta l = \alpha\, l\, \Delta T, \tag{3}$$

where α is called the *coefficient of linear expansion.* This coefficient has different values for different materials. Since we can write

$$\alpha = \frac{\Delta l}{l\, \Delta T}, \tag{4}$$

we see that the dimensions of α are deg^{-1}. Hence, the value of α does not depend on the particular units of length used, provided the same units are used to express Δl and l, but it does depend on the size of units used to measure ΔT. Thus, if a meter bar lengthens 1 mm for a temperature increase of 100 C deg, we have, from (4),

$$\alpha = \frac{10^{-3}\ \text{m}}{1\ \text{m} \times 100\ \text{C deg}} = \frac{10^{-5}}{\text{C deg}}. \tag{5}$$

Since, from (2), 1 C deg = $\frac{9}{5}$ F deg, this same coefficient of linear expansion is

$$\alpha = \frac{10^{-5}}{\text{C deg}} = \frac{10^{-5}}{\frac{9}{5}\ \text{F deg}} = \frac{\frac{5}{9} \times 10^{-5}}{\text{F deg}}.$$

Since the F deg is only $\frac{5}{9}$ as large as the C deg, the solid expands only $\frac{5}{9}$ as much per F deg as per C deg.

For most purposes, α can be considered to be a constant for a given solid material, independent of the particular temperature at which the temperature change takes place, although accurate measurements do show some variation of α with temperature. We shall ignore such variations and assume that if a meter bar lengthens 10^{-5} m when the temperature goes from 0° C to 1° C, it will also lengthen by 10^{-5} m when the temperature goes from 100° C to 101° C. Furthermore, to the accuracy with which most expansion coefficients are known for commercial materials (two significant figures), it does not matter whether the l in (3) and (4) is taken to be the length at the initial temperature or at the final temperature, since these differ only by the quantity Δl, which is very small compared to l itself. Thus, in the example of (5), if $l = 1.0000$ m is the length at 0° C, $l + \Delta l = 1.0010$ m would be the length at 100° C; and if α is known only to the accuracy $\alpha = 1.00 \times 10^{-5}$/C deg, it does not matter whether 1.000 m or 1.001 m is substituted in (3) to find Δl to the accuracy that is justified by the data.

Values of the coefficient of linear expansion of various commercial materials are given in Table I.

TABLE I

Typical Coefficients of Linear Expansion of Commercial Materials Near Room Temperature

Aluminum	24×10^{-6}/C deg	Magnesium	26×10^{-6}/C deg
Bakelite	28	Nickel	13
Brass or bronze	19	Oak (across fiber)	54
Brick	9	Oak (parallel to fiber)	5
Copper	17	Pine (across fiber)	34
Glass (ordinary)	9	Pine (parallel to fiber)	5
Glass (Pyrex)	3	Platinum	8.9
Gold	14	Quartz	0.4
Granite	8	Silver	19
Ice	51	Solder	25
Invar (Ni 36%, Fe 64%)	0.9	Steel	12
Iron (cast)	11	Tin	20
Lead	29	Tungsten	4.3

When an isotropic solid expands thermally, the distance between every two points increases in the ratio α per degree temperature rise, just as in the case of a photographic enlargement except that the solid is three-dimensional. For example, the size of a hole in the solid *enlarges* in the same ratio as an external dimension. That the hole enlarges, rather than shrinks as students have been known to state, should be clear from the analogy with the photographic enlargement which is illustrated in Fig. 4.

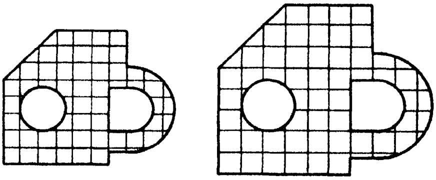

Fig. 4. The thermal expansion of a solid is like a photographic enlargement.

Every line drawn on the solid, whether straight or curved, lengthens in the ratio α per degree temperature rise. For example, if C is the circumference of a circle, $\Delta C = \alpha C \, \Delta T$. One can verify this by noting that the diameter D certainly increases by $\Delta D = \alpha\, D\, \Delta T$. Then

$$C = \pi D,$$

$$C + \Delta C = \pi(D + \Delta D),$$

so

$$\Delta C = \pi\, \Delta D = \pi(\alpha\, D\, \Delta T) = \alpha\, C\, \Delta T.$$

Let us now consider what happens to the *area* of a figure drawn on the surface of a solid or to the area of a sheet of solid. We consider first the change in area of a rectangle (Fig. 5) of sides a and b, area $A = ab$. When the temperature increases by ΔT, a lengthens by $\Delta a = \alpha\, a\, \Delta T$, b lengthens by $\Delta b = \alpha\, b\, \Delta T$, and the area increases by

$$\Delta A = a\, \Delta b + b\, \Delta a + \Delta a\, \Delta b,$$

as may be seen algebraically from the equation $A + \Delta A = (a + \Delta a)(b + \Delta b)$, or geometrically from Fig. 5. On substituting the values of Δa and Δb in the equation above, we find

$$\Delta A = \alpha ab\, \Delta T + b\alpha a\, \Delta T + \alpha^2 ab\, (\Delta T)^2$$
$$= \alpha ab\, \Delta T(2 + \alpha\, \Delta T).$$

Now we see from Table I that

$$\alpha \sim 10^{-5}/\text{C deg};$$

so even if ΔT is as large as 1000 C deg, $\alpha\, \Delta T \sim 10^{-2}$, which is negligible in comparison with 2. Therefore, we can drop the $\alpha\, \Delta T$ term in this equation and write $\Delta A = 2\alpha\, ab\, \Delta T$, or

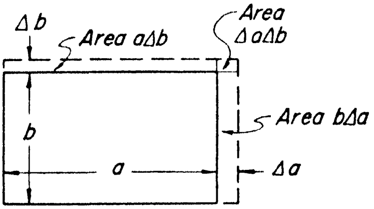

Fig. 5. Thermal increase in area. The sizes of Δa and Δb are highly exaggerated.

$$\Delta A = 2\alpha\, A\, \Delta T. \tag{6}$$

The reader can readily verify that what we have neglected is the area $\Delta a\, \Delta b$ of the small rectangle in Fig. 5 in comparison with the area of the two strips $a\, \Delta b$ and $b\, \Delta a$. In equation (6), 2α expresses the change in area per unit area per degree and is known as the *coefficient of area expansion.*

> The coefficient of area expansion is twice the coefficient of linear expansion.

If we consider that (6) applies to an infinitesimal rectangle and then use calculus methods, we can easily see that formula (6) will give the change in area for a surface with a boundary of any shape and that it will apply equally well to plane and curved surfaces. For example, it will give the change in surface area of a sphere or in the surface area of a hole in a solid.

Now let us make a similar computation for the change in volume of a rectangular parallelepiped of edges a, b, c, volume $V = abc$. We have

$$\begin{aligned} V + \Delta V &= (a+\Delta a)(b+\Delta b)(c+\Delta c) \\ &= abc + ab\ \Delta c + ac\ \Delta b + bc\ \Delta a \\ &\quad + a\ \Delta b\ \Delta c + b\ \Delta a\ \Delta c + c\ \Delta a\ \Delta b + \Delta a\ \Delta b\ \Delta c. \end{aligned}$$

In this equation the terms on the last line, which contain products of two or three of the small quantities Δa, Δb, Δc, are negligible compared to those in the line before, by the same kind of argument we used above. Dropping these terms, subtracting $V = abc$ from both sides, and remembering that $\Delta a = \alpha\, a\, \Delta T$, $\Delta b = \alpha\, b\, \Delta T$, and $\Delta c = \alpha\, c\, \Delta T$, we find

$$\begin{aligned} \Delta V &= ab\ \Delta c + ac\ \Delta b + bc\ \Delta a \\ &= ab\alpha c\ \Delta T + ac\alpha b\ \Delta T + bc\alpha a\ \Delta T \\ &= 3\alpha(abc)\ \Delta T, \end{aligned}$$

or

$$\Delta V = 3\alpha\ V\ \Delta T. \tag{7}$$

In this equation 3α expresses the change in volume per unit volume per degree and is known as the *coefficient of volume expansion.*

> The coefficient of volume expansion is three times the coefficient of linear expansion.

Equation (7) can readily be shown to give the change in volume of a solid of any shape, the change in volume of any hole in the solid, or, therefore, the change in gross volume of a porous or hollow solid.

The difference in expansion coefficient of different metals, usually brass ($\alpha = 19 \times 10^{-6}$/C deg) and steel ($\alpha = 12 \times 10^{-6}$/C deg), is utilized in thermometry and more particularly in the common *thermostat* by welding strips of these materials together to form a *bimetallic strip.* As illustrated in Fig. 6, if the strip is straight at a certain temperature, it will bend one way at higher temperatures, the other way at lower temperatures, because the brass has a greater tendency than steel to lengthen or shorten with temperature change. Bimetallic strips are also used in the rim of the balance wheel of a watch to move weights in as temperature rises. This movement compensates

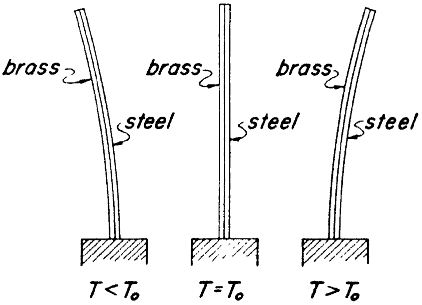

Fig. 6. Principle of the bimetallic thermometer or thermostat element. In the thermostat, the motion of the strip opens or closes electrical contacts.

for the increased length of the radius arms and for the decrease in torque constant of the spring (see Fig. 7).

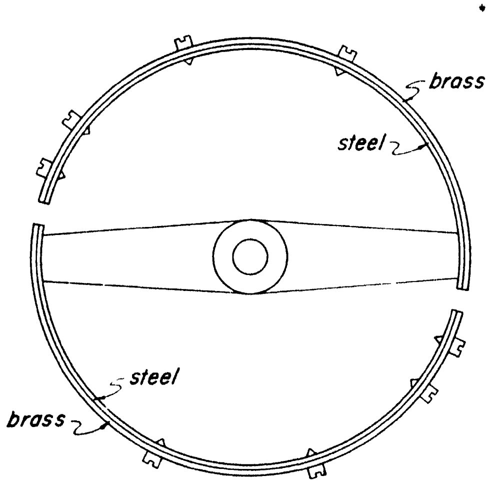

Fig. 7. Compensated balance wheel of a watch. A variable number of weights in the form of screws are placed at various positions around the rim to give the correct variation of moment of inertia with temperature, as the spokes expand and the bimetallic rim bends in, in order to maintain the period constant as a torsion pendulum.

PROBLEMS

Note: Use coefficients from Table I unless otherwise directed. In a number of the following problems, expansion relative to the surface of the earth is to be computed, as for bridges and railroad tracks. It can be assumed that there is no change in distance between points on the earth, winter or summer, because the position of each point is determined by that of the underlying stratum, where seasonal temperature changes do not penetrate. The surface material will expand and contract, but this difference must be taken up by localized surface cracks and stresses, and the distance between two points any considerable distance apart will not change.

1. A metal rod 120 cm long increases in length by 1.48 mm when heated from 0° C to 96° C. What is the coefficient of linear expansion of this metal in (C deg)$^{-1}$? in (F deg)$^{-1}$? Ans: 12.8×10^{-6}/C deg; 7.1×10^{-6}/F deg.

2. A glass rod 120 cm long increases its length by 0.72 mm when heated from 0° C to 75° C What is the coefficient of linear expansion of this glass in (C deg)$^{-1}$? in (F deg)$^{-1}$?

3. If steel railroad rails are placed when the temperature is 35° F, how much gap must be left between each standard 39-ft rail section and the next if the rails should just touch when the temperature rises to 120° F? Ans: 0.27 in.

4. How many inches provision for expansion and contraction must be provided for a 400-ft steel bridge section, to allow for temperature extremes of −20° F and +110° F?

5. A locomotive wheel is 48 inches in diameter. A steel tire, with diameter 0.020 inch undersize at 20° C, is to be shrunk on. To what temperature must it be heated to make the diameter 0.020 inch oversize? Ans: 89° C.

6. A blacksmith wants to fit a steel tire on a wooden wheel 36 inches in diameter. He makes the tire 0.1 inch undersize in diameter at 20° C. To what temperature must he heat the tire before it can just be slipped over the wheel?

7. Shrink fits are frequently made by cooling the male part in solid carbon dioxide at the sublimation temperature given in the text of Sec. 3. How many ten-thousandths of an inch oversize can a steel shaft be made at 30° C and still fit a 1-inch hole if it is first cooled with CO_2? Ans: 13.

8. Answer the question in Prob. 7 for the case of brass.

9. In modern practice, railroad rails are sometimes butt-welded into a continuous length. If rails are laid and welded in the summer at 90° F, what is the tensile stress in the winter at −20° F? If the tensile strength is 72,000 lbf/in^2, what is the factor of safety in respect to rupture? Ans: 21,300 lbf/in^2; 3.4.

10. A steel surveyor's tape is accurate at 70° F. With what tension in lbf/in^2 would it have to be stretched at 40° F in order to read correctly?

11. If a steel tape is accurate at 20° C and is used to measure off a mile at 20° F, how many feet should be read on the tape to get a true mile? Ans: 5281.7 ft.

12. If a steel tape is accurate at 20° C and is used to measure off a mile at 100° F, how many feet should be read on the tape to get a true mile?

13. A pendulum clock with an uncompensated brass pendulum is adjusted at 70° F. Considering the pendulum as a simple pendulum, compute the number of seconds the clock will gain or lose per day at 100° F. Ans: 14 sec lost.

14. If a clock with a simple uncompensated brass pendulum is adjusted at 20° C and is expected to be accurate within one second per day, how closely must the temperature of the clock be controlled?

15. The cross-sectional area of a brass tube is exactly 1 mm^2 at 0° C. What will be its area in mm^2 at 300° C? Ans: 1.011.

16. A steam pipe has a cross-sectional area of exactly 10 in^2 at 20° C. What will be its cross-sectional area when carrying superheated steam at 200° C? Neglect expansion arising from the increased pressure.

17. If a graduated vessel made of ordinary glass is correct at 20° C, what will be the percentage error in using it at 80° C? Will this error be significant if a volume of 100 cm^3 can be read to an accuracy of 0.05 cm^3? Ans: 0.16 %; yes.

18. Answer the same questions as in Prob. 17 for a graduated vessel made of Pyrex glass.

19. Explain why Pyrex glassware is more resistant to breakage under thermal shock than is ordinary glassware.

20. Why is the alloy called Invar used for the best measuring tapes and in the construction of certain types of scientific apparatus?

21. Why was an intensive effort made to make the blank for the Palomar telescope mirror from fused quartz? When this effort failed, why was Pyrex glass used?

22. Why are pendulums for the most accurate determinations of the acceleration of gravity made of quartz?

23. Prove that the change in density, ρ, of a solid can be obtained from the formula

$$\Delta\rho = -3\alpha\,\rho\,\Delta T.$$

24. Prove that the change in moment of inertia, I, of a solid object is given by

$$\Delta I = 2\alpha\,I\,\Delta T.$$

25. Prove (using the result of Prob. 24) that the change in period, t, of any physical pendulum is given by

$$\Delta t = \tfrac{1}{2}\alpha\, t\, \Delta T.$$

5. THERMAL EXPANSION OF LIQUIDS

In the case of a *fluid* (a liquid or a gas) we are not concerned with linear or area expansion because the *shape* of a fluid is not well defined; only the *volume* of a fluid is of significance. The response of liquids and gases to changes in temperature or pressure is quite different. Gases respond strongly, whereas the change of volume of liquids with changes in temperature or pressure is very small, of the same order as that of solids. Gases will be considered in detail in Chap. 17. The effect of pressure on liquids has already been considered in Chap. 9; here we consider the effect of temperature.

Experiment shows that for liquids it is possible to define a coefficient of volume expansion of the same type as we defined for a solid. In the case of a solid we denoted the coefficient of volume expansion in (7) by 3α, since it was directly related to the coefficient of linear expansion. In the case of a liquid we have no coefficient of linear expansion, so we shall denote the *coefficient of volume expansion* by β, writing

$$\Delta V = \beta\, V\, \Delta T. \tag{8}$$

For most liquids the value of β is relatively independent of the temperature at which the temperature change ΔT takes place. Typical values of β are given in Table II; it is seen that these values are in general of the order of 10 times as great as the coefficient of volume expansion 3α for a solid. The behavior of water is distinctly anomalous, and will be considered separately.

TABLE II

COEFFICIENTS OF VOLUME EXPANSION OF LIQUIDS NEAR ROOM TEMPERATURE

Liquid	β at 20° C	Density at 20° C
Alcohol, ethyl	112×10^{-5}/C deg	791 kg/m^3
Alcohol, methyl	120	792
Benzene	124	877
Carbon tetrachloride	124	1,595
Ether, ethyl	166	714
Glycerin	51	1,261
Mercury	18.2	13,546
Turpentine	97	873

One is frequently concerned with the change of density of a liquid, since density is a well-defined property that can readily be measured by a hydrometer. The density is defined by the equation

$$\rho = \frac{m}{V}, \tag{9}$$

where m is the mass of liquid that occupies a volume V. In this equation both ρ and V depend on temperature. If we differentiate with respect to temperature, we find

$$\frac{d\rho}{dT} = -\frac{m}{V^2}\frac{dV}{dT} = -\frac{\rho}{V}\frac{dV}{dT},$$

or

$$\frac{1}{\rho}\frac{d\rho}{dT} = -\frac{1}{V}\frac{dV}{dT}.$$

To the approximation in which (8) is valid, these differentials can be replaced by finite changes, and we can write

$$\frac{1}{\rho}\frac{\Delta\rho}{\Delta T} = -\frac{1}{V}\frac{\Delta V}{\Delta T} = -\beta.$$

Hence,

$$\Delta\rho = -\beta\,\rho\,\Delta T. \tag{10}$$

The minus sign expresses the fact that since volumes increase with increasing temperature, densities decrease with increasing temperature. From (10), changes in density can be computed from the coefficients of volume expansion given in Table II.

The use of an equation such as (8) or (10) implies that the change in volume or density is so small that it does not matter whether the volume or density inserted on the right is the value at temperature T or that at $T+\Delta T$. This condition is not satisfied over so large a temperature range for volumetric expansion of liquids as it is for solids; it is satisfied over a much larger range for mercury than for other liquids such as alcohol (see Tables I and II). When this condition is not satisfied, one must consult detailed data on density as a function of temperature as found in handbooks and physical tables.

Anomalous behavior of water. The density of water does not decrease at all regularly with temperature according to an equation like (10). Rather, the density of water increases slightly from 0° C to 4° C, where it has its maximum value of 1000 kg/m^3 or 1.0000 g/cm^3;* then the density decreases, at first slowly and later more rapidly. The density of liquid water is plotted against temperature in Fig. 8.

To get an idea of how the expansion coefficient of water, at the higher temperatures where the expansion is fairly regular, compares with those of other liquids, we might compute an average expansion coefficient between 70° and 90° C. For this range we have $\Delta T = 20$ C deg, $\Delta\rho = 0.9653 - 0.9778 = -0.0125$ g/cm^3, $\rho_{av} = 0.971$ g/cm^3. Substituting these in the relation (10) gives

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* It is to this value of *unit* maximum density of water, in g/cm^3, that the relative sizes of the kilogram and meter were originally supposed to correspond. However, accurate measurements show that the international standards of mass and length do not correspond exactly to this value, but that the maximum density of water is actually 0.999973 g/cm^3 at 3.98° C (see inset to Fig. 8).

$$\beta = \frac{-\Delta\rho}{\rho\,\Delta T} = \frac{0.0125\ \text{g/cm}^3}{0.971(\text{g/cm}^3)\times 20\ \text{C deg}} = 64\times 10^{-5}/\text{C deg},$$

of the same order of magnitude as the values for other liquids in Table II.

The peculiarity of the expansion of water lies in its behavior at low temperatures as shown in the inset to Fig. 8. *Between* 0° C *and* 4° C, *water contracts as the temperature rises.* Such a contraction with increasing temperature is not observed in any other common liquid or for any solid, except for rubberlike substances. Salt solutions, such as sea water, behave in the same way for a few degrees above the freezing point. This behavior has an important bearing on the manner in which lakes and the polar ocean freeze, as will be noted in the discussion of convection in Chap. 16.

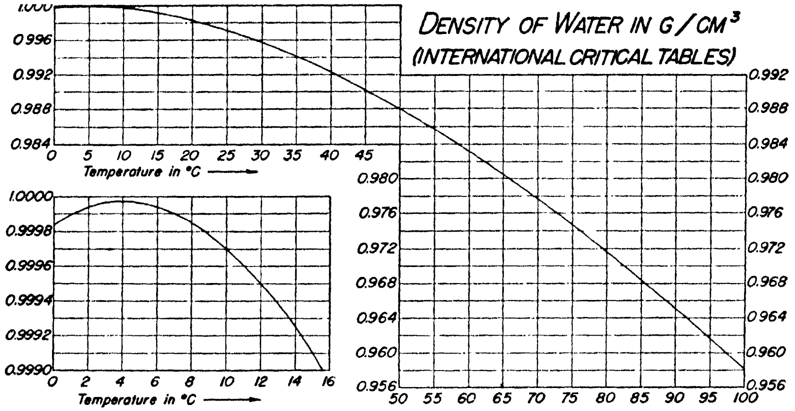

Fig. 8.

PROBLEMS

1. From the data of Table II, compute the density of mercury at 0° C.

Ans: 13,595 kg/m^3.

2. From the data of Table II, compute the density of mercury at 100° C.

3. A quantity of mercury occupies 150 cm^3 at 20° C. What will be its change of volume when cooled to 0° C?

Ans: -0.546 cm^3.

4. A quantity of mercury occupies 950 cm^3 at 20° C. What will be its change of volume when heated to 100° C?

NOTE: When pressure (for example, atmospheric pressure) is expressed in mm of mercury, what is meant is the height of mercury column, at the 0° C density of 13,595.0 kg/m^3 and under the pull of standard gravity, that will balance the pressure being measured. A barometer reading must therefore be corrected for both temperature of mercury and for local gravity (latitude and altitude of location). It is easy to show that the correction from local acceleration of gravity, g_{local}, to standard gravity, g_0, is made by multiplying the barometric height by g_{local}/g_0. From Table III, p. 110, we see that at sea level this correction can amount to as much as 8 parts in 3000 or 2 mm in 760 mm. In the following problems we assume that the barometer is located where the pull of gravity is standard, and we shall consider the correction for temperature.

5. Show that when the temperature of a liquid in a barometer changes by ΔT degrees, the pressure remaining constant, the height h changes by

$$\Delta h = \beta\, h\, \Delta T, \tag{11}$$

where β is the coefficient of volume expansion. *Suggestion:* since the pressure is $h\rho g$ in appropriate units, write $(h+\Delta h)(\rho+\Delta\rho)g = h\rho g$, and solve for Δh by the methods we have been using.

6. Show that equation (11) will give the dependence on liquid temperature for any manometer (see Fig. 9) that reads a pressure difference in terms of differential height h of a liquid column.

7. The mercury in a barometer stands at a height of 759.6 mm in a room at 20° C. What would be the mercury height at 0° C? Ans: 756.8 mm.

8. When the Weather Bureau gives a barometric height of 29.54 inches, what would be the height in inches of a mercury barometer at 25° C?

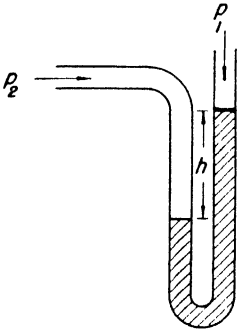

Fig. 9.

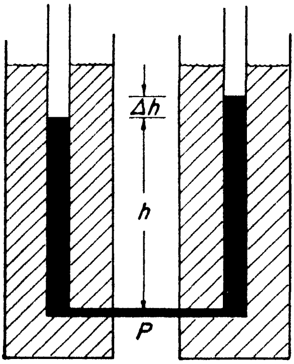

Fig. 10. Apparatus first used by Dulong and Petit in 1816.

9. The height of mercury in a barometer is read on a brass scale as 759.6 mm. Mercury and brass are at 20° C, but the scale is inaccurate because it was calibrated at 0° C. What is the barometric height corrected to 0° C? Ans: 756.5 mm.

10. For the same pressure as in Prob. 8, what height would be read on a brass scale when mercury and brass are at 25° C, the scale having been calibrated at 0° C?

Note: In the next two problems, equation (11) can be used, because the height of each column is a measure of the same pressure difference, that between point P and the atmosphere. Apparatus based on the principle of Fig. 10 furnishes the most accurate measurement of coefficients of volume expansion of liquids.

11. In Fig. 10, the U-tube is filled with a liquid whose volumetric expansion is to be determined. The water bath on the left is kept at 0°C by melting ice. The bath on the right is at a temperature of 15.0° C. h is 116 cm and Δh is observed as 1.50 cm. Determine β. Ans: 86×10^{-5}/C deg.

12. In Fig. 10, if the water bath on the left is at 10° C, that on the right at 25° C, $h = 153$ cm, and $\Delta h = 1.43$ cm, determine the coefficient of volume expansion of the liquid in the U-tube.

NOTE: In the following problems, the expansion of the container as well as that of the liquid must be considered.

13. A narrow-necked bottle of ordinary glass is just filled with mercury at 20° C. The volume of mercury is then 80.0 cm^3. How many cm^3 spill over when the bottle is placed in steam at 100° C? Ans: 0.99 cm^3.

14. A narrow-necked brass bottle is just filled with 116 cm^3 of ethyl alcohol at 0° C. How many cm^3 spill over when the bottle is brought to a room temperature of 19° C?

15. Show that the volume of liquid that spills over in problems such as 13 and 14 is given by

$$\Delta V = (\beta - 3\alpha)\ V\ \Delta T,$$

where α is the coefficient of linear expansion of the container.

16. Show that if a thread of liquid of length l is contained in a capillary tube of uniform bore and the temperature of tube and liquid is raised by ΔT, then the increase in length of the thread will be

$$\Delta l = (\beta - 2\alpha)\ l\ \Delta T,$$

where α is the coefficient of linear expansion of the tube.

17. A thermometer is made of a capillary tube of ordinary glass of 0.0100 mm^2 cross section at 0° C. At 0° C, there is 0.600 cm^3 of mercury in the reservoir and capillary up to the 0° mark. How far does the mercury move up the tube at 15° C? Ans: 14.0 cm.

18. A thermometer is made of a capillary tube of ordinary glass of 0.0150 mm^2 cross section at 0° C. At 0° C there is 0.400 cm^3 of mercury in the reservoir and capillary up to the 0° mark. How far does the mercury move up the tube at 30° C?

NOTE: Use data read from Fig. 8 in working the following problems.

19. A brass cube 4 cm on an edge at 0° C is weighed in water at 0° C and at 50° C. Find the difference in the weights. Ans: 0.586 g.

20. A cube of ordinary glass 6 cm on each edge at 20° C is weighed in water at 20° C and at 70° C. Find the difference in the weights.

21. If a column of water a meter high at 0° C were balanced against a column at 4° C in the apparatus of Fig. 10, which column would stand higher and what would be the height difference in mm? Ans: column at 0°; 0.13 mm.

22. If a column of water a meter high at 0° C were balanced against a column at 10° C in the apparatus of Fig. 10, which column would stand higher and what would be the height difference in mm?

23. Derive equation (10) by writing

$$\rho + \Delta\rho = m/(V + \Delta V) = m(V + \Delta V)^{-1},$$

then expanding the last factor by the binomial formula to obtain

$$\Delta\rho = -\rho\ \Delta V/V = -\rho\beta\ \Delta T.$$

CHAPTER 15

CALORIMETRY

Calorimetry refers to the laboratory science of making measurements of quantities of heat—that is, of quantities of thermal energy added to or subtracted from a quantity of matter. Under this heading we shall not discuss laboratory techniques but rather the principles of calorimetric methods and the numerical results of calorimetric determinations of thermal constants of matter.

We start with definitions of the energy units that are peculiarly convenient to the study of heat exchange. These thermal energy units have definite sizes in terms of the mechanical energy units we have already studied, but so long as attention is confined to the exchange of heat between different bodies, with no introduction of mechanical or electrical energy, the relation of the thermal energy units to the mechanical energy units does not enter the considerations.

Everything we shall study in this chapter is based on the equation

loss of heat by hot bodies = gain of heat by cold bodies,

which applies when bodies at various temperatures are placed in juxtaposition and come to thermal equilibrium in a properly insulated enclosure, called a *calorimeter*. Thermal units were invented and this equation was applied in the days when heat was considered a kind of substance (caloric), and indeed it is seen that this equation is consistent with conservation of quantity of a substance. Only with the study of interchange of energy between thermal and other forms was it recognized that heat is not a substance but mechanical energy of a particular form.

Much of the modern calorimetric work is done by electrical methods, and the most accurate values of the thermal constants are so obtained. In these methods, heating is done electrically, and the quantities of energy furnished are measured by electrical means. We shall have to postpone discussion of these methods until we have studied electrical energy.

1. DEFINITION OF THE KILOCALORIE AND BRITISH THERMAL UNIT

Common usage based on the historical course of development of the subject makes it necessary to introduce new units for the measurement of quantities of thermal energy. It is unfortunate that we may not merely measure quantities of heat in joules or ft·lbf, but it is necessary to establish access to the large body of experimental data expressed in thermal units. There is a tendency in modern scientific practice to dispense with the thermal units. We shall be able to change freely from thermal units to ordinary mechanical units by means of conversion factors where this is desirable.

The basic metric unit of thermal energy is the *kilocalorie* (kcal):

> The kilocalorie (kcal) is the amount of energy required to raise the temperature of one kilogram of water from 14.5° C to 15.5° C.

The *calorie* (cal) is $\frac{1}{1000}$ of the kilocalorie, and is thus the energy required to raise one gram of water from 14.5° C to 15.5° C.

This is the usage among physical scientists. We note, however, that physiologists in discussing food metabolism use the term *calorie* for what we call the kilocalorie. Popular cookbook tables that say that there are "120 calories in one wedge of angel cake" or "250 calories in two codfish balls" are really referring to the approximate number of *kilo*calories of heat developed when one wedge of angel cake or two codfish balls are burned (in a bomb calorimeter, in pure oxygen, after thorough drying).

The only other heat unit in use in the United States is the *British thermal unit* (BTU). This unit is essentially the amount of energy required to raise the temperature of one pound of water by one Fahrenheit degree. This statement cannot be taken as a precise definition because the required energy varies by as much as 1 per cent according to where, between 32° F and 212° F, the one Fahrenheit degree is chosen. No precise definition seems to have been generally agreed upon, probably because the BTU is little used in precision work. For our purpose, to get a precise relation to the calorie, it will be best to choose the degree centered around 15° C = 59° F and use the definition:

> The BTU is the amount of energy required to raise the temperature of one pound of water from 58.5° F to 59.5° F.

To relate the BTU and the kcal, we use the experimental fact that for small temperature intervals the heat required to raise the temperature of water is proportional to the temperature interval. Thus, since 1 C deg = $\frac{9}{5}$ F deg,

$\frac{9}{5}$ BTU will raise the temperature of 1 lb of water from 14.5° C to 15.5° C.

Since 1 lb = 0.4536 kg,

$\frac{9}{5}$ BTU will raise the temperature of 0.4536 kg of water from 14.5° C to 15.5° C;

1 BTU will raise the temperature of $\frac{5}{9} \times 0.4536 = 0.2520$ kg of water from 14.5° C to 15.5° C.

But it requires 0.2520 kcal to do just this, so

$$1\ \text{BTU} = 0.2520\ \text{kcal}. \qquad (1)$$

The determination of the relation of these heat units to the mechanical units will be discussed in Chap. 19. We give the results here so that we can make use of them as occasion arises:

$$\left.\begin{aligned} 1\ \text{kcal} &= 4186\ \text{joules}, \\ 1\ \text{BTU} &= 778\ \text{ft}\cdot\text{lbf}. \end{aligned}\right\} \quad (2)$$

We note that the calorie and the joule are energy units of the same order of magnitude, since

$$1\ \text{cal} = 4.186\ \text{joules}.$$

PROBLEMS

1. How many kcal of heat are given off per hour by a 1-kw electric heater?
Ans: 860.

2. How many kcal of heat per hour are given off by a 60-watt lamp bulb? All the electrical energy goes into heat.

3. The mechanical output of an electric motor is $\frac{1}{2}$ hp. This is 70 per cent of the electric power input, the balance of the input being 'lost' as heat. How many BTU of heat are developed per minute? Ans: 9.09.

4. The electrical input to a motor is $\frac{3}{4}$ hp, of which 90 per cent is delivered as useful mechanical work, the balance being wasted as heat. At what rate in kcal/sec is heat being liberated?

2. SPECIFIC HEAT

> The amount of heat energy necessary to increase the temperature of unit mass of a substance one degree is called the specific heat of the substance, and is denoted by c.

In the metric system the specific heat is expressed in kcal/kg·C deg; in the English system in BTU/lb·F deg. Since

$$\frac{1\ \text{BTU}}{1\ \text{lb} \times 1\ \text{F deg}} = \frac{0.2520\ \text{kcal}}{0.4536\ \text{kg} \times \frac{5}{9}\ \text{C deg}} = 1\ \frac{\text{kcal}}{\text{kg}\cdot\text{C deg}}, \quad (3)$$

the value of the specific heat is the same in either set of units. We notice that the heat units, kcal and BTU, were defined in such a way that the specific heat of water would be exactly unity in either system. It takes 1 kcal to raise the temperature of 1 kg of water 1 C deg, and 1 BTU to raise 1 lb of water 1 F deg. Since the specific heat of water is the same in both systems of units and since the specific heat of any other substance must bear a constant ratio to that of water, independent of the system of units, the specific heat of any substance must be the same in both systems, as demonstrated numerically by (3). We note also that the specific heat will still have the same value when expressed in cal/g·C deg.

There is a commendable tendency, in modern scientific work, to express specific heat in joules/kg·C deg.

For most purposes, the specific heat of a substance may be considered to be independent of the temperature at which the temperature change takes place. This condition does not always hold, however. The specific heat of water in thermal units is exactly 1 at 15° C because of the way the thermal units, kcal and BTU, are defined; but it varies by almost 1 per cent over the range from the freezing to the boiling point, as shown by Fig. 1. Other substances show similar small variations of specific heat with temperature.

In our present discussion, we shall ignore these small variations and assume that specific heat is a constant independent of temperature. Then we can determine the heat (customarily denoted by Q) necessary to raise the temperature of a mass m of a substance by a temperature ΔT degrees by multiplying the specific heat (which is the heat added per unit mass per degree) by m and by ΔT. Hence,

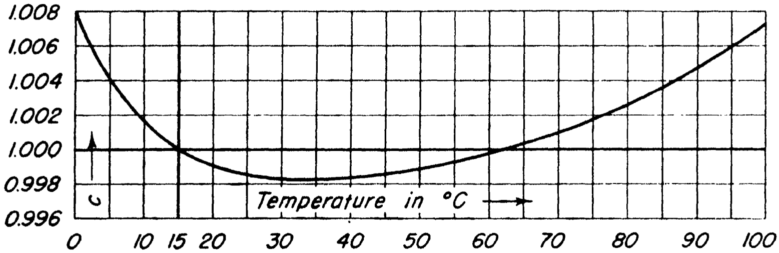

Fig. 1. Specific heat of liquid water, in kcal/kg·C deg or BTU/lb·F deg, as a function of temperature. These values were determined by measuring the electrical energy required to heat the water one degree at various temperatures. Source of data: N. E. Dorsey, *Properties of Ordinary Water Substance*, p. 258 (Reinhold, New York, 1940).

$$Q = c\, m\, \Delta T. \tag{4}$$

In this equation, if m is in lb and ΔT in F deg, Q will be in BTU; if m is in kg and ΔT in C deg, Q will be in kcal; also, if m is in g and ΔT in C deg, Q will be in cal; c will have the same *numerical* value in each of these three cases.

Values of the specific heat of various solids and liquids are given in Table I; the specific heats of gases will be considered in Chap. 17. It will be noted that water has the highest specific heat on this list. Water, in fact, has the highest specific heat of any known substance except gaseous hydrogen and helium.

Specific heats of many substances can be determined by the method of mixtures. A discussion of this method will illustrate the principle on which all calorimetric computations are made, namely,

loss of heat by hot bodies = gain of heat by cold bodies. (5)

Let us determine the specific heat of copper by using a copper calorimeter of 0.25-kg mass, containing 0.4 kg of water. The calorimeter is well jacketed to prevent heat flow to or from the surroundings. (Since such heat flow cannot be entirely eliminated, elaborate methods are used in accurate work to estimate, and correct for, the actual heat transfer to the surroundings. We shall not discuss these methods here.) Let us say that the copper calorimeter and the water are initially at room temperature of 20.0° C as measured by a thermometer in the water. Let a block of copper of 1-kg mass be heated to 100° C by placing it in boiling water. It is then quickly removed from the boiling water and placed in the water of the calorimeter. The copper block cools, the water and the calorimeter become warmer, and the final temperature, as read on the thermometer in the water, is found to be 34.5° C. From these data we wish to determine the specific heat c_{Cu} of Cu.

TABLE I

SPECIFIC HEATS OF SOLIDS AND LIQUIDS

(In kcal/kg·C deg or BTU/lb·F deg)

Metallic solids		Nonmetallic solids		Liquids	
Aluminum	0.212	Ice	0.48	Water	1.00
Brass	0.090	Clay	0.22	Ethyl alcohol	0.58
Copper	0.094	Coal	0.3	Gasoline	0.5
Gold	0.031	Concrete	0.16	Mercury	0 033
Iron and steel	0.11	Glass	0.12–0.20	Mineral oil	0.5
Lead	0.031	Limestone	0.22	Olive oil	0.47
Platinum	0.032	Marble	0.21	Turpentine	0.41
Silver	0.056	Paraffin	0.69		
Tin	0.055	Rubber	0.48		
Zinc	0.094	Wood	0.3 0.7		

The fundamental equation (5) becomes

$$\begin{Bmatrix}\text{heat lost by}\\ \text{copper block}\end{Bmatrix} = \begin{Bmatrix}\text{heat gained}\\ \text{by water}\end{Bmatrix} + \begin{Bmatrix}\text{heat gained by copper}\\ \text{calorimeter}\end{Bmatrix}.$$

The value of each of these terms is given by (4) as $mc\,\Delta T$. Substituting the known values of m in kg, of c in kcal/kg·C deg, and of ΔT in C deg, we have

$$1\text{ kg}\cdot c_{Cu}\cdot 65.5\text{ C deg} = 0.4\cdot 1\cdot 14.5\text{ kcal} + 0.25\text{ kg}\cdot c_{Cu}\cdot 14.5\text{ C deg},$$

whence

$$c_{Cu} = 0.094\text{ kcal/kg}\cdot\text{C deg}.$$

In the above substitution, 14.5 C deg is the temperature increase of the calorimeter ($34.5 - 20.0 = 14.5$), and 65.5 C deg is the temperature decrease of the copper block ($100.0 - 34.5 = 65.5$).

Methods similar to the above should be used in the following calorimetric problems.

PROBLEMS

NOTE: Use values of specific heat from Table I.

1. How many kcal are required to raise the temperature of 8 kg of iron from 35° C to 44° C? from 35° F to 44° F? Ans: 7.9 kcal; 4.4 kcal.

2. How many BTU are required to raise the temperature of 15 kg of copper from 25° F to 43° F? from 25° C to 43° C?

3. If 100 lb of water at 40° F is mixed with 50 lb at 70° F, what is the final temperature? Ans: 50° F.

4. When 15 kg of methyl alcohol at −20° C is mixed with 40 kg of methyl alcohol at 25° C, what is the final temperature?

5. A copper calorimeter of 63-g mass contains 79.5 g of a liquid at 18° C. Into this is placed a 120-g block of copper at 100° C. The final temperature is 39.0° C. What is the specific heat of the liquid? Ans: 0.338 kcal/kg·C deg.

6. A copper calorimeter of mass 40 g contains 106 g of water at 15° C. A 100-g nugget of gold at a temperature of 99° C is placed in the calorimeter. The final temperature is found to be 17.6° C. What value does this experiment give for the specific heat of gold?

7. A 1.5-kg platinum ball is removed from an oven and placed in a copper calorimeter of 0.5-kg mass containing 1.6 kg of water. The temperature of the water rises from 20.0 to 25.8° C. What was the temperature of the oven? Ans: 220° C.

8. A blacksmith throws a red-hot horseshoe weighing 2 lb at a temperature of 1900° F into a pail containing 40 lb of water at 70° F. If all the heat goes into warming the water, what is its temperature rise?

3. LATENT HEATS OF FUSION AND OF VAPORIZATION

If one puts enough 0°-C ice into water at room temperature, he observes that the ice does not all melt before the water and ice have reached thermal equilibrium at 0° C. In reaching this thermal equilibrium, the ice has not changed temperature, although some of it has melted. Since the water has lost thermal energy in cooling down, the ice must have gained thermal energy in melting, without change in temperature.*

If one starts then with a piece of ice at −20° C, one must add heat to it (0.48 kcal/kg·C deg according to Table I) in order to raise its temperature to 0° C. Then one must add further heat to melt the ice. During the melting process no temperature rise takes place. However,

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* These remarks are probably not so convincing to contemporary students as they would have been to those of previous generations. Nowadays we usually cool our drinking water with ice fresh from a mechanical refrigerator at a temperature of about −15° C. But not so long ago the ice ordinarily used had lain in an icehouse since the previous winter, melting on its surface all the while, and had certainly reached thermal equilibrium at 0° C throughout. Zero-degree ice may be obtained nowadays by fishing pieces out of ice water in which they have remained 5 or 10 minutes in order to reach thermal equilibrium with the 0°-C water.

80 kcal must be added to each kg of ice to melt it. Not until sufficient heat has been added to melt all the ice does any temperature rise of the water take place.

The quantity of heat, 80 kcal/kg, that is required for melting is known as the *latent heat of fusion** of ice.

> The latent heat of fusion of any substance is defined as the heat that must be added to unit mass of the solid at its melting point to change it to liquid at the same temperature and the same pressure.

The units in which latent heats of fusion are specified will be considered after we have defined the latent heat of vaporization.

If we put a pan of water on the stove and turn on the gas, the temperature of the water rises fairly rapidly to its boiling point of about 100° C. After it has reached this temperature, we must continue to supply heat for a very much longer time if we want to evaporate all the water. Supplying heat at a constant rate causes the water to evaporate at a constant rate. During the process of evaporation no temperature rise takes place; a thermometer in the steam and one in the water will each read 100° C continuously. But we have to add 540 kcal to evaporate 1 kg of water. This value, 540 kcal/kg, is known as the *latent heat of vaporization* of water.

> The latent heat of vaporization of any substance is defined as the heat that must be added to unit mass of the liquid at its boiling point to change it to vapor at the same temperature and the same pressure.

The boiling point of a liquid varies with pressure, and the latent heat of vaporization varies also. Furthermore, evaporation of a liquid takes place at temperatures below its boiling point and a latent heat is involved in this process also. These matters will be treated in Chapter 18. For the present we shall confine ourselves to ordinary boiling taking place at normal atmospheric pressure.

We shall also point out in Chapter 18 that only part of the latent heat goes into increasing the thermal energy of the substance; part goes into doing the external work required to effect the volume expansion that always accompanies vaporization and usually accompanies fusion. For this reason it would be incorrect to define latent heat as the difference between the thermal energies of the vapor and the liquid or of the liquid and the solid.

Either of the two latent heats can be expressed in kcal/kg, cal/g, or

---

* The term *latent,* which according to the dictionary means "not visible or apparent," is used here to indicate that the addition of this latent heat does not result in any change in temperature, so that its addition is not apparent in the reading of a thermometer.

BTU/lb. Of course 1 kcal/kg = 1 cal/g, but

$$\left.\begin{aligned} &1\,\frac{\text{BTU}}{\text{lb}} = \frac{0.2520\text{ kcal}}{0.4536\text{ kg}} = 5/9\,\frac{\text{kcal}}{\text{kg}}, \\ \text{and}\qquad &1\text{ kcal/kg} = 9/5\text{ BTU/lb}. \end{aligned}\right\}\qquad (6)$$

Hence, the latent heat of fusion of water is

$$80\text{ kcal/kg} = 80 \times 9/5\text{ BTU/lb} = 144\text{ BTU/lb},$$

and the latent heat of vaporization of water is

$$540\text{ kcal/kg} = 970\text{ BTU/lb}.$$

These are approximate values; more accurate values are given in Table II.

The reason why a factor 9/5 occurs in converting latent heats from metric to English units even though no such factor occurs in specific heats can be seen from the following argument: It takes 1 kcal to raise 1 kg of water 1 C deg, 1 BTU to raise 1 lb of water 1 F deg. But the BTU does not raise the temperature of the pound by as much as the kcal raises the temperature of the kg because 1 F deg is smaller than 1 C deg. To

TABLE II

TEMPERATURES AND LATENT HEATS OF FUSION AND VAPORIZATION AT NORMAL PRESSURE

Substance	Melting point (°C)	Latent heat of fusion (kcal/kg)	Boiling point (°C)	Latent heat of vaporization (kcal/kg)
Water	0	79.70	100	539.2
Ammonia	−75	108.0	−34	327.1
Helium	−272	.....	−269	5.97
Hydrogen	−259	15.0	−253	106.7
Methane	−182	14.5	−161	138
Nitrogen	−210	6.2	−196	47.8
Oxygen	−219	3.3	−183	51
Ethyl alcohol	−115	24.9	78	204.3
Methyl alcohol	−98	22.0	65	262.8
Aluminum	660	93.0	2056	2000
Copper	1083	50.6	2595	1760
Gold	1063	16.1	2966	446
Iron	1539	65	3000	1620
Lead	327	6.3	1744	222
Mercury	−39	2.7	357	71
Platinum	1774	27.1	4407	640
Silver	960	24.3	2212	552
Tin	232	14.4	2270	650
Tungsten	3400	44	5927	1180
Zinc	419	24.1	907	362

raise 1 kg of water from its freezing to its boiling point requires 100 kcal, whereas to raise 1 lb of water from its freezing to its boiling point requires 180 BTU. It takes 180 BTU to have the *same physical effect* on a lb as 100 kcal have on a kg. Hence, it takes 144 BTU to have the same effect on a lb as 80 kcal has on a kg—for example, in melting ice.

To change 4 kg of ice at $-20°$ C into steam at 100° C, we must, then, supply

$$4\times0.48\times20 = 38 \text{ kcal} \quad \text{to heat the ice to } 0° \text{ C},$$

$$4\times80 = 320 \text{ kcal} \quad \text{to melt the ice},$$

$$4\times1\times100 = 400 \text{ kcal} \quad \text{to heat the water to } 100° \text{ C},$$

$$4\times540 = 2160 \text{ kcal} \quad \text{to evaporate the water},$$

making a total of 2920 kcal. In calorimetric problems involving melting, freezing, vaporization, or condensation, one must include the latent heats among the quantities of heat lost or gained.

Table II gives the melting and boiling points and the corresponding latent heats for various substances.

PROBLEMS

1. At what rate will ice be melted by a 200-watt electric heater submerged in a mixture of ice and water? Ans: 36.0 g/min.

2. At what rate will water at 100° C be evaporated by a 1-kw electric heater if all the energy supplied goes into latent heat?

3. If a 50-g aluminum freezing tray containing 600 g of water is at 20° C when placed in the refrigerator, how much heat must the refrigerator remove before half the water is frozen? Ans: 36.2 kcal.

4. If a Dewar flask containing liquid oxygen loses mass at the rate of 150 g/hr, at what rate is heat leaking in from the outside?

5. A 200-g copper calorimeter contains 300 g of water at 20° C. How much steam at 100° C must be added to raise the temperature of the water to 65° C? Ans: 25.0 g.

6. A 1-kg copper ball is heated in a steam bath at 100° C and is then placed on a large cake of ice at 0° C. How much ice will be melted?

7. If a total of 30 kg of molten lead at 327° C is dropped into 4 liters of water initially at 20° C, how much water is evaporated, assuming no heat loss? Ans: 148 g.

8. If 10 kg of steam at 120° C is passed into a vessel containing 50 kg of ice at $-20°$ C, what will be the temperature of the mixture when it has reached equilibrium? Assume no heat loss, and take the specific heat of steam as 0.5 kcal/kg·C deg.

9. An electric motor is loaded by means of a Prony brake until it is delivering 20 hp. The Prony brake initially contains 15 lb of water at 72° F. Assume that the whole 20 hp delivered by the motor goes into heating the water. How many pounds of water are boiled away during the first 10 minutes of operation of the motor? Ans: 6.58 lb.

10. A 500-watt electric heater is immersed in 3.50 lb of water at 62.0° F contained in a 1.20-lb aluminum vessel. Neglecting loss of heat and the heat capacity of the electric heater, how long will it take to heat the water to boiling and boil off 0.250 lb?

11. A continuous-flow steam generator takes in 1 liter of water per minute at 10° C and changes it to steam at 100° C. How many kilowatts of electrical power are necessary to furnish the required heat? Ans: 43.9.

12. Bismuth melts at 271° C. Its specific heat is 0.032 kcal/kg·C deg. To determine its latent heat of fusion, 2 kg of molten bismuth at its melting point is dropped into a 1-kg copper calorimeter containing 3 kg of water at 15° C. The temperature rises to 22.1° C. What is the latent heat of fusion of bismuth?

13. To determine the latent heat of fusion of water, 1 lb of ice at 32° F is dropped into 5 lb of water in a 1-lb copper calorimeter at 100° F. The final temperature is observed as 64° F. What value does this experiment give for the latent heat of fusion of water? Ans: 151 BTU/lb.

4. HEAT OF COMBUSTION

When a chemical reaction takes place, it is ordinarily either *exothermic*, meaning that it gives out heat, or *endothermic*, meaning that it takes in heat. A certain amount of chemical binding energy is changed into thermal energy, or vice versa. The amount of this energy, per unit quantity of material reacting, is known as the *heat of reaction.*

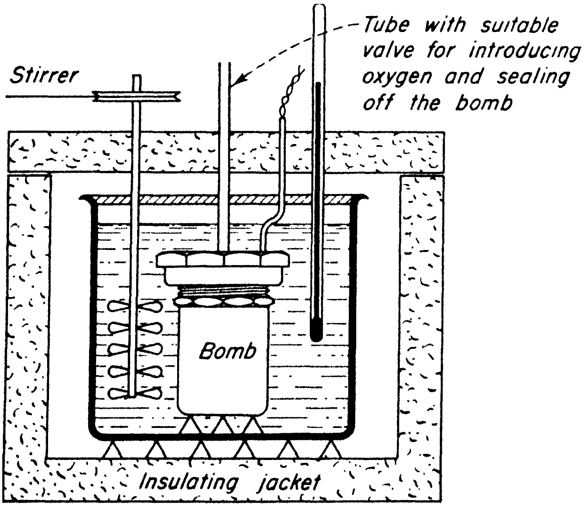
Tube with suitable valve for introducing oxygen and sealing off the bomb
Stirrer
Bomb
Insulating jacket

Fig. 2. Bomb calorimeter (schematic).

The principal chemical reaction that is used for heating purposes is the process of combination with oxygen—burning or *combustion.* In this case the heat of reaction is known as *heat of combustion* and is specified in kcal/kg, or BTU/lb. The mass in the denominator refers to the mass of material burned but does not include the mass of the oxygen that enters the reaction. Thus, the statement that the heat of combustion of anthracite is 8000 kcal/kg, means that one kilogram of anthracite, when it reacts completely with whatever mass of oxygen is necessary to ensure complete combustion, gives out 8000 kcal of heat.

The relations (6) connect the metric and British units, for the same reason as in the case of latent heats. For example,

$$10{,}000 \text{ kcal/kg} = 18{,}000 \text{ BTU/lb};$$

if combustion of 1 kg of a substance gives 10,000 kcal, combustion of 1 lb will give 18,000 BTU.

Heats of combustion are ordinarily measured in a *bomb calorimeter.* The 'bomb' (Fig. 2) is a massive gun-metal cylinder, fitted with a gas-tight screwed cover, and capable of withstanding the high pressures of the gaseous combustion products. In this calorimeter a measured mass of material burns in an atmosphere of pure oxygen under pressure so that complete combustion is assured. The ignition is by an electric current through a fine wire, and the combustion takes place with explosive violence. The heat of combustion is determined by the final temperature rise of the water and of the calorimeter and its contents, after thermal equilibrium is reached.

TABLE III

Typical Values of Heat of Combustion

Solid fuels		
Anthracite	8000 kcal/kg	14,500 BTU/lb
Bituminous coal	7500	13,500
Coke	6000	11,000
Pine wood	4500	8,000
Liquid fuels		
Gasoline	11,400	20,500
Kerosene	11,200	20,000
Diesel oil	10,500	19,000
Alcohol	6,400	11,500
Foodstuffs		
Proteins	4000	
Carbohydrates	4000	
Fats	9500	
Gaseous fuels		
Manufactured gas	500–550 BTU/ft³	
Natural gas	1000–1100 BTU/ft³	

Table III gives heats of combustion of various solid and liquid fuels. As noted earlier, heats of combustion of food products are also determined in a bomb calorimeter, where a well-dried sample of the food is burned. These heats are ordinarily specified in the 'calorie' of the physiologist, which is the same as the kcal of Table III. Heats of combustion of gaseous fuels are ordinarily given in BTU per ft^3 of gas, the volume being measured at 1 atm and 0° C. Typical values for gaseous fuels are given in Table III.

PROBLEMS

1. When delivering 100 hp, a Diesel engine burns 1 lb of oil per minute. Find the over-all efficiency of the engine. Ans: 22.3%.

2. Compare the costs of heating with bituminous coal at \$9.00 per ton, natural gas at 65 cents per 1000 ft^3, and electricity at 1.5 cents per kwh.

3. In a determination of the heat of combustion of gasoline, 6 grams of gasoline is burned in a bomb calorimeter containing 2 kg of water and 7.5 kg of steel. The resulting rise in temperature is observed to be 24 C deg. What value does this give for the heat of combustion of the gasoline? Ans: 11,000 kcal/kg.

4. In one hour an internal-combustion engine uses 25 kg of gasoline. If 12 per cent of the heat is converted into mechanical energy, find the average horsepower developed by the engine.

5. At what price per kwh would the cost of heating with electricity be the same as the cost of heating with anthracite at \$18 per ton if it is assumed that all the electrical energy but only 50 per cent of the heat of combustion of the anthracite is delivered as useful heat? Ans: 0.424 cent.

6. To determine the caloric content of a porkchop, the edible portion of the porkchop is dried and burned in a bomb calorimeter containing 5 kg of water and 20 kg of steel. The resulting temperature rise is 35 C deg. What is the 'food value' of the porkchop in 'calories' (kcal)? Compare with the cookbook value for '1 medium porkchop.'

CHAPTER 16

HEAT TRANSFER

In this chapter we consider rather briefly the methods by which thermal energy is transferred from one body to another or from one material medium to another. The computation of heat transfer in practical cases is frequently very complex and mathematically difficult. We confine ourselves here to an introductory discussion of fundamental principles and to simple examples of technical importance. We also discuss briefly the important role played by heat transfer in the terrestrial atmosphere, particularly in the determination of climate.

1. METHODS OF HEAT TRANSFER

There are three distinct methods by which thermal energy is transferred from one point to another when temperature differences exist. These three are

(1) *conduction, in which the energy is passed from molecule to molecule of any kind of material medium: solid, liquid, or gaseous;*

(2) *convection, which occurs only in liquid or gaseous media and involves mass motion of the molecules of the fluid from one point to another, the thermal energy being carried by the molecules;*

(3) *radiation, which is a transmission of energy by electromagnetic waves, no material medium playing an essential role in the transmission.*

Wherever two different portions of material media have different temperatures, heat will be transferred from the hot portion to the cool, so that the temperature tends to become equalized. By two portions of material media we can mean various things, such as the hot sun and the cool earth, the hot oven and the cool roast, the hot radiator and the cool furniture, the hot air in a room and the cold air outside in winter, or the hot inside portion of the wall of a house and the cold outside portion.

Radiation involves emission, transmission, and absorption of electromagnetic energy (*radiant energy*) in a form exactly analogous to light, as will be explained in more detail in Chap. 30. All bodies at temperatures above absolute zero are continuously radiating energy, at a rate proportional to the fourth power of the absolute temperature, but depending also on surface condition, bodies with dull or black surfaces radiating at a higher rate than those with polished or white surfaces. Like light, radiant energy is transmitted *best* through a vacuum, but fairly well through a gaseous medium and through certain *transparent* liquids and solids. Radiation is the process involved in the transmission of heat from the sun through the vacuum of outer space to the earth. Within a *non-transparent* solid medium, heat can be transferred from point to point only by conduction. Where there is a material medium available to transmit the heat and where temperatures are low (less than 200° C), radiation is usually of little importance in comparison with conduction and convection. In liquid and gaseous media, convection by means of circulating currents of fluid is ordinarily the most important process of heat transfer.

In any given problem in heat transfer, two or three of these processes may be active simultaneously. The total heat transferred will be obtained by adding the amounts transferred by the different processes. It is convenient to consider the different processes one at a time, as we shall proceed to do.

2. LAWS OF HEAT CONDUCTION

Suppose that we have a slab of solid material (Fig. 1) in which there is a *temperature gradient,* by which we mean a space rate of change of temperature as we go from one point to another in the material. A temperature gradient is set up in Fig. 1 because one side of the slab is in contact with hot water, the other with cold water. It is found experimentally that as one passes through the slab there is a *uniform* space rate of change

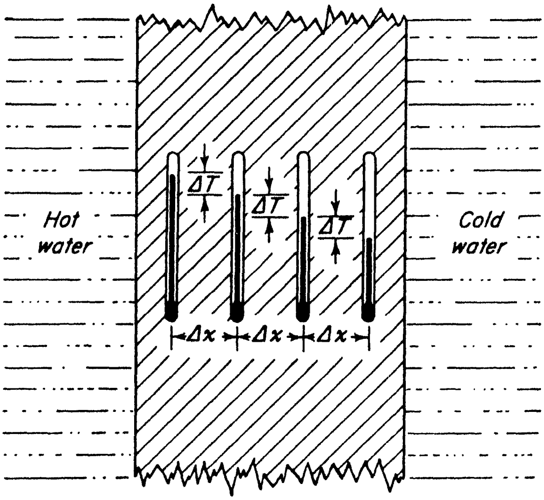

Fig. 1. The rate of heat flow in proportional to the temperature gradient $\Delta T/\Delta x$.

of temperature, that is, a constant temperature gradient. This uniformity is indicated schematically by the thermometer readings in Fig. 1; in practice one uses very tiny thermocouples (see Chap. 34) buried in the material at various points to determine the temperature gradient without substantially interrupting the continuity of the solid material. The temperature gradient is defined as the derivative dT/dx, in C deg/m for example. Under steady conditions thermometers at equal distances Δx will show equal temperature differences ΔT, so that the value of dT/dx is the same as $\Delta T/\Delta x$.

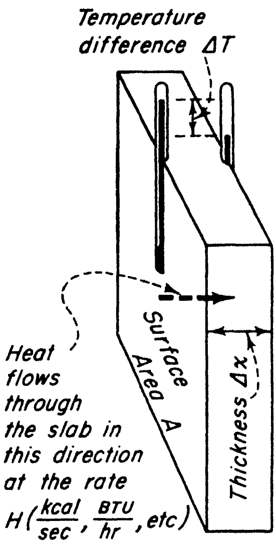
Temperature difference ΔT
Surface Area A
Thickness Δx
Heat flows through the slab in this direction at the rate $H\left(\frac{\text{kcal}}{\text{sec}}, \frac{\text{BTU}}{\text{hr}}, \text{etc}\right)$

Fig. 2. Definition of the symbols in equation (2). The thermometers indicate schematically the surface temperatures of the two sides of the slab.

The temperature gradient in Fig. 1 is associated with a heat flow through the solid slab, from left to right (from the region of higher temperature to the region of lower temperature). The heat flow per unit time per unit area can be measured by calorimetric methods. The area referred to here is the area of surface of the slab in a plane normal to the paper in Fig. 1—the area of surface in contact with the hot water, for example. If we let H denote the *rate of heat flow* through the slab in kcal/sec, BTU/sec, BTU/min, or any other appropriate unit, then the heat flow per unit time per unit area will be H/A, where A is the area defined above.

It is found experimentally that H/A, the rate of heat flow per unit area, is proportional to the temperature gradient $\Delta T/\Delta x$. The proportionality constant, which differs from material to material, is called the *thermal conductivity*, or *coefficient of heat conduction*, and is denoted by the symbol k. Analytically, then,

$$\frac{H}{A} = k\,\frac{\Delta T}{\Delta x}. \tag{1}$$

If, as in Fig. 2, we have a section of slab of area A and thickness Δx, with temperature difference ΔT between the two faces of the slab, the rate of heat flow will be given by

$$H = kA\,\frac{\Delta T}{\Delta x}. \tag{2}$$

The thermal conductivity k has the same dimensions as $H\,\Delta x/A\,\Delta T$. Various systems of units are used in specifying k. If H is in kcal/sec, Δx in m, A in m^2, and ΔT in C deg, the unit of k is the kcal/sec·m·C deg. In the English-speaking countries the thermal conductivity of structural and insulating materials is usually specified with H in BTU/hr, Δx in inches,

A in ft², and ΔT in F deg. In this case k is in BTU·in/hr·ft²·F deg. Conversion of k from one system of units to another can be done in the usual way by using the conversion factors for each unit. Thus,

$$1\ \frac{\text{BTU}\cdot\text{in}}{\text{hr}\cdot\text{ft}^2\cdot\text{F deg}} = \frac{(0.252\text{ kcal})(0.0254\text{ m})}{(3600\text{ sec})(0.0929\text{ m}^2)(0.556\text{ C deg})} = \frac{1}{29{,}000}\ \frac{\text{kcal}}{\text{sec}\cdot\text{m}\cdot\text{C deg}}$$

In the British engineering units specified above, the numerical values of the thermal conductivities of nonmetallic structural and insulating materials are not far from unity. Thus, the typical thermal conductivity of concrete is 12 BTU·in/hr·ft²·F deg, and that of most insulating materials (fiber blanket, glass wool, mineral wool, or corkboard) is about 0.25 BTU·in/hr·ft²·F deg. In kcal/sec·m·C deg the numerical values of k would be smaller than these by a factor of 29,000.

Metals have much larger coefficients of heat conduction than nonmetals, greater by a factor of 100 or more. The thermal conductivity of metals is closely related to the electrical conductivity, since both heat and electricity are transferred in metals principally by the agency of free electrons (see Chap. 31). There is an almost constant ratio between the coefficients of heat conductivity and the coefficients of electrical conductivity for different metals, a fact that is known as the *Wiedermann-Franz law*. The thermal conductivity of a metal varies with temperature in the same way as does the electrical conductivity (see Chap. 33); here we shall be content with using an average value satisfactory for temperatures in the neighborhood of room temperature (20° C). The best electrical conductors, and also the best heat conductors, are silver, copper, and gold, in that order. Silver has a thermal conductivity of 0.1 kcal/sec·m·C deg = 2900 BTU·in/hr·ft²·F deg.

Table I gives typical values of thermal conductivity of various materials in both the systems of units we have discussed. It will be noticed that the thermal conductivities of gases are very low. In general the heat transferred by conduction in gases is negligible compared to that transferred by convection. Exceptions arise, however. If air is confined in very small spaces so that it cannot circulate, as in the pores of corkboard or wool clothing, convective processes are inhibited; and since conduction is also low, we have a poor heat transmitter called a *heat insulator*. Such an insulator is said to contain 'dead-air' spaces, meaning that the air cannot circulate readily so that it cannot transmit much heat by convection.

In determining the amount of heat conducted through a wall or a window, it is not correct to give the outer surface the outdoor temperature and the inner surface the indoor temperature. On a winter day when the room temperature is 60° F, the inner surface of a pane of single window glass may be below 32° F, as witnessed by the familiar frosting of the inside of windows in cold climates. Similarly, the outer surface of the glass may be well above the outdoor temperature; for example, it may be 0° F when it is $-20°$ F out-of-doors. Only part of the temperature drop occurs through the glass, the rest occurs through the layers of air close to the glass inside and outside. In applying (2) to conduction through the glass, one must use the actual temperature difference between the inner and outer surfaces as ΔT; this makes the practical engineering computation of heat losses subject to a good many semi-empirical rules which we shall not discuss here.

TABLE I

TYPICAL VALUES OF THE THERMAL CONDUCTIVITY k NEAR 20° C

Substance	$\dfrac{\text{kcal}}{\text{sec}\cdot\text{m}\cdot\text{C deg}}$	$\dfrac{\text{BTU}\cdot\text{in.}}{\text{hr}\cdot\text{ft}^2\cdot\text{F deg}}$
Metals		
Silver	0.101	2930
Copper	0.092	2680
Gold	0.070	2030
Brass	0.026	750
Steel	0.011	320
Cast iron	0.011	320
Aluminum	0.048	1390
Mercury, liquid	0.0015	44
Nonmetallic solids		
Brick, common	1.7×10^{-4}	5.0
Concrete	4.1×10^{-4}	12.0
Wood (across grain)	0.3×10^{-4}	0.9
Glass	1.4×10^{-4}	4.0
Enamel	2.0×10^{-4}	6.0
Hollow and porous materials		
Hollow tile, 4 inches thick	....	4.0*
Concrete block, 8 inches thick	....	8.0*
Cinder block, 8 inches thick	....	5.0*
Fiber-blanket insulation	0.09×10^{-4}	0.27
Glass wool or mineral wool	0.09×10^{-4}	0.27
Sawdust	0.14×10^{-4}	0.41
Corkboard	0.10×10^{-4}	0.30
Liquids		
Water	1.43×10^{-4}	4.15
Ethyl alcohol	0.42×10^{-4}	1.23
Gases		
Air	0.056×10^{-4}	0.16
Hydrogen	0.400×10^{-4}	1.16

* These are effective coefficients applicable only to a wall of thickness mentioned, since the material is not homogeneous. For an extensive listing of building materials see Eshbach, *Handbook of Engineering Fundamentals*, p. 12-34 (Wiley, 1936).

For a discussion of practical details of the apparatus used for the measurement of thermal conductivity, the reader is referred to specialized books on the subject of heat. The conductivity of a solid is usually determined by measuring the rate at which heat enters or leaves a rod or slab. This is equal to the rate at which heat flows through the rod or slab. If the heat is supplied by circulating hot water, the temperature drop and rate of flow of the water determine the amount of heat entering the solid. If the heat is extracted by means of cold water, the amount of heat leaving is similarly determined. The temperature gradient is usually computed from the readings of thermocouples buried in the material in the manner indicated by the thermometers of Fig. 1. This method of determining temperature gradient is desirable because it is usually incorrect to assume that the surface of the slab or the end of the rod is at the same temperature as the bulk of the liquid or gas used to supply or remove heat from these surfaces. In the measurement of thermal conductivity of a liquid or gas, the sample is in the form of a horizontal sheet which is heated from above and cooled from below so that no heat is transferred by convection.

PROBLEMS

1. How much heat is conducted in 24 hours through a pane of glass 15 ft^2 in area and $\frac{3}{16}$ inch thick if the surface temperatures are 60° F and 10° F?
Ans: 384,000 BTU.

2. How much heat is conducted in 24 hours through a solid concrete building wall 8 inches thick and 400 ft^2 in area if the surface temperatures are 60° F and 30° F?

3. A domestic refrigerator has a wall area of 52 ft^2 and is insulated with corkboard 3 inches thick. The room temperature is 75° F, the inside temperature is 40° F, and it may be assumed that the temperature drop all occurs through the corkboard. Twenty pounds of food at room temperature, with average specific heat 0.9 BTU/lb·F deg is placed in the refrigerator and cooled during one hour. How much heat must be extracted during this hour by the cooling coils in the refrigerator? How many pounds of ice would have to melt (from ice at 32° F to water at 32° F) to extract this same heat?
Ans: 812 BTU; 5.64.

4. A certain freezing-plant building is 50 ft long, 25 ft wide, and 20 ft high. Walls and roof are insulated with corkboard 1 ft thick. Let the inside temperature of the corkboard average 0° F and the outside temperature average 50° F over a 24-hr period. The heat conduction through the floor is negligible. Let 12 tons of produce of average specific heat and latent heat the same as water and ice be placed in the plant at 65° F to be frozen to 0° F during this period. What must be the 'tonnage' of the refrigerating machine required for this plant, if a 'one-ton' refrigerating machine is capable of extracting as much heat as one ton of ice at 32° F melting to water at 32° F? (This is the unit actually used in rating refrigerating machines.)

5. The wall of a freezing plant is composed of 1 ft of corkboard inside 8 inches of solid concrete. If the temperature of the inner wall of the corkboard is 0° F and that of the outer wall of the concrete is 65° F, find the temperature of the corkboard-concrete interface and the heat flow in BTU/ft^2·hr.
Ans: 63.9° F; 1.60 BTU/ft^2·hr.

6. A large cast-iron steam pipe $\frac{1}{4}$ inch thick is covered with 1 inch of asbestos insulation of conductivity 0.27 BTU·in/hr·ft^2·F deg. If the temperature of the inner wall of the pipe is 230° F and that of the outer wall of the asbestos is 80° F, find the temperature of the iron-asbestos interface and the heat loss per square foot per hour.

7. Under ideal combustion conditions, how much coal with heat of combustion 14,000 BTU/lb must be burned per month (30 days) to heat a building with wall area 1500 ft^2 of hollow clay tile 4 inches thick, roof area 500 ft^2 of built-up material 2 inches thick, of effective conductivity 2.0 BTU·in/hr·ft^2·F deg, and window area 300 ft^2 of glass $\frac{1}{8}$ inch thick, if it is assumed that inside surfaces are at 60° F, outside surfaces at 30° F? Ans: 8.95 tons.

8. What thicknesses of (a) wood, (b) solid concrete, and (c) cast iron have the same insulating value as $\frac{1}{2}$ inch of corkboard?

3. CONVECTION AND RADIATION; NEWTON'S LAW OF COOLING

At low temperatures, the principal method of heat transfer in liquids and gases is *convection*. Liquids and gases are comparatively poor heat *conductors*, as will be seen from Table I; and if temperatures are low, radiation does not play an important role. The mechanism by which heat is transferred from a hot-water or steam 'radiator' to the air, walls, and objects in a room is almost entirely convection. The term 'radiator' is a misnomer. One reason for the inefficiency of the open fireplace is that it depends almost entirely on radiation for heat transfer to the room. While the temperature difference here, between the fire and the room, is much larger than in the case of a radiator, a large proportion of the heat is lost by convection *up the chimney* to the out-of-doors. One gets much more heat from the same fire burning in a stove, where heat can be transferred to the room by convection.

The process of heating a room by a radiator can be described as follows: The air molecules that collide with the radiator leave with more energy than they had before impact because the radiator is at higher temperature than the air. This process raises the temperature of a thin layer of air in immediate contact with the hot radiator, and the density of this air layer decreases. The hydrostatic balance in the room is thus upset. Archimedean buoyant forces come into play to cause the rarefied air to rise to the top of the room. Similarly, when room air comes into contact with the cooler outside walls of the room, the room air transfers heat to these walls; its temperature decreases, and it becomes more dense than the surrounding air. Again Archimedean forces come into play to cause this cooled air to sink to the floor. Thus a temperature gradient is set up, with warmer air near the ceiling, cooler air near the floor. The temperature gradient is maintained by a continuous circulation in which cool floor air is heated by the radiator and rises, while warm ceiling air is cooled by the walls and sinks. It is this circulation that effects the heat transfer from the radiator to the cool room walls and maintains an average air temperature in the room that is somewhere between that of the hot radiator and that of the cool walls and window glass.

The same process of convection is depended on to effect the circulation of hot water from furnace to radiator in older installations. (In more recent central-heating installations a circulating pump is usually included to increase the rate of circulation.) This process is illustrated schematically in Fig. 3. The water in the furnace coils is heated and expands somewhat. This expansion upsets the hydrostatic balance in the piping of Fig. 3. The lighter water which has been heated rises, the denser water which has been cooled by loss of heat in the radiator sinks, and thus a circulation is set up. This circulation depends on the fact that the two connections to the furnace coils are made at different heights. If both connections were made at the same height, the circulation would not be efficiently set up. The circulation is aided, for the same reason, if the circulating water enters the radiator at the top and leaves at the bottom.

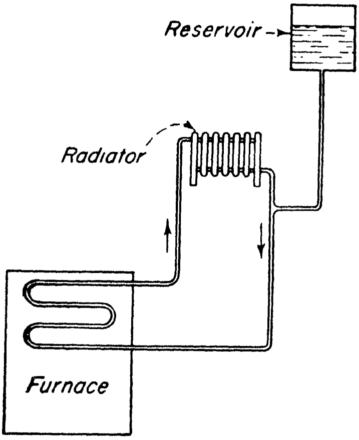

Fig. 3. Convective circulation of heat from furnace to radiator. (A reservoir or a pressure tank partly filled with air must be provided to allow for the net expansion or contraction of the water in the system.)

The laws governing the rate of heat transfer by convection are very complex because complex fluid-dynamic problems are involved. One law which was discovered empirically by Newton relates to the rate of cooling (or warming) of a given body when it is at a temperature different from that of the surrounding air. It states:

> *The rate of heat transfer to the surrounding air is proportional to the difference in temperature between the body and the air.*
>
> (Newton's law of cooling)

This law is found to agree with observed behavior very well when the cooling process is predominantly convective, even though a certain amount of the heat may be transferred by conduction and radiation.

If the rate of heat loss is proportional to the difference $T-T_{am}$ between the temperature T of the body and the temperature T_{am} of the surrounding air (called the *ambient temperature*), the rate of decrease of temperature of the body $(-dT/dt)$ will also be proportional to $T-T_{am}$. The temperature of a body 100° above the ambient temperature will start to fall rapidly, but when the body has cooled to 50° above its surroundings, its temperature will be falling only half as fast, since by Newton's law, heat will be removed from the body only half as fast.

We can write
$$-dT/dt=K(T-T_{am}), \tag{3}$$

where K is a constant for a given body. The *same* equation will hold for the rate of temperature rise (dT/dt) when the body is below the ambient temperature, $T < T_{am}$: $$dT/dt = K(T_{am} - T). \quad (3)$$

By integration of (3) we can determine the shape of the *cooling curve* (or warming curve). We write (3) in the form

$$\frac{dT}{T - T_{am}} = -K\,dt, \quad \text{or} \quad -K\int dt = \int \frac{dT}{T - T_{am}}.$$

Integration gives $$-Kt = \log_e(T - T_{am}) + C.$$

At $t = 0$, let $T = T_0$; then we evaluate the constant of integration as $C = -\log_e(T_0 - T_{am})$, and the above equation becomes

$$-Kt = \log_e(T - T_{am}) - \log_e(T_0 - T_{am}) = \log_e\left(\frac{T - T_{am}}{T_0 - T_{am}}\right).$$

Now the equation $x = \log_e y$ means that $e^x = y$, so

$$e^{-Kt} = \frac{T - T_{am}}{T_0 - T_{am}},$$

and $$T = T_{am} + (T_0 - T_{am})e^{-Kt}. \quad (4)$$

Because $e^0 = 1$, it is seen that this equation does give $T = T_0$ at $t = 0$. As time goes on, e^{-Kt} approaches zero, and T approaches T_{am}. A plot of equation (4) is shown in Fig. 4.

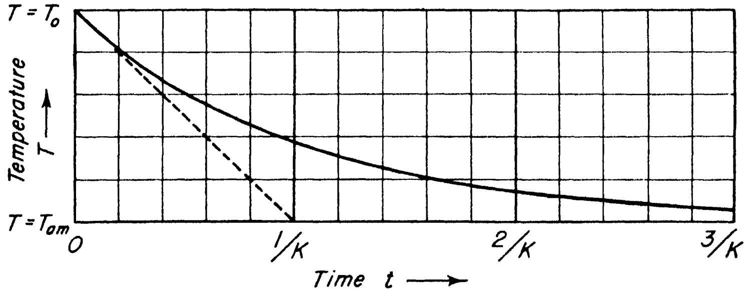

Fig. 4. Cooling curve given by Newton's law. The body starts at temperature T_0 at $t = 0$ and cools to the temperature T_{am} of its surroundings.

Because (3) has exactly the same form for the warming as for the cooling process, equation (4) applies to warming as well as to cooling. However in the case of warming, since T_0 is below T_{am}, it looks better to write (4) in the form

$$T = T_{am} - (T_{am} - T_0)e^{-Kt}. \quad (4)$$

The warming curve is plotted in Fig. 5.

The constant $\tau = 1/K$ is called the *time-constant* of the process because, as shown in Figs. 4 and 5, if the temperature of the body continued to fall or rise at the rate at which it started initially [$dT/dt = -K(T_0 - T_{am})$ initially], it would reach ambient temperature in the time

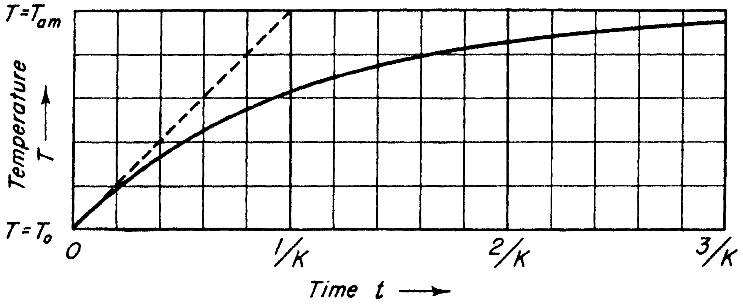

Fig. 5. Warming curve given by Newton's law. The body starts at temperature T_0 at $t=0$ and warms up to the temperature T_{am} of its surroundings.

$\tau = 1/K$. Actually, 63 per cent of its temperature change occurs in this time τ, and 99 per cent in the time 4.6 τ.

PROBLEMS

1. If a body cools at the rate of 0.8 F deg/sec when it is 40 F deg above its surroundings, how fast does it cool when it is 30 F deg above its surroundings? Ans: 0.6 F deg/sec.

2. If a body cools at the rate of 5 C deg/sec when it is at a temperature of 40° C and ambient temperature is 20° C, how fast does it warm up when it is at 20° C and the ambient temperature is 25° C?

3. Why do 'fins' increase the rate of heat transfer from a hot body to the surrounding air?

4. If one wants coffee to cool to a drinkable temperature as rapidly as possible, is it best to add the cream first and then wait, or to wait a bit first and then add the cream?

5. A metal can encloses a lighted 100-watt lamp bulb. If the heat capacity of the system is 0.5 kcal/C deg and if $K = 0.001$ sec^{-1}, how much is the temperature of the can above ambient? Ans: 48 C deg.

6. If a closed metal can filled with water and containing an electric heating element rises to a temperature of 100 F deg above ambient when 120 watts is supplied to the heating element, what will be the initial rate of cooling when the heater is turned off? Take the heat capacity as 4.5 BTU/F deg.

7. If it takes a body 5 min to cool from 100° C to 60° C, how long does it take it to cool from 60° C to 20° C when the ambient temperature is 10° C? Ans: 13.7 min.

8. If a body is X degrees above ambient at $t=0$, how many degrees does the temperature drop in each of eight successive time intervals each of length $\frac{1}{2}\tau$?

9. If a cup of coffee cools from 200° F to 120° F in 8 min when the ambient temperature is 70° F, determine K in Newton's law of cooling. Ans: $K = 0.119$/min.

10. Why is a Dewar flask or Thermos bottle (a) made double-walled, (b) evacuated, (c) silvered?

11. Verify that equation (4) is a correct solution of the differential equation (3) by computing dT/dt from (4) and substituting this and the value of T given by (4) in (3).

4. THE COOLING AND FREEZING OF WATER

If a pail of melted paraffin is placed on the floor in a room where the air temperature is below the melting point of paraffin, the paraffin freezes from the sides and bottom of the pail inward, shrinking as it freezes. This is the normal behavior of almost all materials.

If a pail of water is set outdoors on a day when the temperature is below 32° F, the water freezes in a way different from most other substances. It freezes in a layer at the top and the layer gradually thickens. It does not freeze inward from the sides or bottom of the pail.

The peculiarity of the freezing of water is due to the combined action of two unusual characteristics of water: (1) liquid water does not have its maximum density at its freezing point but at the slightly higher temperature of 4° C—this behavior is not characteristic of any other familiar liquid; (2) in its solid form (ice), water is less dense than in its liquid form, whereas the density of most materials is greater when they are in the solid form.

Let us now see why a lake freezes from the top down. The lake water is cooled by transfer of heat to cold air above the surface. As long as the water temperature is above 4° C, normal convection takes place; cooler water sinks, warm water rises, and the temperature of the lake water gradually approaches 4° C at all depths. Once this temperature is reached, further cooling of the surface layer does not result in convection, since the density of the surface layer now decreases as its temperature becomes lower. Surface water cooled below 4° C remains at the surface. Cooling of the surface layer continues until a thin layer of ice is formed; this layer of ice remains at the top, since the density of ice is less than the density of water. Immediately below the layer of ice is a layer of water at temperatures below 4° C; below this layer of water the main body of lake water remains at 4° C. If cooling continues the layer of ice gradually becomes thicker, but the water in the depths of the lake remains at 4° C.

Now consider the processes occurring when a pail of water freezes. Suppose water in contact with the sides of the pail is cooled below 4° C. Instead of waiting there to be frozen, it will, as a result of Archimedean forces, rise to the top of the pail. All water below 4° C will collect at the top of the pail, and only there will it freeze. The water can be made to freeze at the sides or bottom of the pail only if heat is extracted so rapidly that the convective process does not have time to carry the water away before it is frozen.

If water did not have the two unusual properties mentioned above, lakes would freeze from the bottom up rather than from the top down;

some would freeze solid; melting would be very slow. Such behavior would not only be bad for the fish and make the lakes less suitable for swimming and for skating; it also probably would result in altered climatic conditions in the vicinity of great lakes and in the polar regions.

5. HEAT TRANSFER IN THE ATMOSPHERE*

The heat transferred from the sun by *radiation* furnishes almost all of the heat of the earth† and is the source of all energy used by man except for a small amount of nuclear energy and tidal energy. The rate of solar radiation is such that 19.4 kcal per minute falls on each square meter of projected area at the top of the earth's atmosphere.‡ The projected area is the area measured normal to the sun's rays. The power received on a square meter of horizontal area (perpendicular to the earth's radius) will be seen from Fig. 6 to be 19.4 $\cos\psi$ kcal/min, where ψ is the angle between the radius to the point on the earth and that to the point where the sun is directly overhead. The angle ψ varies from 0 to 90° over the half of the earth that is exposed to solar radiation and for which this formula is applicable. No radiation is received directly from the sun by the dark side of the earth.

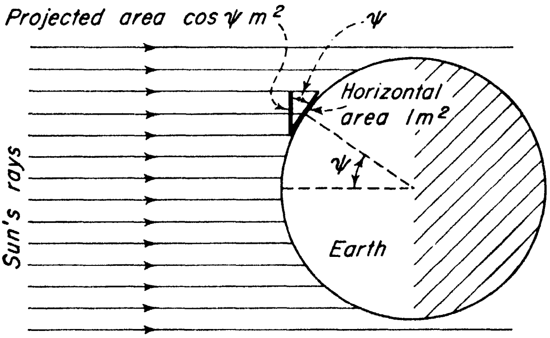

Fig. 6. Only the amount of solar energy passing through $\cos\psi$ m² of projected area falls on 1 m² of horizontal area at a point on the earth where the sun is at angle ψ away from the zenith.

There is a seasonal variation of the radiation received at any given latitude on the earth. The *total* radiation received on a square meter of

---

* This section may be omitted without loss of continuity.

† A small amount of heat comes from natural radioactivity in the interior of the earth. This is negligible in total compared to the heat received from the sun, but it accounts for the fact that the earth's interior is hotter than its surface. Small amounts of thermal energy are also generated by tidal effects, and small amounts are received from cosmic rays, meteors, and the light of the stars.

‡ The data on solar radiation given here are taken from Bulletin No. 79 of the National Research Council (Washington, 1931).

horizontal surface at the top of the atmosphere during the course of each half-year and the whole year is given in the following table:

Total Incident Solar Energy
(In millions of kcal/m^2)

Latitude	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°
Spring and summer	1.60	1.69	1.74	1.73	1.69	1.60	1.49	1.39	1.34	1.33
Autumn and winter	1.60	1.47	1.29	1.09	0.84	0.59	0.33	0.13	0.04	0
Whole year	3.20	3.16	3.03	2.82	2.53	2.19	1.82	1.52	1.38	1.33

This table contains in itself a rough qualitative explanation of the gross over-all winter and summer temperature variations over the surface of the globe. The gross temperature structure is modified in detail by ocean and air currents, rain, snow, and other climatic factors, which are in turn influenced by the physical features—oceans, continents, mountain ranges, vegetation—of the earth's surface.

Of the total radiation received at the top of the atmosphere, about 10 per cent is absorbed by the air and about 53 per cent by the earth, and about 37 per cent is reflected directly back into space. The surface of the earth, like that of all bodies above absolute zero, is in turn radiating energy. In fact, approximately the same amount of energy that is received from the sun must be reradiated into space, since the earth is not becoming hotter. The balance is not exact because some of the sun's energy is stored in the vegetation of the earth by the process of photosynthesis which changes light into chemical energy, and some of the previously stored energy is released by the combustion of wood, coal, oil, and natural gas.

As we shall see in Chap. 30, the sun, being very hot (6000° K), radiates mostly visible light. The sun's radiation is readily transmitted to the earth's surface, only about 10 per cent of it being absorbed by the earth's atmosphere. The earth, on the other hand, being comparatively cool (only about 300° K), radiates mostly at longer wavelengths in the infrared. Such wavelengths are strongly absorbed by the earth's atmosphere, to the extent of about 90%. Of such radiation absorbed by the atmosphere, much is returned to the earth's surface. The effect of the atmosphere is similar to that of the glass walls of a *greenhouse,* which transmit the sun's visible light but are opaque to the infrared light radiated by the earth and the plants. Similarly, the earth's atmosphere transmits most of the sun's radiation but absorbs most of the earth's radiation. Thus, the earth is hindered from reradiating into space but, being in a steady thermal state on the average, *must* radiate at a certain rate. The net result is that the earth, in order to accomplish the necessary reradiation in spite of this hindrance, has become considerably warmer than it would be if it were not for the *greenhouse effect* of the atmosphere.

The unequal heating of different parts of the earth's surface by solar radiation produces convectional processes resulting in a general pattern of atmospheric circulation that is one of the factors determining the types of climate existing in various regions. This convective circulation is largely limited to the lower portion of the atmosphere, which is called the *troposphere* and is some 9 to 12 km thick. Above the troposphere lies the *stratosphere;* the dividing line between troposphere and stratosphere is called the *tropopause.*

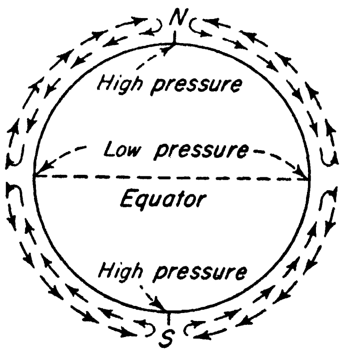

Fig. 7. The pattern of atmospheric circulation which would exist if effects of the earth's rotation were negligible and if the earth's surface were uniform.

If the effects of the earth's rotation could be ignored and if the earth's surface were either all land or all water, the convective circulation pattern would be simple. Warmed air near the earth's surface in the equatorial regions would be forced upward to high altitudes and would travel toward the polar regions, where it would descend and begin its return trip toward the equator along the surface of the earth. The general circulation pattern would be that shown schematically in Fig. 7. In the Northern Hemisphere the surface winds would be *north* winds (blowing *from the north* toward the south); in the Southern Hemisphere the surface winds would be *south* winds. The equatorial region would be a region of low pressure and the polar region would be a region of high pressure. The pressures mentioned here are surface pressures as read by a barometer, representing the total weight of a column of atmosphere above unit area on the surface. Surface winds tend to blow from a region of higher pressure to one of lower, but in the Northern Hemisphere they are deflected to the right by the effects of the earth's rotation, in the Southern Hemisphere to the left, as we shall discuss below.

Actually, there is a permanent equatorial low-pressure belt on the earth (called the *doldrums*), and the polar regions are regions of high pressure, but the observed wind pattern between these two regions bears little resemblance to the simple north-south pattern indicated in Fig. 7. The gross differences become understandable when the effect of the rotation of the earth is taken into account. Consider an air mass in the upper troposphere that is just north of the equator and is moving toward the north under the forces set up by the convection phenomenon. To an observer in space who is not on the rotating earth, this air mass has not only a small velocity to the north but has a large velocity to the east (~1500 ft/sec) equal to the eastward velocity of the surface of the earth in its rotation. The air mass thus has a large angular momentum about the axis of the earth. This angular momentum tends to be conserved as the air mass moves to the north. But as the air mass moves north it also moves closer to the earth's axis. The earth under it has less eastward velocity; the air mass tends to acquire more eastward velocity to conserve angular momentum, so that relative to the earth it becomes a wind *toward the east* (a *west wind*).

The result of this effect is that the winds in the upper troposphere veer toward the east as the air moves away from the equator. This effect is shown schematically in Fig. 8. When these upper winds reach a latitude of about 30°, they are primarily westerly winds (blowing eastward), and the air moving northward or southward from the equator tends to 'pile up' at these latitudes. The result is that high-pressure regions known as the *subtropical highs* develop near the 30° latitudes (called the *horse latitudes*). Air from the subtropical highs flows away along the earth's surface in the manner indicated in Fig. 9. As a result of the conservation of angular momentum, the resulting surface winds do not move directly north and south from the subtropical high but are deflected in the manner shown. The easterly winds from the subtropical high toward the equator are called the *trade winds;* these remarkably steady winds have been used for centuries by sailing vessels. The surface winds blowing northward from the subtropical high are deflected toward the east and are therefore termed *westerly winds.* The winds observed at latitudes near 40° are sometimes very violent and are responsible for the nautical term 'roaring forties' applied to these latitudes.

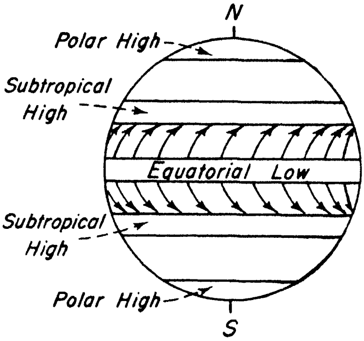

Fig. 8. Winds in the upper troposphere between the equatorial low and the subtropical highs.

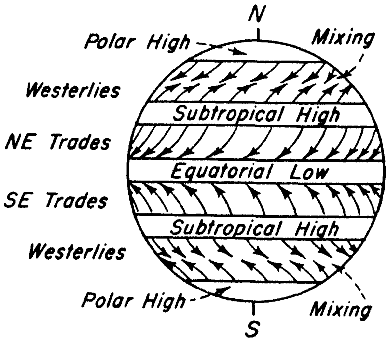

Fig. 9. Surface-wind belts.

Relatively warm air masses moving northward from the subtropical high encounter much colder air masses moving southward from the polar high-pressure area. The 'line' between the warm air masses from the subtropical high and the cold air masses from the polar regions is quite irregular. The succession of high- and low-pressure 'centers' present on the daily weather maps result from the irregularity of this line. These centers generally move eastward across the United States. The warm low-pressure centers come from the subtropical high and the cold high-pressure centers from the polar front.

The general pattern we have described undergoes a seasonal migration. The equatorial low is located between the equator and the Tropic of Cancer (23½° N. latitude) during the northern spring and summer and lies between the equator and the tropic of Capricorn during the northern autumn and winter; the rest of the circulation pattern shows a similar annual migration.

The simplified explanation of atmospheric circulation patterns given above would be expected to be adequate only for an earth with no surface irregularities. Actually, irregularities of land and sea areas exert so great an influence on the actual circulation pattern that their effects in some localities outweigh the effects to be expected from the general circulation we have described.

One of the major effects of surface irregularities results from the fact that land areas are heated or cooled much more quickly than surrounding bodies of water. The results of this differential heating are most important for large land masses. For example, southern Asia becomes much hotter than the neighboring ocean during summer; a local low pressure area is produced over the land, causing winds from the surrounding sea areas to blow toward the land during the summer months. On the other hand, southern Asia becomes much colder than the surrounding ocean areas during the winter and a local high pressure area is produced; winds from this land-centered high pressure area blow toward the sea during the winter months. These seasonal winds are known in India as *monsoons*.

The monsoon effect is but one of many phenomena arising from surface irregularities. It should be emphasized once more that these effects are superposed upon the general pattern we have described and are of great importance in the determination of the climate of any locality.

PROBLEM

1. Assuming that heat is absorbed at equal rates by equal areas of land and water, give two reasons why land areas are heated much more quickly than surrounding bodies of water.

CHAPTER 17

PROPERTIES OF GASES

The laws that govern the behavior of most of the common gases under ordinary conditions are remarkably simple. These *ideal gas laws* are obeyed extremely closely by such gases as air, oxygen, nitrogen, hydrogen, and helium under all conditions except those of extremely low temperature or extremely high pressure. They are very closely obeyed, too, by carbon dioxide under ordinary conditions of temperature and pressure. However Dry Ice—solid CO_2—can exist at normal pressure but at the low temperature of $-80°$ C; and liquid carbon dioxide—used to carbonate beverages—can exist in tanks at normal temperature but at the high pressure of about 65 atmospheres. Clearly, under these conditions the CO_2 is not behaving like an ideal gas. It turns out that the vapor of any substance behaves like an ideal gas if the pressure is sufficiently low and the temperature sufficiently high. If the temperature is decreased, or the pressure raised, to a value near to that required for condensation to a liquid, then large departures from the ideal gas laws are observed.

The ideal gas laws furnish such a good approximation to the behavior of most gases under most conditions encountered in scientific and engineering work that it will be profitable to devote this chapter to a study of *the ideal gas.* In the succeeding two chapters we shall discuss cases of departure of gases from ideal behavior. We shall not continue to repeat the word *ideal* in this chapter because to ordinary engineering and technical accuracy most gases of interest *are* ideal under most conditions.

1. THE GAS LAWS

There are three experimental laws, known as Boyle's law, Charles' law, and Avogadro's law, which together enable us to compute at once the density of any gas at any temperature and pressure.

The density of a gas depends on only three variables: the pressure, the temperature, and of course the kind of gas.

Boyle demonstrated experimentally that

> *If the temperature of a given kind of gas is held constant, its density is directly proportional to the pressure.* (BOYLE'S LAW)

If the pressure on a given sample of gas is doubled, its volume becomes one-half, and hence its density is doubled; and so on.

Charles and Gay-Lussac demonstrated experimentally that

> *If the pressure on a given kind of gas is held constant, its density is inversely proportional to its absolute temperature.* (CHARLES' LAW)

Thus, if a given sample of gas at atmospheric pressure is heated from 0° C to 273° C (273° K to 546° K), its absolute temperature doubles, its volume doubles, and its density becomes one-half.

Avogadro demonstrated experimentally that

> *At the same temperature and pressure, different kinds of gases have densities proportional to their molecular weights.* (AVOGADRO'S LAW)

This law implies that equal volumes of different gases at the same temperature and pressure contain the same numbers of molecules—since if they contain the same numbers of molecules they will have masses and hence densities proportional to the average masses of the individual molecules, that is, proportional to the molecular weights.*

If we let ρ stand for the density, p for the absolute pressure, T for the absolute temperature, and M for the molecular weight (*the molecular weight is to be considered as dimensionless: it is 16 times the ratio of the average mass of a molecule of the gas to the mass of a single* O^{16} *atom*),† we can combine the above three laws into a single law which says that

$$\rho \propto pM/T.$$

This is called the *general gas law.* In this proportionality, the proportionality constant must be a universal constant since we have taken into account all factors influencing the density of a gas. Hence if we write the general gas law in the form

$$\boxed{\rho = \frac{1}{R}\frac{pM}{T},} \tag{1}$$

---

* The reader is presumed to have learned about molecular weight, moles (frequently called gram-molecules), Avogadro's number, and similar concepts from previous work in chemistry. Hence, these ideas are introduced very informally in this chapter. However, a complete and accurate set of definitions of these quantities is given in italics in the following text.

† Molecular weight is discussed in some detail in Chap. 34. Natural oxygen contains atoms of three different masses, called *isotopes* O^{16}, O^{17}, *and* O^{18}. The ratios of the masses of these isotopes are 16.0000:17.0045:18.0049, and their relative abundances are in the ratios 2500:1:5. We shall employ the *physical scale* of atomic and molecular weights, in which the atomic weight of the predominant isotope O^{16} is taken as 16 exactly. This makes the molecular weight of natural O_2 come out as 32.0087. The difference between this value and the value of exactly 32 that is assigned on the *chemical scale* is unimportant for most applications of the gas laws.

The molecular weight of a gas must be defined in terms of the *average* mass of the molecules because even a 'pure' gas such as H_2 contains isotopes of different masses.

A selected list of atomic weights will be found in Sec. 2 of the Appendix to this volume; a complete list will be found in the Appendix to vol. II.

as is customary, the constant R must be a universal constant independent of the kind of gas considered. R is called *the universal gas constant.* We shall compute its value in a moment. Equation (1) is known as the *equation of state* of a gas.

Since density is mass per unit volume, if a mass m of gas occupies volume v, we have

$$\rho = m/v.$$

If we substitute this in (1), we obtain

$$\frac{m}{v} = \frac{1}{R}\frac{pM}{T}.$$

This enables the equation of state to be written in the form

$$v = \frac{RT}{pM}m, \tag{2}$$

which gives the volume occupied by a mass m of any kind of gas under any pressure and temperature; in the form

$$m = \frac{pM}{RT}v, \tag{3}$$

which gives the mass of gas contained in volume v under any conditions of pressure and temperature; or in the form

$$p = \frac{m}{M}\frac{RT}{v}, \tag{4}$$

which determines the pressure of mass m of gas in volume v at temperature T.

We can use equation (2) to compute the value of R. It is observed that *one mole of any gas at normal temperature and pressure* occupies a volume of* 22,421 cm³. We shall translate this statement into MKS units and substitute in (2). The volume in question is

$$v = 22{,}421 \times 10^{-6}\ \text{m}^3.$$

Normal temperature is $T = 273.16°\ \text{K}$,

as we have seen on p. 323. Normal pressure is the pressure exerted by a column of mercury exactly 0.76 m high when it is at 0° C and under standard gravity. As we have seen on p. 83 this corresponds to

$$p = 10{,}332\ \text{kgf/m}^2 = 1.0132 \times 10^5\ \text{nt/m}^2.$$

A mole of gas is, by definition, M grams *of the gas, or* $10^{-3}M$ kg *of the gas;*

---

* 'Normal temperature and pressure' are defined as 0° C and 1 atm. We shall abbreviate this frequently recurring expression as NTP.

hence $$m = 10^{-3}M \text{ kg}.$$

Solving equation (2) for R and substituting these values, we obtain

$$R = \frac{vpM}{Tm} = \frac{22{,}421 \times 10^{-6} \text{ m}^3 \times 1.0132 \times 10^5 \text{ nt}\cdot\text{m}^{-2} \times M}{273.16 \text{ K deg} \times 10^{-3}M \text{ kg}}$$
$$= 8317 \text{ nt}\cdot\text{m/kg}\cdot\text{K deg},$$

or $$\boxed{R = 8317 \text{ joule/kg}\cdot\text{K deg} = 8317 \text{ m}^2/\text{sec}^2\cdot\text{K deg},} \tag{5}$$

since the nt·m is a joule, and since the nt is a kg·m/sec². This is the value of R to be used in (1), (2), (3), and (4) when all other quantities are expressed in MKS units.

If we are interested in the variation of density of a given kind of gas as the temperature and pressure vary, we note from (1) that

$$\frac{\rho T}{p} = \frac{M}{R},$$

and that the right side of this equation is constant so long as we consider only one kind of gas. Hence, if the density is ρ_1 when the temperature is T_1 and the pressure is p_1, and is ρ_2 when the temperature is T_2 and the pressure p_2, we have

$$\frac{\rho_1 T_1}{p_1} = \frac{\rho_2 T_2}{p_2}. \tag{6}$$

Equation (6) holds so long as we consider only one kind of gas.

If we are interested in the variation of volume of a given mass of a certain gas as we change the temperature and pressure, we note from (2) that

$$\frac{pv}{T} = \frac{Rm}{M},$$

and that the right side of this equation is constant so long as we consider a given mass of a certain gas. Hence, *for a given mass of gas under two different conditions,*

$$\frac{p_1 v_1}{T_1} = \frac{p_2 v_2}{T_2}. \tag{7}$$

Let us now examine in more detail the properties of a gas that contains a mixture of molecules of different types, say a mass m_X of molecules of molecular weight M_X, a mass m_Y of molecules of molecular weight M_Y, and a mass m_Z of molecules of molecular weight M_Z. The molecular weight of such a mixture is defined as an average over the individual molecules of the gas, so that if the gas contains n_X molecules of type X, n_Y of type Y, and n_Z of type Z, the molecular weight is

$$M=\frac{n_X M_X+n_Y M_Y+n_Z M_Z}{n_X+n_Y+n_Z}. \tag{8}$$

It is now convenient to introduce a new unit of mass, the *atomic mass unit* (amu), which is becoming of increasing usefulness in modern physics and which we shall have frequent occasion to use in Chap. 44. The atomic mass unit is defined as one-sixteenth the mass of a single atom of O^{16}; in other words, the mass of a hypothetical atom of atomic weight unity is 1 amu. Then *the average mass of an individual molecule of molecular weight M is M* amu (keep in mind that M itself is a dimensionless pure number). Since one mole has a mass of $10^{-3}M$ kg and contains 6.0251×10^{23} molecules (*Avogadro's number*), the average mass of an individual molecule is

$$M\text{ amu}=(10^{-3}M\text{ kg})/(6.0251\times 10^{23}),$$

whence

$$1\text{ amu}=1.6597\times 10^{-27}\text{ kg}. \tag{9}$$

Since the number of molecules in mass m_X of gas is the mass of gas divided by the average mass of an individual molecule, for our mixture of gases above we can write

$$n_X=\frac{m_X}{M_X\text{ amu}};\qquad n_Y=\frac{m_Y}{M_Y\text{ amu}};\qquad n_Z=\frac{m_Z}{M_Z\text{ amu}}. \tag{10}$$

If we now substitute these three expressions in (8), we obtain

$$M=\frac{m_X+m_Y+m_Z}{\dfrac{m_X}{M_X}+\dfrac{m_Y}{M_Y}+\dfrac{m_Z}{M_Z}}=\frac{m}{\dfrac{m_X}{M_X}+\dfrac{m_Y}{M_Y}+\dfrac{m_Z}{M_Z}}, \tag{11}$$

where $m=m_X+m_Y+m_Z$ is the total mass of gas.

From (11) we can determine the ratio m/M for a mixture of gases and can write the equation of state (4) in the form

$$p=\left(\frac{m_X}{M_X}+\frac{m_Y}{M_Y}+\frac{m_Z}{M_Z}\right)\frac{RT}{v}. \tag{12}$$

We now notice that the first term on the right side, namely $(m_X/M_X)(RT/v)$, is, according to (4), the pressure that molecules of type X would exert if they alone occupied the whole volume v. The second and third terms are similarly the pressures that molecules of type Y and Z would exert if alone in v. The separate terms are known as the *partial pressures* of the different types of molecules. Equation (12) expresses

DALTON'S LAW OF PARTIAL PRESSURES: *If several types of gas are put into the same container, the total pressure p exerted is the sum of the partial pressures which each type of gas would exert if it alone occupied the container.*

Dalton's law is useful in determining the pressure resulting when gases of different types are mixed. Comparison of (12) with (10) shows that *the partial pressures of the different types of gas are proportional to the numbers of molecules of each type in the container.*

In normal *dry air*, 78.09 per cent of the molecules are N_2, 20.95 per cent are O_2, 0.93 per cent are A (argon), and the other 0.03 per cent are CO_2, Ne, Kr, Xe, He, and H_2. For this composition, (8) gives a molecular weight of 28.97. Thus, *for all ordinary purposes, dry air behaves like an ideal gas of molecular weight* 29.0. At NTP, 1 mole (29.0 g) of dry air would occupy 22.4 liters, so dry air would have the density

$$\rho = \frac{m}{v} = \frac{0.0290\ \text{kg}}{0.0224\ \text{m}^3} = 1.293\ \frac{\text{kg}}{\text{m}^3}. \tag{13}$$

This is a convenient number to remember, since from it the density of any other gas can be computed by noting that density is proportional to molecular weight. Oxygen would be $^{32}\!/_{29}$ as dense, hydrogen $^{2}\!/_{29}$ as dense, carbon dioxide $^{44}\!/_{29}$ as dense, etc. It is noted that the density of air at NTP is a little over $^{1}\!/_{1000}$ the density of water, which is 1000 kg/m³.

All the relations of this section remain true in the British engineering system of units commonly used by mechanical and aeronautical engineers. In this system we express

m in slugs, p in lbf/ft²,

v in ft³, T in Rankine deg,

ρ in slug/ft³, R in ft·lbf/slug·R deg.

There is no alteration of molecular weight M, since this is dimensionless.

If the units in (5) are changed to the British system, the universal gas constant has the value

$$R = 49{,}690\ \text{ft·lbf/slug·R deg} = 49{,}690\ \text{ft}^2/\text{sec}^2\text{·R deg}. \tag{14}$$

It is this value of R that must be used in the equations of this section when English units are employed.

PROBLEMS

NOTE: Pressure is *absolute* unless otherwise specified. *Gauge pressure,* as read by most pressure gauges, is pressure above actual atmospheric pressure as it exists on the outside of the gauge. Pressures in cm refer to cm of mercury unless otherwise specified. Use the values 273° K = 0° C; 460° R = 0° F.

1. If 1 liter of a certain gas at 76 cm and 0° C has a mass of 1.87 g, what would be the mass of a liter of the same gas at 60 cm and 47° C? Ans: 1.26 g.

2. If air has a density of 1.29 kg/m³ at sea level when the pressure is 76 cm and the temperature is 0° C, what will be its density at 30,000 ft altitude where the pressure is 20 cm and the temperature −50° C?

3. The measured volume of a quantity of hydrogen collected over mercury as in Fig. 1 is 700 cm^3 when the temperature is 23° C and the mercury stands $h = 5$ cm above the reservoir. The barometer reads 72 cm. What is the volume of H_2 at NTP, and what is its mass? Ans: 569 cm^3; 0.0512 g.

4. The measured volume of a quantity of CO_2 collected over mercury as in Fig. 1 is 450 cm^3 when the temperature is 20° C and the mercury stands at a height $h = 8$ cm above the reservoir. The barometer reads 74.6 cm. What is the volume of CO_2 at NTP, and what is its mass?

5. An 'empty' bottle is corked at 70° F and put in an oven. At what temperature will the pressure inside have increased by 50 per cent? Ans: 335° F.

6. What happens to the volume of a quantity of gas if the temperature is doubled and the pressure is tripled?

7. Air flasks used for charging torpedoes hold 25 ft^3 of air at 3000 lbf/in^2 and 75° F. How many ft^3 of air is this at NTP and what is its mass in pounds? Ans: 4390; 354 lb.

8. An oxygen tank holds 2 ft^3 at a pressure of 30 atm and a temperature of 70° F. After some oxygen is withdrawn, the pressure is 6 atm and the temperature 60° F. How many pounds of oxygen were withdrawn?

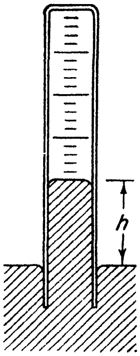

Fig. 1.

9. A carelessly made barometer has some air trapped above the mercury. The mercury stands at 73.2 cm when the air space is 10 cm long. When the barometer tube is pushed down into the reservoir so that the air space is only 5 cm long, the mercury stands at 71.6 cm. What is the true barometric reading? Ans: 74.8 cm.

NOTE: A schematic diagram of a constant-volume gas thermometer is shown in Fig. 2. The volume of gas in the bulb is kept constant by raising or lowering the mercury reservoir so that the mercury in the right-hand tube always stands at the etched mark. This type of thermometer gives very accurate temperature measurements over a wide range in terms of the simple law that pressure is proportional to temperature if volume is constant. For accurate work, corrections which we shall ignore are applied for the thermal expansion of the bulb, for the part of the gas that is in the narrow stem and is not at the same temperature as the rest, and for the change in density of mercury in the manometer with change in room temperature.

10. In an experiment with a constant-volume gas thermometer the mercury level in the reservoir is 7.8 cm below the etched mark when the bulb is in melting ice; it is 17.2 cm above the etched mark when the bulb is in steam from boiling water. The barometer reads 76.0 cm. From these data and the definitions of the fixed points of the centigrade thermometer, determine the value of the absolute zero of temperature on the centigrade scale.

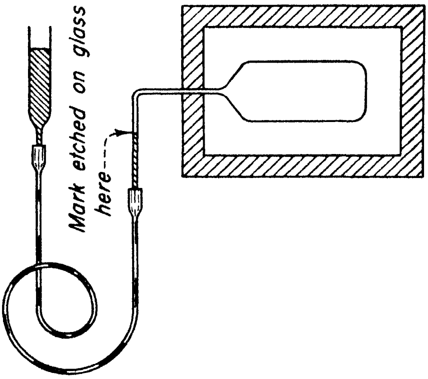

Fig. 2. Constant-volume gas thermometer (prototype).

11. Hydrogen behaves like a perfect gas down to liquid-air temperatures. If a constant-volume hydrogen thermometer (see Fig. 2) has mercury level 28.5 cm

above the etched mark when the temperature of the bulb is 0° C, 41.4 cm below the etched mark when the bulb is immersed in boiling liquid oxygen, and 46.4 cm below the etched mark when the bulb is immersed in boiling liquid nitrogen, and if the barometer reads 76.0 cm, determine the boiling points of oxygen and nitrogen.
Ans: −183° C; −196° C.

12. The airtight bag of a radiosonde balloon is partially inflated with 100 m^3 of He at the ground under conditions of NTP. What is the maximum load (weight of bag plus suspended gear) that can be lifted?

13. As the balloon of Prob. 12 rises, the helium expands to fill the bag more and more completely. If the bag is not yet filled out taut at a height of 10 km where the pressure is 20 cm and the temperature is −55° C, what is the maximum load that can be supported at this altitude? Ans: 111 kgf.

14. If one makes the crude assumption that the temperature of the air does not vary as one ascends vertically, the variation of pressure with altitude h is given by the formula

$$p = p_0 e^{-gMh/RT}$$

where p_0 is the pressure at sea level where $h = 0$, M is the molecular weight of the atmosphere, and T and g are assumed constant. Verify this equation by showing that the decrease in pressure $-dp$ when h is increased by dh equals the weight $\rho g\, dh$ of a unit column of air of thickness dh.

15. Using the formula of Prob. 14, compute the pressures in an isothermal atmosphere at heights of 1000, 2000, 4000, 8000, and 16,000 ft, taking $p_0 = 76$ cm, $T = 0°$ F. (These are not far from the observed pressures at these altitudes.)
Ans: 73.0, 70.0, 64.5, 54.8, 39.5 cm.

16. Show that from a continuous plot of temperature vs. pressure it is possible to determine accurately by numerical computation the height to which a balloon ascends. The only assumption that need be made is in regard to the average molecular weight of the air, which actually varies slightly (but only slightly) with changing water-vapor content.

17. How many cubic feet of air at atmospheric pressure and 20° C is required to increase the gauge pressure of a tire of 2 ft^3 capacity from 20 lbf/in^2 to 35 lbf/in^2 at 20° C when the atmospheric pressure is 15 lbf/in^2? Assume that the tire volume does not change.

18. A vessel contains 39.9 g of argon (A) and 4.032 g of hydrogen (H_2) under a total pressure of 76 cm. Find the partial pressures of the argon and the hydrogen and the volume of the vessel.

19. A vessel contains 16.03 g of methane (CH_4) and 42.03 g of nitrogen (N_2). The total pressure is 200 cm, the temperature 100° C. Find the partial pressures of the methane and of the nitrogen and the volume of the vessel.
Ans: 80 cm; 120 cm; 29.1 liters.

20. A diving bell contains 125 ft^3 of air at normal pressure when it is at the surface of the water. After it has been lowered to a depth of 150 ft, what are the pressure and volume of the air in it? What volume of additional air, measured at normal pressure, must be pumped in to again fill the bell? Neglect temperature change and water-vapor content.

21. Two glass bulbs, of volumes 400 cm^3 and 200 cm^3, are connected by a capillary tube. This container is sealed when it contains dry air at 20° C and 76 cm. The larger bulb is then immersed in steam at 100° C, the smaller in ice water at 0° C. Neglecting thermal expansion of the glass, find the resulting pressure. Ans: 86.2 cm.

22. Derive (14) from (5) by conversion of units.

23. Using (1) with the gas constant in British units, determine the density of air in slug/ft^3 at NTP (14.7 lbf/in^2 and 32° F). For check, compare your answer with (13) by conversion of units.

24. Using (3) with the gas constant in British units, determine the mass in pounds of 25 ft^3 of O_2 gas at a pressure of 2000 lbf/in^2 and a temperature of 72° F.

2. THE EARTH'S ATMOSPHERE*

Problems 14 and 15 of the previous section concern an approximation to the earth's atmosphere in which the temperature is assumed to be constant. For such an atmosphere the pressure would vary with altitude h according to the relation

$$p = p_0 e^{-(gM/RT)h} \tag{15}$$

where p_0 is the sea-level or ground-level pressure. Such a pressure curve,

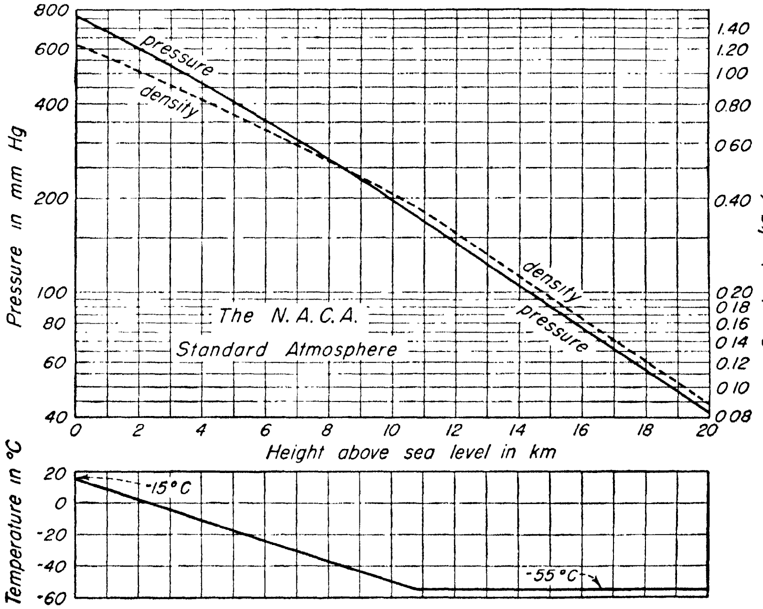

Fig. 3.

if plotted on semilog paper as in Fig. 3, would give a straight line, since

$$\log_e p = \log_e p_0 - (gM/RT)h, \tag{16}$$

so that $\log_{10} p = 0.434 \log_e p$ would decrease in proportion to h.

Since the actual atmosphere is not strictly isothermal, the pressure variation departs somewhat from the above equation. The atmospheric temperature is typically represented by a curve in which the temperature falls linearly up to the top of the troposphere at a height of about 10 km and is then roughly constant at about −55° C up to the bottom of the ozone layer at about 35 or 40 km. The fall of temperature in the troposphere, which is the region of thermal circulation of the atmosphere (see p. 366) is associated with the cooling of the rising air which undergoes adiabatic expansion and the warming of the sinking air which undergoes adiabatic compression. These effects will be explained in Chap. 19.

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* This section may be omitted without loss of continuity.

Typical curves for the variation in temperature, pressure, and density with height up to 20 km above sea level are given in Fig. 3. These are the National Advisory Committee for Aeronautics (N.A.C.A.) standard curves which are used for aircraft performance specification and testing. They are intended to represent a mean for various seasons of the year and various locations on the earth. It will be seen that the pressure and density curves are straight lines on semilog paper for the isothermal region above 10.7 km but are not straight in the lower region where temperature varies.

The variation of atmospheric temperature at still higher altitudes, as shown in Fig. 4, is interesting. At about 50 km, above the approximately isothermal region, is a layer of air containing considerable ozone which strongly absorbs the sun's ultraviolet light. (It is largely this ozone layer that prevents the physiologically dangerous rays of wave length below 300 mμ from reaching the earth's surface.) This layer is warm, having temperatures usually higher than surface temperatures. Above this layer the temperature falls again (to very low values on the basis of V-2 tests) and then above 80 km starts to rise again to temperatures much higher than those elsewhere in the atmosphere. The two regions in which temperature rises with altitude are known as the lower and upper 'temperature inversions.' These temperature inversions have an important bearing on the long-range transmission of sound, as we shall discuss in Chap. 22.

Figure 4 shows, by the breadth of the crosshatched region at any altitude, the maximum and minimum temperatures to be expected at any season and at any point on the earth. These are N.A.C.A. estimates which were made before V-2 data became available. They are well established by radiosonde balloon measurements up to something over 30 km. Above this height they were predicted from all available indirect evidence on meteors, radio and sound transmission, atmospheric absorption and emission of light, and so on. Data from two V-2 flights are available and confirm the N.A.C.A. predictions as to the form of the curves, although they fall somewhat outside the N.A.C.A. limits.

The methods of obtaining temperatures on the V-2 flights is of interest. No ordinary thermometer will work at the supersonic speeds of the V-2 rocket because the very motion causes compression and heating of the air for the same reason that meteors or 'shooting stars' become incandescent and usually burn up. However, it is possible to make pressure and altitude measurements from which the ambient temperature can be determined. The pressure is automatically radioed back to the earth. At the same time the height of the V-2 is continuously measured by radar, so that a pressure vs. height curve is obtained.

Now let the pressure at height h be p and the pressure a little while later at height $h+\Delta h$ be $p-\Delta p$. The pressure difference must be the weight of a unit column of air of height Δh, or

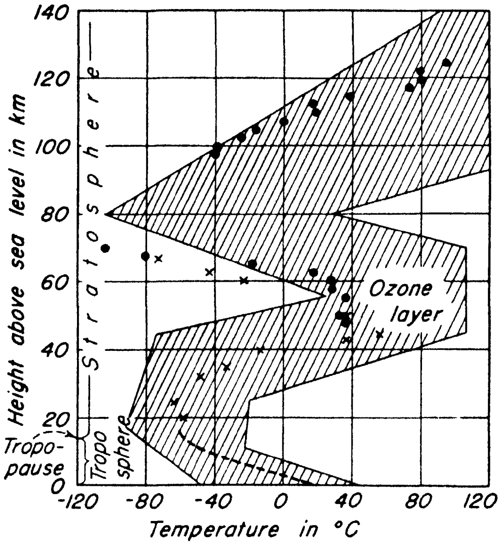

Fig. 4. Temperatures in the upper atmosphere. The extreme ranges predicted by the National Advisory Committee for Aeronautics are shown by the limits of the cross-hatched area. The dots and crosses show the first actual measurements in the very high atmosphere—those of the V-2 rocket flights of 7 March 1947 (dots) and of 22 January 1948 (crosses). The broken line shows data obtained from meteorological radiosonde balloons at the time and location of the V-2 tests at White Sands, N. M.

$$\Delta p = \rho_m g \, \Delta h,$$

if ρ_m is the mean density in the column. From this equation the value of ρ_m can be determined. Let p_m be the mean pressure in the column, which is known. Then, from (1),

$$p_m = \rho_m RT/M,$$

where M is the molecular weight of the air, which is known (V-2 samples and other evidence show that this does not vary very much with altitude). This equation can be solved for

$$T = p_m M / \rho_m R. \tag{17}$$

It is in this way that temperature is obtained on the V-2 flights.

3. EXTERNAL WORK; INTERNAL ENERGY

If a gas expands by pushing apart the walls of its container, as in the case of the expansion of the gas in a balloon or in the cylinder of Fig. 6 (p. 383) when the piston moves to the right, the gas does mechanical work on the walls of the vessel. Such work is done only if the walls of the container move, since work is done only if a force acts through a *distance*. This work is called the *external work* done *by* the gas. In the case of Fig. 6, the external work done by the gas is the product of the distance the piston moves to the right by the average force exerted by the gas on the piston because of the gas pressure. If the piston moves to the *left* in Fig. 6, external work is done *on* the gas; in this case the external work done *by* the gas is considered as a negative quantity.

The total microscopic mechanical energy (thermal energy) of the molecules of a gas is called the *internal energy of the gas.* The internal energy includes all the energy of thermal motion—kinetic energy of translation and rotation, and kinetic and potential energy of vibration of the individual molecules—as well as any potential energy which may arise from forces between the different molecules of the gas.

If one adds *heat* to a gas, this heat all goes into increasing the internal energy of the gas, *provided that* the gas does no external work while the heat is being added. But if the gas pushes back the walls of the container and hence does external work while the heat is being added, conservation of energy requires that some of the added heat be used to supply the energy needed for this external work. This is the case, for example, if the pressure on the gas is kept constant while heat is being added. In Fig. 6, if the net force F which pushes on the piston from the right is kept constant, the pressure of the gas will remain constant since the piston is in static equilibrium. If heat is added under these conditions, the gas will expand and do external work; part of the added heat goes into increasing the internal thermal energy and part into doing external work.

From the above arguments we conclude that conservation of energy requires the following energy balance:

$$\textbf{(heat added to a gas)} = \textbf{(increase in internal energy)} + \textbf{(external work done by the gas)}. \tag{18}$$

This fundamental relation is known as the *first law of thermodynamics* as applied to gases. It is noted that each of the three terms in the above equation may be either positive or negative. Heat subtracted from the gas is considered a negative amount of heat added. A decrease in internal energy is a negative increase. External work done *on* the gas when the gas contracts is expressed as negative external work done *by* the gas.

The internal energy per unit mass of gas, for example the number of joules/kg, kcal/kg, or BTU/lb, we shall call the *specific internal energy* and denote by e. Since the state of a gas is completely determined when temperature, pressure, and the kind of gas are specified, the specific internal energy can depend only on these three quantities.

Everything that has been said in this section so far holds rigorously for real gases. We now state an important experimental law that is found to hold *extremely closely* for all real gases for which the general gas law is closely obeyed, and which is assumed to hold *exactly* for our ideal gas. This experimental law is that

> *The specific internal energy of a gas depends only on the kind of gas and the temperature; it is independent of the pressure.*

This means that there is no variation of specific internal energy of a given kind of gas with variations in pressure, and consequent variations in density, so long as the temperature does not change. It means that the internal energy of a given mass of a given kind of gas is known once the temperature is given, no matter what the pressure or volume may be.

The simplest demonstration of this law is by means of Joule's free-expansion experiment, which employs the apparatus shown in Fig. 5. Two copper vessels are connected by a pipe fitted with a stopcock. Initially one vessel is filled with a gas and the other vessel is evacuated. The two vessels are then immersed in a carefully insulated water calorimeter and allowed to come into thermal equilibrium with the water. After thermal equilibrium has been established, the stopcock is opened and the gas is allowed to rush from the full vessel into the empty one. After thermal equilibrium has been established again throughout the system, it is found that *the temperature of the water has not changed.* Since the gas is in thermal equilibrium with the water both before and after the expansion, *the temperature of the gas has not changed.* Since the temperature of the water does not change, no heat is gained or lost by the water, so *no heat has been added to or subtracted from the gas.*

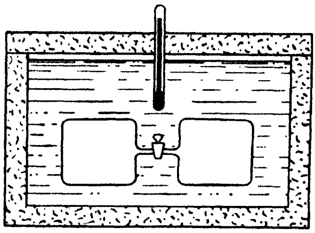

Fig. 5. Joule's free-expansion experiment (schematic).

The rest of the argument consists in applying the first law of thermodynamics, equation (18), to the over-all effects of this expansion process. As noted above, the experimental result indicates that *no heat is added* to the gas. It is also clear that *no external work is done* by the gas, because the walls of the vessel do not move. Hence, according to (18), *the internal energy of the gas does not change.* We have noted above that *the temperature of the gas does not change.* So here we have the same mass of gas at two quite different states of pressure, volume, and density but at the same temperature, and have demonstrated experimentally that the internal energy in these two states is the same.

It is by this argument that one derives from Joule's experiment the conclusion that the specific internal energy is independent of gas density if the temperature is constant, and completely determined by kind of gas and temperature.

By a more accurate type of experiment, in which gas expands continuously through a porous plug, Kelvin and Joule were able to measure the very slight departures from the above law that real gases do exhibit.

4. SPECIFIC HEAT OF A GAS AT CONSTANT VOLUME

If we place a sample of gas in a container of fixed volume and add heat, all the heat added goes into increasing the internal energy of the gas. (This statement is true only if the volume of the gas is held constant.)

> The amount of heat that must be added to a unit mass of gas to increase its temperature by one degree when the volume is held constant is called the *specific heat at constant volume* and is denoted by c_v.

Like the specific heats for solids and liquids which were defined in Chap. 16, this specific heat has the same numerical value in kcal/kg·C deg, in cal/g·C deg, and in BTU/lb·F deg. The amount of heat that must be added to a mass m of gas to increase its temperature ΔT degrees at constant volume is

$$Q = mc_v \, \Delta T. \tag{19}$$

Since this heat all goes into internal energy, c_v is a measure of the increase in internal energy when unit mass is heated one degree. The increase in specific internal energy (internal energy per unit mass) will be given by multiplying the specific heat at constant volume by the change in temperature:

$$\boxed{\Delta e = c_v \, \Delta T.} \tag{20}$$

This equation is of particular importance because *it gives the increase in specific internal energy that is associated with a temperature rise ΔT whether or not the volume changes while the temperature is rising.* This statement follows from the fact that the change in internal energy is dependent purely on the change in temperature. Thus, by a measure of the specific heat at constant volume, where we know that all the heat goes into internal energy, we can determine the coefficient of ΔT in the expression for the change in internal energy for any type of process. For a mass m of gas, the change in internal energy will be

$$m\,\Delta e = mc_v\,\Delta T \tag{21}$$

for any type of process. Numerical values of c_v will be considered in the next section.

5. WORK DONE BY AN EXPANDING GAS; SPECIFIC HEAT AT CONSTANT PRESSURE

If the gas in the cylinder of Fig. 6 is at pressure p, it exerts a force F on the piston equal to pA, where A is the area of the piston. We imagine the piston to be frictionless but still gastight. Then for equilibrium, a resultant force F, equal and opposite to that exerted by the gas, must be exerted externally on the piston.

By varying the external force, or by adding or subtracting heat, we can cause the gas to expand or contract. We shall now derive a general equation for the work done by the gas when it expands from volume v_1 to volume v_2.

In Fig. 6, if the piston moves to the right a distance dx, the work that the gas does on the piston is

$$dW = F\,dx = pA\,dx.$$

But $A\,dx$ is the increase of volume of the gas when the piston of area A moves to the right a distance dx. We write $A\,dx = dv$ and obtain

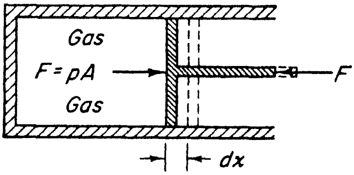

Fig. 6. Work done by a gas in expanding. The force F acting on the right side of the piston is the resultant of the force exerted by the pressure of the external atmosphere and any force that may be applied to the piston rod directly.

$$dW = p\,dv.$$

It can be readily shown that this relation gives the work done whenever a gas exerting a pressure p increases its volume by dv, whether the volume changes in the geometrical configuration of Fig. 6 or in any other geometrical configuration, such as, for example, the expansion of the gas in a balloon or of a gas bubble under water, or even of a quantity of air in a house which must push aside the outdoor atmosphere as it expands when it is heated.

If, then, a quantity of gas expands from volume v_1 to volume v_2, the external work done by the gas is

$$W = \int_{v_1}^{v_2} p\,dv. \tag{22}$$

This relation holds no matter how the pressure may vary as the volume changes, but in order to evaluate the integral, we must know the pressure p as a function of the volume v.

An especially simple case is that of an *expansion* at *constant pressure*, in which p is constant and can be taken outside the integral sign. In this case we have

$$W = p \int_{v_1}^{v_2} dv = p(v_2 - v_1). \tag{23}$$

We are now in a position to compute the amount of heat that must be added to a gas to increase its temperature from T_1 to T_2 *when the pressure is kept constant.* Let the mass of gas be m. Then its volume at T_1 will be, by (2),

$$v_1 = (Rm/pM)T_1,$$

and that at T_2 will be $\quad v_2 = (Rm/pM)T_2.$

During the process of adding heat, the volume will increase by

$$v_2 - v_1 = \frac{Rm}{pM}(T_2 - T_1).$$

Hence, according to (23), the gas will do external work of amount

$$W = (Rm/M)(T_2 - T_1).$$

The heat that must be added to the gas is, according to (18), the sum of the increase in internal energy and the external work done by the gas. According to (21), the increase in internal energy is

$$mc_v(T_2 - T_1),$$

so the total heat that must be added is

$$Q = mc_v(T_2 - T_1) + (Rm/M)(T_2 - T_1). \tag{24}$$

> The amount of heat that must be added per unit mass per degree increase in temperature when a gas is heated and the pressure is kept constant is called the *specific heat at constant pressure* and is denoted by c_p.

Thus, to increase the temperature of a mass m of gas by ΔT degrees at constant pressure, we must add heat

$$Q = mc_p\,\Delta T. \tag{25}$$

From equation (24), we can compute the value of the specific heat at constant pressure as

$$c_p = \frac{Q}{m(T_2 - T_1)} = c_v + \frac{R}{M}.$$

The specific heat at constant pressure is greater than the specific heat at constant volume: in heating a gas at constant pressure we must not only add heat to supply the increase in internal energy that is associated with the temperature rise but also must add heat to supply the energy for doing external work in the expansion.

In the equation $$c_p = c_v + R/M, \tag{26}$$

we must use the same units in all three terms. If we use R in mechanical energy units, joules/kg·K deg or ft·lbf/slug·R deg, we must express c_p and c_v in the same units. It is frequently more convenient to use c_p and c_v in thermal units, kcal/kg·C deg or BTU/lb·F deg, and to change R to these thermal units. If the value of R given by (5) is converted to thermal units by using the relation 1 kcal = 4186 joules, it becomes

$$R = 1.987 \text{ kcal/kg·C deg} = 1.987 \text{ BTU/lb·F deg}. \tag{27}$$

It is convenient to remember that R has a magnitude very close to 2 in these thermal units.

We shall now discuss the magnitudes of c_v and c_p. First we note that since c_v is a measure of the rate of increase of internal energy with increase of temperature, and since the internal energy is independent of pressure and depends only on the temperature of the gas, c_v must be *independent of pressure* but may depend on temperature. Since, for a given gas, the difference between c_p and c_v is a constant, c_p must also be *independent of pressure.*

Although the general tendency is for c_v and c_p to increase with increasing temperature, over a very wide range of temperature from values close to the boiling point of the liquid up to at least 500° C, these specific heats are approximately constant for monatomic and many diatomic gases and have values given by remarkably simple formulas which we shall discuss further in connection with the kinetic theory of gases.

For a monatomic gas, c_v *has the value* $\frac{3}{2}R/M$; *hence, from* (26), c_p *has the value* $\frac{5}{2}R/M$. Thus,

for He, $c_v = \frac{3}{2}(1.987)/(4.00) = 0.745$ kcal/kg·C deg,

for Ne, $c_v = \frac{3}{2}(1.987)/(20.19) = 0.148$ kcal/kg·C deg,

for Hg vapor, $c_v = \frac{3}{2}(1.987)/(200.7) = 0.015$ kcal/kg·C deg.

These values are in complete agreement with experiment.

For a light diatomic gas, c_v *has the value* $\frac{5}{2}R/M$, c_p *the value* $\frac{7}{2}R/M$. Thus,

for H_2, $c_v = \frac{5}{2}(1.987)/(2.017) = 2.463$ kcal/kg·C deg,

for O_2, $c_v = \frac{5}{2}(1.987)/(32.0) = 0.155$ kcal/kg·C deg,

for N_2 or CO, $c_v = \frac{5}{2}(1.987)/(28.0) = 0.178$ kcal/kg·C deg.

These values are also in complete agreement with experiment. The specific heat of H_2 gas is the highest of all known specific heats, liquid water coming next. For heavy diatomic gases such as Cl_2 and Br_2, the specific heats are slightly greater than those given by the above expressions.

For a *polyatomic* gas, one can say only that c_v *has a value greater than* $3R/M$, c_p *a value greater than* $4R/M$. For most polyatomic gases the value of specific heat increases rapidly with temperature. In the following tabulation, the measured values of c_p for certain polyatomic gases at 20° C are given, along with the value of $4R/M$ for comparison (values are in kcal/kg·C deg or in BTU/lb·F deg):

CO_2:	$c_p = 0.202$	$4(1.987)/(44.0) = 0.181$
SO_2:	$c_p = 0.154$	$4(1.987)/(64.1) = 0.124$
NH_3:	$c_p = 0.520$	$4(1.987)/(17.03) = 0.467$
CH_4:	$c_p = 0.593$	$4(1.987)/(16.04) = 0.496$
C_2H_6:	$c_p = 0.386$	$4(1.987)/(30.1) = 0.264$

The specific heat of a mixture of gases can best be computed as follows: Let masses m_X, m_Y, and m_Z of gases X, Y, Z, with specific heats c_{vX}, c_{vY}, c_{vZ}, be mixed. Then to raise the temperature by one degree requires the addition of heat:

$$m_X c_{vX} + m_Y c_{vY} + m_Z c_{vZ}.$$

The heat required per unit mass of the mixture is, then,

$$c_v = \frac{m_X c_{vX} + m_Y c_{vY} + m_Z c_{vZ}}{m_X + m_Y + m_Z}.$$

For such a mixture, the relation (26) between c_p and c_v is valid in terms of the molecular weight M of the mixture, given by (11).

If we ignore the argon content of air, it is a mixture of two diatomic gases, for which $c_{vO_2} = \frac{5}{2}R/M_{O_2}$ and $c_{vN_2} = \frac{5}{2}R/M_{N_2}$ at ordinary temperatures. Substitution in the above equation for c_v then gives

$$c_{v\ \text{air}} = \frac{m_{O_2}/M_{O_2} + m_{N_2}/M_{N_2}}{m_{O_2} + m_{N_2}} \cdot \tfrac{5}{2}R.$$

But by (11), the fraction here is just the reciprocal of the molecular weight of air, so

$$c_{v\ \text{air}} = \tfrac{5}{2}R/M_{\text{air}}.$$

Still ignoring the correction for the one per cent of argon in the air, this would give

$$\text{for air,}\quad c_v = \tfrac{5}{2}(1.987)/29.0 = 0.171\ \text{kcal/kg·C deg},$$

$$c_p = \tfrac{7}{2}(1.987)/29.0 = 0.239\ \text{kcal/kg·C deg},$$

$$c_p/c_v = \tfrac{7}{5} = 1.40.$$

These values are in agreement with experiment at ordinary temperatures. The presence of the argon introduces only a negligible correction.

Specific heats of gases are rather difficult to measure directly, because the mass of gas and hence the quantity of heat involved are necessarily small. Continuous flow methods are applicable to the determination of the specific heat at constant pressure. The ratio c_p/c_v may be accurately determined by indirect methods involving measurement of temperature change in an adiabatic expansion or compression (see Chap. 19) or the measurement of the speed of sound in the gas (see Chap. 22). This ratio alone is sufficient to determine both c_p and c_v if the gas is assumed ideal so that relation (26) is satisfied.

PROBLEMS

1. Krypton is a monatomic rare gas. Compute its specific heat at constant pressure and its specific heat at constant volume. Compare with the observed value: $c_p/c_v = 1.67$. Ans: 0.060, 0.036 kcal/kg·C deg.

2. Xenon is a monatomic rare gas. Compute its specific heat at constant pressure and its specific heat at constant volume. Compare with the observed value: $c_p/c_v = 1.67$.

3. Compute the specific heat of O_2 gas at constant volume; at constant pressure. Compare with the observed values: $c_p = 0.217$ kcal/kg·C deg; $c_p/c_v = 1.40$. Ans: 0.155, 0.217 kcal/kg·C deg.

4. Compute the specific heat of F_2 gas at constant volume; at constant pressure. Compare with the observed value: $c_p = 0.182$ kcal/kg·C deg.

5. From measurements of the speed of sound in CO_2, it is found that $c_p/c_v = 1.30$ at 0° C. Assuming CO_2 to behave like a perfect gas, compute c_p and c_v at 0° C. Ans: 0.196, 0.151 kcal/kg·C deg.

6. From measurements of the speed of sound in methane gas (CH_4), it is found that $c_p/c_v = 1.32$ at 20° C. Assuming that methane behaves like an ideal gas, compute c_p and c_v at 20° C.

7. ✓ Show that if we use the notation $\gamma = c_p/c_v$, then

$$c_v = \frac{R/M}{\gamma - 1}, \tag{28}$$

and

$$c_p = \frac{\gamma R/M}{\gamma - 1}. \tag{29}$$

8. How much heat must be added to 0.3 kg of air to raise its temperature from 20° C to 110° C at constant pressure? What is the increase of internal energy? How much external work is done by the gas? How much heat would be needed to accomplish the same temperature rise at constant volume? (Treat air as a perfect diatomic gas of molecular weight 29.0.)

9. How much heat must be added to 0.2 kg of H_2 gas to raise its temperature from −100° C to 25° C at constant pressure? What is the increase in internal energy? How much external work is done? How much heat would be needed to accomplish the same temperature rise at constant volume? Ans: 86.2, 61.6, 24.6, 61.6 kcal.

10. Eight liters of O_2 at 0° C and 2 atm pressure is heated to 273° C at constant pressure.

(a) Use (23) to determine the external work in joules done by the gas.

(b) Use the value of c_v in joules/kg·C deg to determine the increase in internal energy.

(c) Use the corresponding value of c_p to determine the heat added.

(d) Verify that the first law of thermodynamics (18) is satisfied.

11. Six liters of H_2 gas at NTP is heated to 100° C at constant pressure. Make the same computations as in (a) through (d) of Prob. 10.

Ans: (a) 223 joules; (b) 556 joules; (c) 779 joules.

12. In the manufacture of lead pipes, the solid lead is forced continuously through an annular die by applying a pressure of 1400 kgf/cm². Given that the specific heat of lead is 0.0313 and its density is 11.4 g/cm³, how much will the temperature of the lead rise in passing through the die, assuming that all work goes into heating the lead?

13. Prove that the external work done by a gas in expanding from pressure p_0 and volume v_0 to pressure p_1 and volume v_1 *isothermally* (at constant temperature) is

$$W = p_0 v_0 \log_e(v_1/v_0) = (m/M)RT \log_e(v_1/v_0). \tag{30}$$

How much heat must be added to the gas during this expansion? Ans: W.

14. If 1 m³ of air at NTP expands isothermally to a volume of 2 m³, how much external work is done by the gas? How much heat must be added?

15. If 5 ft³ of He at pressure of 25 lbf/in² is compressed isothermally to a pressure of 50 lbf/in², how much external work is done on the gas? How much heat must be extracted from the gas during the compression? Ans: 12,500 ft·lbf; 16.0 BTU.

6. THE KINETIC THEORY OF GASES*

During the latter half of the nineteenth century, it was realized by Maxwell, Boltzmann, and others that the extreme simplicity of the experimental behavior of gases which we have sketched so far in this chapter implied an extreme simplicity in the structure of gases on a molecular scale. Only by making a very simple picture of an ideal gas can one derive these very simple laws. This picture and the derivation of the experimental laws from it constitute the subject matter of the *kinetic theory of gases.*

In order to derive results that are consistent with the fact that the specific internal energy of a gas depends on temperature alone, it is necessary to assume that the molecules of a gas are essentially *free particles,* that is, that most of the time they are acted on by no forces. They do, however, make frequent collisions with each other and with the walls of the vessel, and during these collisions very large forces act to change the direction of the molecular motion. Between collisions the molecules move in straight lines in random directions at very high speeds.†

The following picture is convenient and accurate: The molecules of a monatomic gas behave as if they were hard rigid spheres with diameters of the order of 10^{-8} cm. At NTP there are 2.7×10^{19} of them per cm³. They are on the average about 30 diameters apart. They rush about madly at speeds of the order of 1 mi/sec, making collisions each time they have gone about 1000 diameters along a given path. The molecules of a monatomic gas have no energy of rotation—only energy of translation. The molecules of a diatomic gas behave like rigid dumbbells moving in the same frantic manner except that now rotational energy may be interchanged at each collision. Of course the molecules are not really hard spheres or dumbbells, but the intramolecular forces hold the electrons and nuclei so firmly in position that the molecules behave during collisions as if they were such rigid bodies.

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* The remainder of the chapter may be omitted without loss of continuity.

† The effect of the force of gravity on the paths is insignificant. The molecules are on the average traveling so fast (with speeds of the order of 1 mi/sec) that the curvature of path introduced by gravity has a negligible effect over the ordinary distance between collisions, and the molecules may be assumed to move in straight lines between collisions.

The pressure of the gas on the walls of the container arises from the collisions of the molecules with the walls. In Fig. 7, p represents the average force that a unit area of the wall exerts on the gas molecules, which is equal and opposite to the average force the gas molecules exert on a unit area of the wall (the gas pressure). In accordance with the discussion of Chap. 10, the force p of the wall equals the total change of x-component of momentum of all molecules colliding with unit area of the wall in unit time. Since the number of collisions is enormous ($\sim 10^{14}$ per cm² per sec at NTP), the force p appears as a steady pressure.

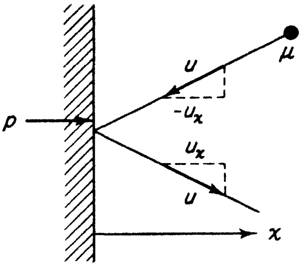

Fig. 7.

Now let us make a computation of the manner in which p depends on the density and the speed of the molecules. This computation will be made with very crude assumptions to avoid analytical complications. The answer obtained will certainly be correct in form but might be expected to be off by a numerical factor. Actually, the numerical factor happens to come out correctly, as we shall show in a footnote.

When an individual molecule of mass μ (we have used m for mass of gas, M for molecular weight, so we shall use μ for the mass of an individual molecule) that has x-component of velocity $-u_x$ (we have used v for gas volume, so we shall use v for molecular velocity) collides with the wall and rebounds with reversed x-component of velocity, as in Fig. 7, the x-component of momentum before the collision is $-\mu u_x$, after the collision it is $+\mu u_x$, so the change in x-component of momentum is $2\mu u_x$. We have assumed the collision to be elastic, that is, that the molecule does not lose kinetic energy in the collision. This assumption will hold true on the average if the gas is not gaining or losing energy.

Since molecules of different masses occur in most gases, we shall compute the partial pressure p_i arising from those molecules of mass μ_i. The total pressure can then be obtained by summation over i.

Let us make the crude assumption that *half* the molecules of mass μ_i have the same x-component of velocity u_{xi}, the other half the opposite component $-u_{xi}$. Then *the partial pressure will be given by the product of the number of collisions per unit area per second of these molecules by the change $2\mu_i u_{xi}$ in x-component of momentum per collision.* How many of these molecules collide per second? Let N_i be the number per unit volume of gas. Then in the volume $u_{xi}\,\delta t$ of Fig. 8 there will be $N_i u_{xi}\,\delta t$ molecules. Of these, half will be moving to the left, half to the right. We can take the time δt sufficiently small so that all the molecules in this volume that are moving to the left may be validly assumed to reach the wall and collide with it without colliding with any other molecule. Since in time δt a molecule moves a distance $u_{xi}\,\delta t$ in the x-direction, we see that just half of the molecules contained in the sheet of thickness $u_{xi}\,\delta t$ at any given instant strike the wall during the succeeding time interval δt. Hence, the number of molecules striking unit area of the wall in time δt will be $\frac{1}{2}N_i u_{xi}\,\delta t$, so the number of molecules striking unit area in unit time will be $\frac{1}{2}N_i u_{xi}$.

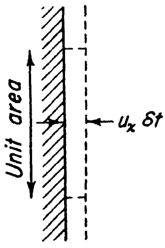

Fig. 8.

Consequently, if we refer to the statement in italics in the previous paragraph, we see that the partial pressure will be $\frac{1}{2}N_i u_{xi}\times 2\mu_i u_{xi}=N_i\mu_i u_{xi}^2$. Now, since the molecules are moving in random directions, they do not really all have the same value of x-component of velocity. However, if we let $\overline{u_{xi}^2}$ stand for the average value of the square of the x-component of velocity, it is reasonable to write

$$p_i = N_i\mu_i\overline{u_{xi}^2}. \qquad (31)$$

This formula happens to be exact.*

Now, if u_i is the magnitude of the velocity of a molecule, note that

$$u_i^2 = u_{xi}^2 + u_{yi}^2 + u_{zi}^2.$$

If we average over all the molecules, we obtain

$$\overline{u_i^2} = \overline{u_{xi}^2} + \overline{u_{yi}^2} + \overline{u_{zi}^2}.$$

But since no direction is preferred over any other in a truly random

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* To derive this formula rigorously, let $N_i(u_{xi})\,du_{xi}$ be the number of molecules per unit volume with *absolute magnitude* of x-component of velocity between u_{xi} and $u_{xi}+du_{xi}$. On striking the wall, these molecules will change momentum by $2\mu_i u_{xi}$. The number of such molecules striking a unit area per second will be $\frac{1}{2}N_i(u_{xi})\,u_{xi}\,du_{xi}$. These will contribute $N_i(u_{xi})\mu_i u_{xi}^2\,du_{xi}$ to the partial pressure. The whole partial pressure will be

$$\mu_i\int_0^\infty N_i(u_{xi})u_{xi}^2\,du_{xi} = \mu_i N_i\overline{u_{xi}^2},$$

as in (31).

motion, we must have $\overline{u_{xi}^2}=\overline{u_{yi}^2}=\overline{u_{zi}^2}.$

Hence, $\overline{u_i^2}=3\overline{u_{xi}^2}$, or $\overline{u_{xi}^2}=\frac{1}{3}\overline{u_i^2}.$

So, from (31), $$p_i=\tfrac{1}{3}N_i\mu_i\overline{u_i^2}. \tag{32}$$

Now $N_i\mu_i$, the product of the number of molecules per unit volume by the mass of a single molecule, is just the density ρ_i of the molecules of type i—the mass per unit volume. So we can write

$$p_i=\tfrac{1}{3}\rho_i\overline{u_i^2}. \tag{33}$$

This theory shows that the partial pressure is proportional to the density and the mean square speed. Now, in the first term of equation (12), if we write i for X, and ρ_i for m_i/v, we find that experiment gives

$$p_i=(R/M_i)\rho_iT \tag{34}$$

for the partial pressure of the molecules of molecular weight M_i. Thus experiment shows that the partial pressure is proportional to the density and the temperature. For a given kind of molecule, the mean square speed must, then, be a measure of the absolute temperature.

To secure agreement between our kinetic theory and experiment, we must equate (33) and (34) to obtain

$$\overline{u_i^2}=3RT/M_i \tag{35}$$

for the mean square speed of molecules of molecular weight M_i at absolute temperature T. This equation shows that the mean square speed is proportional to the temperature, but for different kinds of molecules at the same temperature is inversely proportional to the molecular weight.

Light molecules with low M move faster than heavy molecules with high M at the same temperature. This is true whether the molecules are mixed in the same container or are in different containers. Actually, we shall show that

> *molecules of different masses at the same temperature have the same average translational kinetic energy,*

and hence, by (32), have partial pressures that are proportional to their numbers per unit volume, in agreement with Dalton's law.

If we denote the translational kinetic energy of a molecule by $E_i=\frac{1}{2}\mu_iu_i^2$, then the average value of E_i is

$$\overline{E_i}=\tfrac{1}{2}\mu_i\overline{u_i^2}=\tfrac{1}{2}\mu_i\cdot 3RT/M_i. \tag{36}$$

Now, in the notation of p. 373, we can write the mass of a molecule of gas as

$$\mu_i=M_i\,(1\text{ amu}).$$

When this is substituted in (36), the molecular weight M_i cancels out, and we obtain the expression

$$\overline{E}_i = \tfrac{3}{2}\{R(1\text{ amu})\}\,T, \tag{37}$$

which no longer contains any reference to the kind of gas.

The quantity in braces in (37) is a universal constant ordinarily denoted by k:

$$k = R(1\text{ amu}), \tag{38}$$

and called *Boltzmann's constant*, or sometimes the 'gas constant per molecule.' In MKS units, k has the value, from (5) and (9),

$$k = 8317\,\frac{\text{joules}}{\text{kg}\cdot\text{K deg}}\,(1.660\times10^{-27}\text{ kg}) = 1.380\times10^{-23}\,\frac{\text{joule}}{\text{K deg}}.$$

In terms of Boltzmann's constant, the average translational kinetic energy of a gas molecule is

$$\overline{E} = \tfrac{3}{2}kT, \tag{39}$$

the same for any kind of molecule at the same temperature. *The temperature of a gas is precisely a measure of the average translational kinetic energy of a molecule of the gas,* with the proportionality factor given by (39).

From (35), we can obtain the mean square speed of molecules of a given mass at any temperature. Thus, for molecules with $M = 2.017$, which overwhelmingly predominate in H_2 gas, at $T = 0°\text{ C} = 273°\text{ K}$, we find from (35) that

$$\overline{u^2} = 3\times8317\,\frac{\text{m}^2}{\text{sec}^2\cdot\text{K deg}}\times\frac{273\text{ K deg}}{2.017} = 3{,}380{,}000\,\frac{\text{m}^2}{\text{sec}^2}.$$

The root-mean-square speed is thus

$$\sqrt{\overline{u^2}} = 1840\text{ m/sec}, \tag{40}$$

or 1.14 mi/sec. For higher temperatures this value would be greater; for heavier molecules it would be less, in accordance with (35).

The value (40) is not exactly the same as the mean or average speed $\bar{u}$ of the molecules because the process of squaring all speeds, averaging, and then taking the square root will give a slightly different answer from that obtained by averaging the speeds themselves. Maxwell, by a more detailed analysis of the motion of a collection of molecules having the simple mechanical properties we have described, determined that the speed distribution is such that the number of molecules in speed range du_i at speed u_i is proportional to

$$u_i^2\,e^{-\frac{1}{2}M_i u_i^2/RT}\,du_i, \tag{41}$$

for molecules of the particular molecular weight M_i. This speed distribution, which is called the *Maxwellian distribution,* is plotted in Fig. 9. There has been ample experimental check that the molecular speeds are actually distributed in this fashion.

From this distribution, one finds, as indicated in Fig. 9, that the root-mean-square speed is

$$\sqrt{\overline{u_i^2}} = \sqrt{3}\,\sqrt{RT/M_i} = 1.732\,\sqrt{RT/M_i} \qquad (42)$$

as in (35); the *mean* or *average speed* is

$$\overline{u_i} = \sqrt{8/\pi}\,\sqrt{RT/M_i} = 1.596\,\sqrt{RT/M_i}; \qquad (43)$$

and the most probable speed (the speed possessed by the largest number of molecules) is still lower at

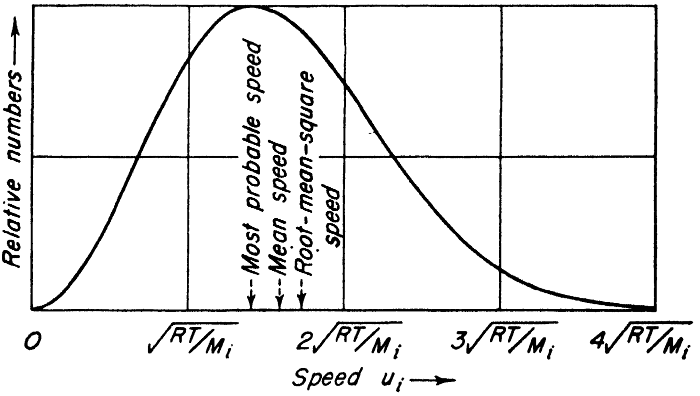

Fig. 9. The Maxwellian distribution. The relative numbers of molecules having speeds in unit speed range at various values of speed are proportional to the ordinates of this curve.

$$\sqrt{2}\,\sqrt{RT/M_i} = 1.414\,\sqrt{RT/M_i}. \qquad (44)$$

Thus the mean speed is 1.596/1.732 = 0.921 times the root-mean-square speed. So, for H_2 at 0°C, the mean speed is

$$0.921 \times 1840 \text{ m/sec} = 1690 \text{ m/sec} = 1.05 \text{ mi/sec}.$$

The molecules are by no means all moving at this speed, but are distributed over a wide range of speeds as indicated in Fig. 9. Only 1 in 10,000 of the molecules has a speed over $3\overline{u_i}$; there is, however, a definite probability that a molecule may have any speed whatsoever, no matter how large.

It is noted that, because of the non-linear dependence of (42)–(44) on M_i, these formulas must be applied to molecules of one mass at a time. We cannot enter these formulas with a mean molecular weight, either of the isotopes of an element or of the gases in a mixture, and obtain correct averages over the molecules.

PROBLEMS

1. Calculate the average translational kinetic energy of an argon atom at 25° C. At what Centigrade temperature does the average energy have half this value?
Ans: 6.18×10^{-21} joule; $-124°$ C.

2. Calculate the average translational kinetic energy of a molecule of benzene vapor (C_6H_6) at a temperature of 25° C; at a temperature of 100° C.

3. Calculate the root-mean-square speed and the mean speed of argon atoms of atomic weight 40.0 at 25° C. Ans: 432, 397 m/sec.

4. Calculate the root-mean-square speed and the mean speed of benzene-vapor molecules of molecular weight 78.0 at 100° C.

5. Calculate the total kinetic energy of translation of all the molecules in 1 liter of an ideal gas at 25° C and 2 atm pressure. Ans: 304 joules or 72.6 cal.

6. Calculate the total kinetic energy of translation of all molecules in 25 ft^3 of an ideal gas at 100° F and 200 atm pressure.

7. Determine the value of k in ft·lbf/R deg. Ans: 0.565×10^{-23} ft·lbf/R deg.

8. Hydrogen gas contains some molecules of molecular weights 3.0 and 4.0, in addition to those of molecular weight 2.0. What are the ratios of the average speeds of these three types of molecules? the ratios of the average kinetic energies?

7. EQUIPARTITION OF ENERGY; SPECIFIC HEATS

The discussion of the previous section shows that the average kinetic energy of translation of gas molecules depends on temperature only and is independent of the pressure of the gas. Hence the specific internal kinetic energy of translation (the energy per unit mass of gas) must depend on temperature alone and be independent of pressure. The experimental fact that the whole specific internal energy depends on temperature alone shows that the other forms of energy which the molecules may possess—energy of rotation and vibration—must also have average values depending on temperature alone. This reasonable conclusion follows from the details of the kinetic theory as worked out by Maxwell and Boltzmann.

The fact that internal energy is independent of pressure and hence of density requires the hard-billiard-ball concept of molecular interaction. Forces between molecules are 'short-range' and come into play only in actual collisions. If there were 'long-range' forces between the molecules, which tended to attract all the molecules toward each other, then in the free-expansion experiment these forces would do work on the molecules as they separated from each other in order to occupy the increased volume. This work would be such as to decrease the kinetic energies of translation and hence to decrease the temperature. The absence of such a decrease in temperature in this experiment requires the picture of free molecules interacting only in actual collision.

Let us now compute the specific heat required to supply the increased translational kinetic energy as the temperature rises. The energy of translation averages $\frac{3}{2}kT$ per molecule, for molecules of any mass. The average mass of a molecule is M amu. The number of molecules per unit mass of the gas is hence $1/(M \text{ amu})$. Hence, by (37), *the energy of translation of all molecules in unit mass of gas is*

$$e_{\text{trans}} = \frac{1}{M(1 \text{ amu})} \cdot \tfrac{3}{2}\{R(1 \text{ amu})\}T = \tfrac{3}{2}\frac{R}{M}T.$$

From its derivation, this result is seen to be valid for a mixture of molecules of different masses, with average molecular weight M.

To increase the temperature of unit mass by one degree, we must supply $\Delta e_{\text{trans}} = \frac{3}{2}R/M$ for the increased kinetic energy. The value of c_v must be at least this great. In fact,

$$c_v = \tfrac{3}{2}\frac{R}{M} + \left\{\begin{matrix}\text{increase in energy of rota-}\\ \text{tion and vibration per unit}\\ \text{mass per degree}\end{matrix}\right\}.$$

As we have stated in Sec. 5, c_v for a monatomic gas has exactly the value $\frac{3}{2}R/M$. The molecules of a monatomic gas change only in translational kinetic energy as the temperature rises. Diatomic and polyatomic gases have specific heats greater than $\frac{3}{2}R/M$, so energy must go into rotation or vibration as the temperature rises.

The classical kinetic theory of Maxwell and Boltzmann shows that there should be energy $\frac{1}{2}kT$ per molecule or $\frac{1}{2}(R/M)T$ per unit mass *for each 'degree of freedom' of the gas molecules*, corresponding to a contribution of $\frac{1}{2}R/M$ to c_v for each degree of freedom.* This division of internal energy equally among the degrees of freedom is known as *equipartition of energy*. This is a useful concept, but it was quickly found that not all degrees of freedom that one would imagine from a classical picture of molecular structure actually made contributions to the specific heat. It remained for quantum mechanics to decide just what degrees of freedom make contributions.

In the case of a monatomic gas we have just three degrees of freedom, corresponding to translation in the three coordinate directions, so we have $c_v = \frac{3}{2}R/M$. This value is in agreement with experiment. There is no rotational contribution to the energy. The specific heat remains constant until the temperature becomes so high that the atomic spectrum of the atoms begins to be excited and the gas becomes luminous. At this stage, the specific heat rises because energy is going into excitation of electron orbits.

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* One degree of freedom is assigned for each independent term appearing in the energy expression. Thus for a single mass point, there are 3 degrees of freedom because there are three independently variable terms in the expression $\frac{1}{2}\mu u_x^2 + \frac{1}{2}\mu u_y^2 + \frac{1}{2}\mu u_z^2$ for translational kinetic energy. An extended rigid body has 6 degrees of freedom—3 associated with translational kinetic energy of the center of mass and 3 with kinetic energy of rotation about the three coordinate axes. For a non-rigid body, two additional degrees of freedom are assigned for *each* characteristic mode of vibration—one associated with the vibrational kinetic energy and the other with the vibrational potential energy.

In the case of a diatomic gas at ordinary temperatures, $c_v = \frac{5}{2}R/M$, corresponding to five degrees of freedom. These are the three translational degrees and two degrees of freedom of rotation of the nuclei. The fact that there are only two degrees of rotational freedom instead of three can be understood from Fig. 10. In this figure the nuclei are considered as mass *points*. Three coordinate axes are drawn through their center of gravity. The nuclei can rotate about the x-axis or the z-axis, but rotation about the y-axis is meaningless so long as the nuclei are considered as points without structure. Certainly, rotation about the y-axis is quite different in character from rotation about the other two, and experimental results show that it makes no contribution to the specific heat.

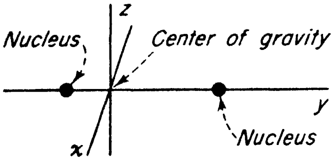

Fig. 10.

At sufficiently high temperatures,* the specific heat of most diatomic gases begins to rise and gradually changes from $\frac{5}{2}R/M$ to $\frac{7}{2}R/M$, corresponding to the excitation of two additional degrees of freedom, associated with the kinetic and potential energy of vibration, respectively. At still higher temperatures, electronic excitation may take place, but the molecules also begin to dissociate into atoms so that we no longer have a purely diatomic gas.

For polyatomic gases we have in general three degrees of translational freedom and *three* of rotational freedom, so that we have a specific heat of at least $\frac{6}{2}R/M$. But since even at room temperature some vibrational degrees are excited or 'partially' excited, the specific heat is greater than this value.

This is the kinetic-theory explanation of the striking regularities observed in the values of the specific heats of gases.

8. BROWNIAN MOTION

In a mixture of gases, kinetic theory gives to the molecules of each type the same average translational kinetic energy $\frac{1}{2}\mu_i\overline{u_i^2}$. Thus, in accordance with (32), each gas makes a contribution to the pressure that is proportional to the number of its molecules per unit volume. This statement is in agreement with Dalton's law of partial pressures. Each gas behaves in every respect as if the other gas were not present at all, except that if two gases at different temperatures are mixed, energy is quickly interchanged in the collisions between molecules so that the two kinds of gases come to the same temperature.

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* The temperature at which vibrational degrees of freedom become excited is different for different diatomic molecules. It is below room temperature for heavy molecules like Cl_2 and Br_2, and far above room temperature for light molecules like H_2.

An interesting application of the kinetic theory occurs in the case of the *Brownian motion.* In 1827 an English botanist, Robert Brown, pointed out that very fine particles of dust, smoke, or pollen contained in air (or in a liquid) seem to execute very irregular motions when observed in a high-power microscope. The correct explanation of these random motions, which almost make the particles seem to be alive as they perpetually dart hither and thither, was not given until much later. The motion is explained by the kinetic theory of gases. The particle of dust behaves as if it were just a very large molecule. In air it has, like all molecules, average translational kinetic energy $\frac{3}{2}kT$. It is repeatedly interchanging energy with the air molecules that collide with it, and in the course of time has velocities ranging over the whole of the Maxwellian distribution with relative probabilities given by the curve of Fig. 9. But unlike the air molecules, the particle is large enough to be watched in a microscope and its average velocity is comparatively slow. Hence, details of the kinetic theory predictions are subject to direct experimental check by observation of the Brownian motion of small particles, and have been accurately confirmed in a number of careful experimental investigations.

Brownian motion furnishes the most direct and convincing check of the kinetic theory.

PROBLEMS

1. Calculate the root-mean-square speed of smoke particles of mass 5×10^{-17} kg in air at NTP. Ans: 1.50 cm/sec.

2. Calculate the root-mean-square speed of oil droplets of diameter 2 microns and density 0.9 g/cm^3 in air at NTP.

CHAPTER 18

SOLIDS, LIQUIDS, AND GASES

In this chapter we shall discuss the three *phases* of matter—solid, liquid, and gaseous—and the conditions that govern transitions from one to the other. We shall confine our attention to the simplest cases of pure substances, such as H_2O or CO_2, or of pure substances in the presence of a foreign gas, such as water vapor in air. We shall further restrict our attention to substances that solidify in *crystalline form.* Ice, solid pure metals, and solid salts fall into this category. The solid is either a *single* crystal or a collection of crystals of various sizes, shapes, and orientations. Such a crystalline material has a definite melting temperature at which a sharp transition from solid to liquid phase occurs. This restriction eliminates from consideration waxlike and glasslike substances in which the solid is not crystalline but *amorphous,* with the molecules arranged in random positions as in a liquid rather than in a regular array as in a crystal. Such amorphous substances do not make a sharp transition from solid to liquid. Rather, they make a gradual transition over a range of temperatures, with the solid first softening and then turning into a thick viscous liquid. The reader is undoubtedly familiar with the great difference between the gradual softening of sealing wax or glass and the sharp transition from hard ice to liquid water.

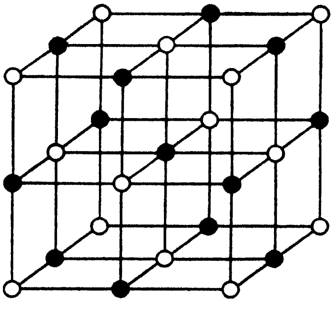

Fig. 1. Crystal structure of NaCl.

1. FUSION, FREEZING

In a crystalline solid, the atoms are arranged in a regular array. For example, in common salt (NaCl), the Na and Cl atoms (actually Na^+ and Cl^- ions) are arranged at the corners of cubes as in Fig. 1. As a result of this cubic atomic arrangement, most of the individual crystals of table salt are perfect cubes, as can be readily observed with a little magnification. In ice, the atoms are arranged in a more complex pattern with hexagonal symmetry, giving rise to the beautiful hexagonal patterns of snow crystals shown in Fig. 2.

The atoms or ions in a crystal are held together by electric and magnetic forces which act exactly like a collection of little springs (see Fig. 1, p. 319). The atoms have definite *equilibrium positions* but are capable of *oscillating or vibrating* about these positions. The internal thermal energy of a solid consists principally of the energy of such vibrations. The average energy and amplitude of vibration increase with an increase in temperature, with a resultant slight *expansion* of the solid.

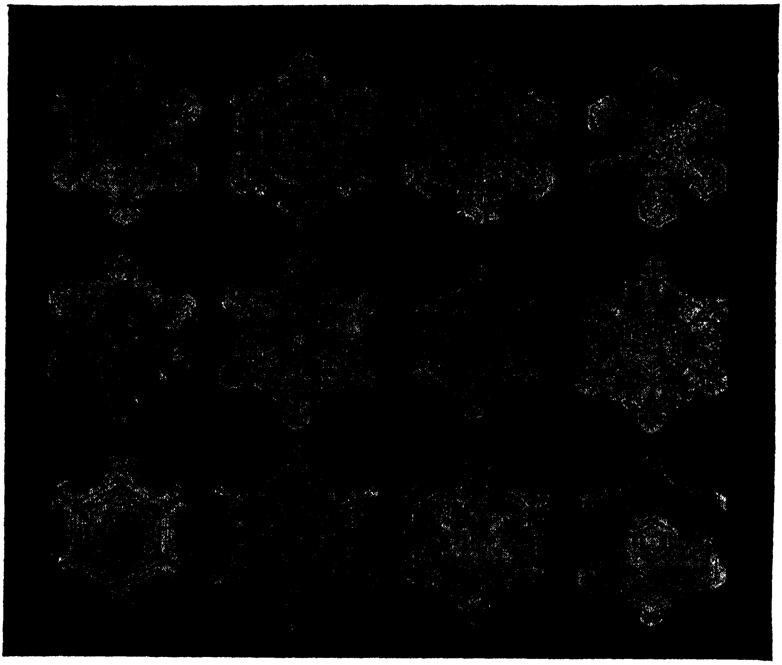

Fig. 2. Photographs of snow crystals. (Reproduced from Thomas A. Blair's *Weather Elements*, 3rd ed., Prentice-Hall.)

If heat is added to a solid, the amplitude of vibration increases until, at a certain definite temperature (0° C for ice), molecular groups at the surface of the solid which happen to have unusually high energy move completely away from their equilibrium positions and enter the *liquid phase.*

In the liquid phase, the molecules no longer have definite equilibrium positions, although the attractive forces between molecules are still sufficiently large to prevent the molecules from getting very far apart. In fact, the distance between neighboring molecules in the liquid is comparable to the distance between neighboring molecules in the solid, but the molecules have sufficient translational kinetic energy to keep sliding around past each other like the particles in a barrel of flour when stirred by the hands. *The molecules of the liquid have a greater average energy than the molecules of the solid at the same temperature (the melting temperature).*

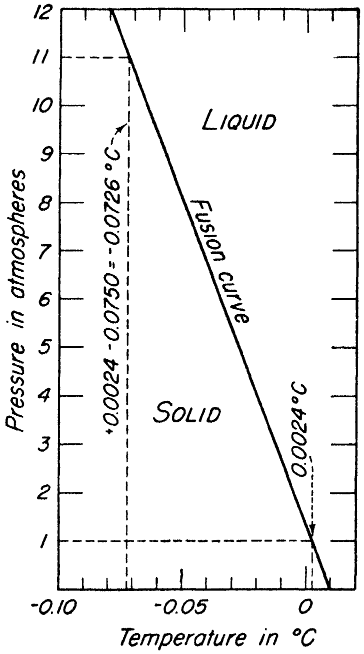

Fig. 3. Fusion curve for pure air-free H_2O. It should be noted that the melting point of pure *air-free* water is not 0° C, but 0.0024° C, at 1 atm. It is the melting point of ice in contact with water saturated with air by being open to the atmosphere that is exactly 0° on the Centigrade scale at 1 atm pressure.

The temperature of a solid cannot be raised above the temperature of fusion, or melting. After the solid has reached this temperature, all added heat goes into fusing (melting) more solid, the heat supplying the energy difference between the liquid and the solid. The temperature will not again commence to rise until *all* of the solid has been melted.

There is thus only one temperature, the temperature of fusion, at which solid and liquid can coexist in thermal equilibrium. Above this temperature the substance is necessarily all liquid; below this temperature it is necessarily all solid.* When solid and liquid coexist at the fusion temperature and no heat is added, there is equilibrium between the number of particularly energetic solid molecules that leave their positions on the surface of the solid and pass into the liquid state and the number of particularly slow liquid molecules that recrystallize on the solid. In the event that heat is added, more molecules pass into the liquid; if heat is taken away, more molecules change from liquid to solid.

The fusion temperature depends slightly on pressure; for example, the fusion temperature of water *decreases* 0.00750 C deg for each atmosphere increase in pressure. Thus we have not a definite fusion temperature but a *fusion curve* on a p, T diagram, such as that shown for water in Fig. 3. It is seen that we can make ice melt not only by increasing the temperature but also by increasing the pressure. Because ice shrinks upon melting, it seems reasonable that increased pressure should tend to make it melt, since this results in a smaller volume. In fact, all substances that shrink on melting (the only known pure substances are water, gallium, and bismuth) have fusion curves that slant to the left like that of Fig. 3, so that increased pressure tends to make them melt. On the other hand, most substances expand on melting and have fusion curves that slant to the right, so that increased pressure tends to make them freeze.

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* More accurately, below the fusion temperature the substance is *all* solid if there is *any* solid. It is possible to *supercool* a liquid below the fusion temperature. Very pure dust-free water has been cooled as low as −40° C without freezing. Apparently the difficulty in freezing lies in the formation of the first crystalline nucleus on which condensation can take place. If a tiny ice crystal is dropped into supercooled water, freezing of a large mass takes place suddenly. There is no similar difficulty in the process of melting, and superheating a solid is impossible.

For a substance to cross the fusion curve from the solid to the liquid phase, the latent heat of fusion must always be supplied. If one has ice in the cylinder of Fig. 4, at a temperature just below the fusion temperature, and increases the pressure *without* adding heat, as soon as the pressure reaches that of the fusion curve of Fig. 3 a little of the ice will tend to melt. But since the latent heat is not supplied from the outside, it will come from the internal energy of the ice itself, so the temperature of the ice will tend to fall. The net result is that once the fusion curve is reached, the state of the substance will follow the fusion curve as the pressure is increased, only so much ice melting as can be accounted for by the heat supplied by the cooling of the material as a whole.

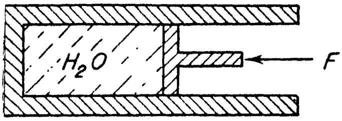

Fig. 4. A cylinder *filled* with H_2O (or other pure substance).

The extremely low friction encountered in skating can be attributed to this tendency for ice to melt under pressure. The sharp blades of the skates make contact only over a very small area, where the pressure is truly enormous. (For example, the weight of a 150-lb man applied to an area of 1 mm^2 gives a pressure of 150 lbf/$\frac{1}{600}$ in$^2 = 15 \times 6000$ lbf/in$^2 =$ 6000 atmospheres.) This pressure causes incipient melting of the ice in this region so that we really skate on a film of water. This theory is supported by the fact that on an extremely cold day, say at $-20°$ F, when the temperature of the surface of the ice is also very low, the ice seems less slippery and skating not so good.

The latent heat of fusion is the heat that must be added to the substance in the cylinder of Fig. 4 to change unit mass of it from solid to liquid with no change in pressure or temperature, that is, at a particular point on the fusion curve. The value of the latent heat will vary somewhat from point to point on the fusion curve.

When unit mass changes from solid to liquid at constant p and T, in general the volume changes. Say that the volume increases by Δv, where

$$\Delta v=\begin{Bmatrix}\text{volume per unit}\\ \text{mass of liquid}\end{Bmatrix}-\begin{Bmatrix}\text{volume per unit}\\ \text{mass of solid}\end{Bmatrix}. \tag{1}$$

To this process, an equation like (18) of Chap. 17, p. 380, is applicable. The latent heat *equals* the increase in internal energy *plus* the external work done. If Δe represents the increase in specific internal energy:

$$\Delta e=\begin{Bmatrix}\text{internal energy per}\\ \text{unit mass of liquid}\end{Bmatrix}-\begin{Bmatrix}\text{internal energy per}\\ \text{unit mass of solid}\end{Bmatrix}, \tag{2}$$

then the latent heat is

$$L=\Delta e+p\,\Delta v, \tag{3}$$

since the mechanical work done at constant pressure is $p\,\Delta v$, as we have seen on p. 384. The volumes per unit mass that occur in (1) are called *specific volumes.*

In the case of water at NTP,

$$\Delta v=1.000\times10^{-3}\ \text{m}^3/\text{kg}-1.086\times10^{-3}\ \text{m}^3/\text{kg}=-8.6\times10^{-5}\ \text{m}^3/\text{kg},$$

negative because ice shrinks on melting. Hence,

$$\begin{aligned}p\,\Delta v&=(1.013\times10^5\ \text{nt/m}^2)(-8.6\times10^{-5}\ \text{m}^3/\text{kg})=-8.7\ \text{joules/kg}\\ &=-2.1\times10^{-3}\ \text{kcal/kg}.\end{aligned}$$

The negative sign indicates that work is actually done *on* the substance *by* the piston when the ice melts. This work is negligible compared to the latent heat $L=79.70$ kcal/kg, so according to (3) the latent heat and the increase in internal energy are very closely the same. This is the usual case in fusion, where volume changes are small, and is quite different from the case in vaporization, where enormous volume changes are involved, as we shall see in the next section.

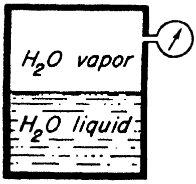

Fig. 5. A vessel of constant volume containing only water and water vapor.

2. VAPORIZATION; CONDENSATION

Consider the vessel of Fig. 5, which contains only H_2O, liquid and vapor. The vessel is at a definite temperature T and we are interested in the factors that determine the absolute pressure p.

A molecule in the interior of the liquid experiences, on the average, no resultant force from the other liquid molecules, since it has, on the average, equal numbers of molecules on all sides. It will have an average translational kinetic energy that depends on the temperature and which within wide limits is independent of the pressure in the liquid. But a molecule that approaches the surface of the liquid experiences, as indicated in Fig. 6, a large resultant force directed back into the liquid. The existence of this force is manifested in the phenomenon of *surface tension.** The surface molecules are continually being pulled back into the liquid and new ones are coming out to take their places. One can think of a molecule that approaches the surface from inside as 'hitting' the surface and 'bouncing' back, because as soon as it gets to within a few molecular layers of the surface it begins to experience a force tending to retard its outward motion and to pull it back into the fluid—it experiences an acceleration component into the fluid that decreases its outward component of velocity to zero and then reverses this velocity component.

But the velocity distribution among the fluid molecules will be such that there are some that approach the surface so fast that in spite of the retarding forces they escape through the surface and enter the vapor. These molecules 'evaporate' or 'vaporize.' *The number of molecules that have sufficient kinetic energy to evaporate (per second per unit area of fluid surface) increases rapidly as the temperature of the liquid is raised and depends only on the temperature of the liquid.*

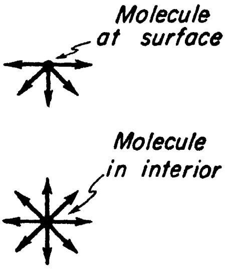

Fig. 6.

Now in Fig. 5 there must be an equilibrium between the rate of evaporation of the molecules of the liquid and the rate of condensation of the molecules of the vapor. *Every vapor molecule that strikes the liquid surface enters the liquid, since it immediately experiences large forces pulling it into the liquid.* At any given vapor temperature, the number of molecules per second striking unit area of a surface is proportional to the vapor pressure. This statement is true whether or not the vapor behaves like a perfect gas.

If we were to start with liquid in the vessel of Fig. 5 and a vacuum in the space above it, the liquid would start evaporating at a rate determined entirely by the liquid temperature. The vapor pressure will build up until there are as many vapor molecules striking the liquid surface and condensing as there are liquid molecules evaporating. When this condition is reached there is no further increase in vapor pressure. *Thus, in Fig. 5, the vapor pressure is determined entirely by the temperature.*

The vapor pressure as a function of temperature is readily measured by an apparatus similar to that shown in Fig. 5. In the case of water this pressure is given by the curve of Fig. 7 and the data of Table I.

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* A molecule at the surface can be considered as having potential energy corresponding to the work that has been done against this force in getting the molecule to the surface. A liquid thus has a surface energy (similar in nature to a potential energy) proportional to its surface area. To increase the surface area requires work equal to the increase in this potential energy. The surface tends, like a stretched membrane, to assume a shape of minimum area (minimum potential energy). Witness, for example, the spherical form of water droplets. The surface behaves in all respects like a membrane stretched with constant tension per unit width.

TABLE I
H_2O: DATA ALONG SUBLIMATION AND VAPORIZATION CURVES

Temperature (° C)	Pressure (mm Hg)	Specific Volume (cm^3/g)*		Latent Heat (kcal/kg)
		Solid	Vapor	Sublimation
−70	0.002	1.080	400,000,000	678
−60	0.008	1.082	90,000,000	678
−50	0.030	1.083	26,000,000	678
−40	0.097	1.085	8,300,000	678
−30	0.286	1.086	2,940,000	678
−20	0.776	1.088	1,130,000	678
−10	1.950	1.089	467,000	678
− 8	2.326	1.090	395,000	678
− 6	2.765	1.090	334,000	678
− 4	3.280	1.090	284,000	678
− 2	3.880	1.091	242,000	677
0.0099	4.579	1.091	206,300	677.3
		Liquid		Vaporization
0.0099	4.579	1.0002	206,300	597.4
1	4.93	1.0001	192,700	597
2	5.29	1.0001	180,000	596
3	5.68	1.0000	168,200	596
4	6.10	1.0000	157,200	595
5	6.54	1.0000	147,100	595
6	7.01	1.0001	137,700	594
7	7.51	1.0001	129,000	594
8	8.04	1.0002	120,900	593
9	8.61	1.0002	113,400	592
10	9.21	1.0003	106,300	592
11	9.84	1.0004	99,800	591
12	10.52	1.0005	93,700	591
13	11.23	1.0006	88,100	590
14	11.99	1.0008	82,900	590
15	12.79	1.0009	77,900	589
16	13.63	1.0011	73,300	588
17	14.53	1.0012	69,100	588
18	15.48	1.0014	65,100	587
19	16.48	1.0016	61,300	587
20	17.53	1.0018	57,800	586
21	18.65	1.0020	54,500	586
22	19.83	1.0022	51,500	585
23	21.07	1.0025	48,600	584
24	22.38	1.0027	45,900	584
25	23.76	1.0030	43,400	583

* NOTE: 1 $cm^3/g = 10^{-3}$ m^3/kg.

TABLE I (*Continued*)

Temperature (° C)	Pressure (mm Hg)	Specific Volume (cm³/g)*		Latent Heat (kcal/kg)
		Liquid	Vapor	Vaporization
26	25.21	1.0032	41,000	583
27	26.74	1.0035	38,800	582
28	28.35	1.0038	36,700	582
29	30.04	1.0041	34,800	581
30	31.82	1.0044	32,900	580
40	55.32	1.0079	19,540	575
50	92.52	1.0121	12,040	569
60	149.4	1.0171	7,678	563
70	233.7	1.0228	5,046	558
80	355.2	1.0290	3,409	552
90	525.9	1.0359	2,361	546
	(atm)			
100	1	1.0435	1,673	539.2
110	1.414	1.0515	1,210	533
120	1.959	1.0603	892	526
130	2.666	1.0697	668	519
140	3.566	1.0798	509	512
150	4.697	1.0906	392.5	505
200	15.35	1.1565	127.2	464
250	39.26	1.2512	50.1	410
300	84.79	1.4036	21.6	335
350	163.2	1.7468	8.8	213
370	207.8	2.23	5.0	107
371	210.3	2.30	4.8	97
372	212.8	2.38	4.5	84
373	215.4	2.50	4.2	68
374	218.0	2.8	3.6	35
374.15	218.4	3.1	3.1	0

* NOTE: 1 cm³/g = 10⁻³ m³/kg.

Now let us return to the cylinder of Fig. 4, which contains only H_2O. Let us assume that we maintain the pressure constant, say at ½ atm (38 cm of Hg) absolute. We have seen that if we start with ice at a temperature below 0° C and add heat, the H_2O will remain solid until we reach +0.0061° C (Fig. 3). At this temperature the ice will change to liquid water as we add sufficient heat to cause fusion. After all the ice has melted, the temperature of the liquid will rise as we continue to add heat. Not until we reach the temperature of 81.6° C given by the curve of Fig. 7 will any vapor be formed. For, suppose that at some lower temperature, say 72° C, an incipient bubble of vapor were formed. The vapor in this bubble would have to be at a pressure of $\frac{1}{2}$ atm if it were to be formed at all. But water at 72° has a surface rate of evaporation which equals the rate of condensation of vapor at a pressure of $\frac{1}{3}$ atm and is much less than the rate of condensation of vapor at $\frac{1}{2}$ atm. So

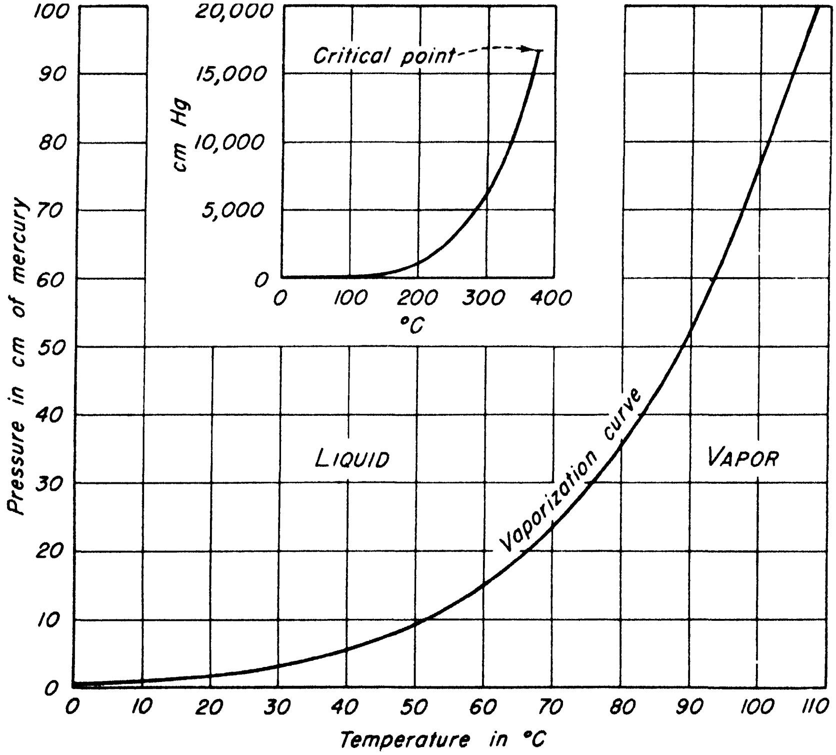

Fig. 7. Vaporization curve for water. This curve gives the conditions under which liquid and vapor can coexist, and hence gives the pressure of vapor saturated as in Fig. 5. No vapor can exist (in stable fashion) to the left of this curve; no liquid to the right. Vapor to the right of this curve is called *superheated steam.* This is also the boiling-point curve.

the vapor in the bubble would condense much faster than the liquid would evaporate into the vapor bubble, and the incipient bubble would collapse immediately.

Not until we reach a temperature of 81.6° C will the liquid evaporate from a surface fast enough to maintain a bubble of vapor at $\frac{1}{2}$ atm pressure. At this temperature liquid and vapor can coexist in equilibrium. If we add heat to the liquid so that the liquid exceeds this temperature ever so slightly, the liquid will evaporate faster than the vapor (at ½ atm) condenses, and the vapor bubbles will grow. The liquid will be topped by a layer of vapor. The rate of change of liquid to vapor will be governed purely by the rate at which heat is added since the latent heat of vaporization must be furnished by the added heat. Since it is only the most energetic liquid molecules that can evaporate, every molecule that evaporates in excess of those that condense results in a decrease in the average energy per liquid molecule which must be made up by the addition of heat if the temperature of the liquid is not to fall.

If the *pressure* exerted by the piston in Fig. 4 is kept constant, no temperature rise above 81.6° C will take place until *all* the liquid has evaporated. During this evaporation process the volume of course increases enormously and the piston of Fig. 4 must be imagined to move about 200 ft to the right on the scale of this figure.

After all the liquid has evaporated, further addition of heat will result in increase in temperature and volume of the vapor in accordance with the value of c_p and the behavior of this imperfect gas which we shall discuss in Sec. 5. At 81.6° C, this vapor is said to be *saturated steam;* above 81.6° C it is said to be *superheated steam.*

We now see that we could make the same arguments at any pressure, and that a cylinder containing H_2O under pressure and temperature conditions falling to the left of the curve of Fig. 7 will contain only liquid. To the right of this curve it will contain only vapor. Only for values of p and T falling *on* the vaporization curve can liquid and vapor coexist in equilibrium; at these values of p and T the vapor is said to be *saturated.*

Of the latent heat of vaporization, some goes into the increased internal energy of the vapor, some into mechanical work in pushing back the piston to increase the volume. In the case of vaporization, the latter is appreciable. Equations like (1), (2), and (3) are applicable, with appropriate modifications. At a pressure of 1 atm (1.013×10^5 nt/m^2) and a temperature of 100° C, we see from Table I that $L = 539.2$ kcal/kg; the vapor has specific volume of 1.673 m^3/kg; the liquid has specific volume of 0.001 m^3/kg. Hence $\Delta v = 1.672$ m^3/kg, and the external work done is

$$p\,\Delta v = 1.013 \times 10^5\ (\text{nt/m}^2)\cdot 1.672\ (\text{m}^3/\text{kg}) = 1.694 \times 10^5\ \text{joules/kg},$$

or, since 1 kcal = 4186 joules,

$$p\,\Delta v = 40.5\ \text{kcal/kg}.$$

Hence, since $L = \Delta e + p\,\Delta v$, where Δe is the increase in specific internal energy,

$$\Delta e = 539.2 - 40.5 = 498.7\ \text{kcal/kg}.$$

Of the latent heat, 7.5 per cent goes into doing external mechanical work, 92.5 per cent into the increase in internal energy of the vapor as compared with the liquid.

If we start with a cylinder containing only superheated steam, say at 100° C and ½ atm pressure, and increase the pressure isothermally, no liquid will appear until we reach the pressure of 1 atm. At this pressure liquid will begin to condense out. At this point, if we do not want the temperature to rise, we must take out the latent heat of 539.2 kcal/kg as the vapor condenses. This heat represents the 40.5 kcal of work that the piston does per kilogram of vapor condensing plus the extra internal energy of 498.7 kcal/kg in the vapor. If we increase the pressure without removing this latent heat when condensation begins, then the heat released by condensation will increase the temperature, and the material will follow up the vaporization curve toward increased pressures and temperatures.

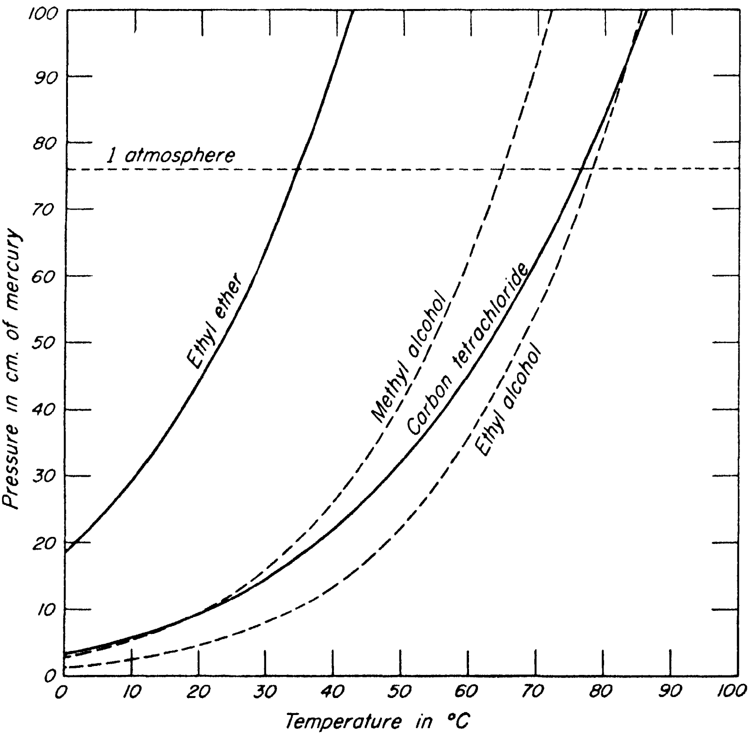

Fig. 8. Vaporization curves for ethyl ether, methyl alcohol, ethyl alcohol, and carbon tetrachloride.

Exactly similar considerations apply to the vaporization of any pure liquid. Vaporization curves for other familiar liquids are given in Fig. 8.

PROBLEMS

NOTE: The saturated vapor pressure of mercury is very low (0.0001 cm at 20° C, 0.028 cm at 100° C) so the mercury vapor present above a column of mercury has a negligible effect at ordinary temperatures.

1. Compute an accurate height in meters at which a water barometer will stand when the temperature is 30° C, the pressure 1 standard atmosphere, and the acceleration of gravity normal. (Take vapor pressure and liquid density from Table I.) Ans: 9.941 m.

2. Repeat Prob. 1 for the water barometer at 20° C.

3. A mercury barometer reads 754 mm. A drop of water is then introduced at the bottom of the mercury column and rises to the surface of the mercury. After evaporation has taken place, a small amount of liquid water remains. By how many mm is the mercury column depressed if the temperature is 30° C? Ans: 31.8.

4. Repeat Prob. 3 for the case where the temperature is 10° C.

5. If a drop of methyl alcohol is introduced into a barometer at 20° C in the same manner as in Prob. 3, the mercury is depressed 89 mm. What is the saturated vapor pressure of methyl alcohol at 20° C? Ans: 89 mm.

6. If a drop of any one of the liquids of Fig. 8 is introduced into a barometer at 30° C in the same manner as in Prob. 3, by approximately how much is the mercury depressed in each case?

7. If water is evaporated at a temperature of 50° C, what percentage of the latent heat goes into increasing the internal energy and what percentage into external work? Ans: 93.8 %; 6.2 %.

8. If water is evaporated at a temperature of 10° C, what percentage of the latent heat goes into increasing the internal energy and what percentage into external work?

9. A vessel of 1 liter volume contains 1 g of H_2O at 100° C. What fraction of the gram is liquid? Ans: $673/1672$.

10. A boiler of 0.5 m^3 volume contains 100 kg of H_2O at 200° C. How many kg of steam are in the boiler?

11. Compute the specific volume of saturated water vapor at 10° C as given by the perfect gas law. Compare with the observed value of Table I. Ans: 106 m^3/kg.

12. Compute the specific volume of saturated water vapor at 100° C as given by the perfect gas law. Compare with the observed value of Table I.

3. SUBLIMATION; THE TRIPLE POINT

A comparison of the fusion curve of Fig. 3 and the vaporization curve of Fig. 7, both for water, shows that these curves intersect. The point of intersection is at

$$p = 4.58 \text{ mm}, \quad T = 0.0099° \text{ C}. \tag{4}$$

Since solid and liquid can coexist along the fusion curve, and since liquid and vapor can coexist along the vaporization curve, the point of intersection of these curves must give a value of p and T at which solid, liquid, and vapor can all three coexist in equilibrium. This point (4) is therefore called the *triple point* for water.

It is very easy in principle to determine the triple point of water. Take a thermally insulated vessel containing only ice and liquid water. Then increase the volume somewhat to make space for vapor. Evaporation will take place—some slight adjustment of the quantity of ice and that of liquid water will occur in arriving at thermal equilibrium, and the system will settle down at the temperature and pressure of the triple point (see Fig. 9). *There is only one temperature and one pressure at which solid, liquid, and vapor phases can coexist in equilibrium.* The triple point of water is found to be readily reproducible experimentally to within 0.005 C deg.

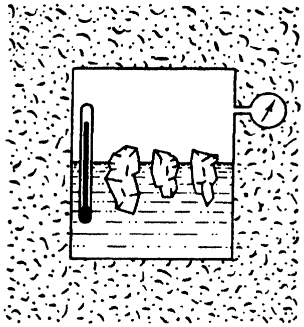

Fig. 9. The triple point.

What is the phase of the substance at pressures below that of the triple point? It is seen (Fig. 10) that the fusion and vaporization curves intersect in such a way that no liquid can exist at lower pressures. But both solid and vapor can exist. Below the pressure of the triple point we have a third curve, called the *sublimation curve,* which represents conditions for equilibrium between solid and vapor. This curve is shown for water in Fig. 10. The diagram of Fig. 10, which shows all three curves and the triple point, is commonly called the *triple-point diagram.* The

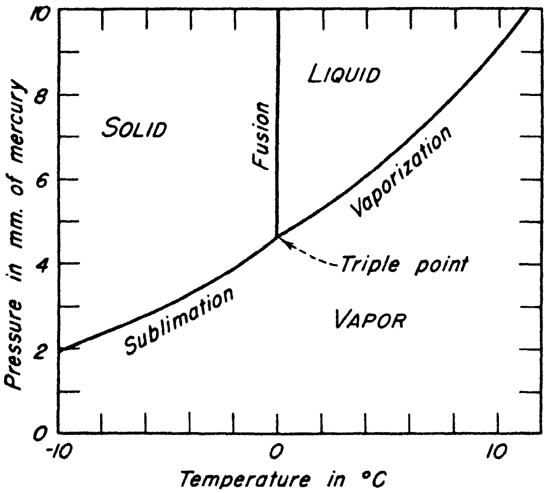

Fig. 10. Triple-point diagram for water.

sublimation curve is not a direct continuation of the vaporization curve. It has a slightly different slope at the triple point.

Sublimation represents a direct evaporation of molecules from a crystalline solid into the vapor phase. At the low pressures and temperatures occurring on the sublimation curve, solid and vapor can coexist, with equilibrium between the rate of sublimation and the rate of condensation. There is a latent heat of sublimation which is exactly analogous to the other latent heats we have considered; it is given for H_2O in Table I.

While the triple point for H_2O lies well below ordinary pressures, the triple points for some substances, notably CO_2 and iodine, lie well above ordinary pressures, so these substances are not commonly observed in the *liquid* phase.

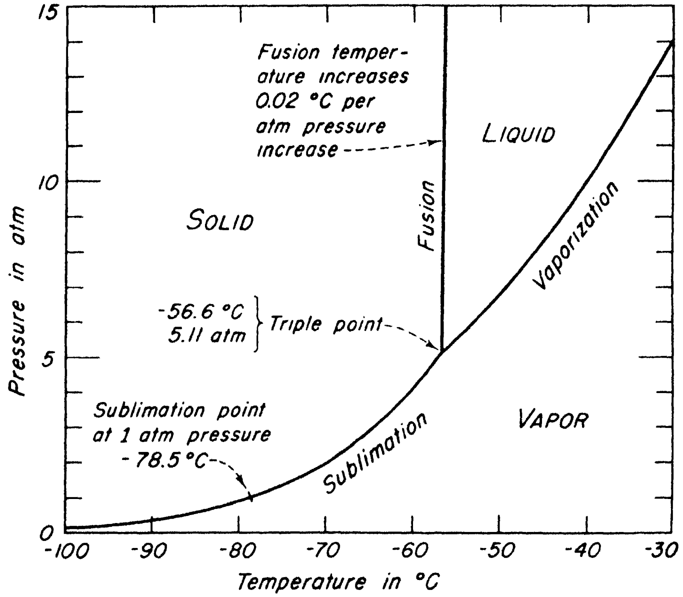

Fig. 11. Triple-point diagram for CO_2.

The triple-point diagram for CO_2 is shown in Fig. 11. The triple point is at −56.6° C and 5.11 atm. We are acquainted with solid CO_2 in equilibrium with its vapor at 1 atm pressure and a temperature of −78.5° C, and with its sublimation. The latent heat of sublimation is 137.9 kcal/kg at 1 atm pressure. We are also acquainted with liquid CO_2 in tanks, used for the carbonation of beverages. This liquid is at room temperature but must be at a pressure above 56.5 atm or it could not exist as a liquid at 20° C. The vaporization curve of Fig. 11 has the following coordinates above the temperatures covered by Fig. 11:

T (° C)	−30	−20	−10	0	10	20	30	31.1
p (atm)	14.1	19.4	26.1	34.4	44.4	56.5	71.1	72.9

PROBLEMS

1. At the triple point, the specific volumes of liquid water, ice, and steam are 1, 1.09, and 206,300 cm^3/g. The pressure is 4.58 mm. The latent heat of fusion is 79.7 kcal/kg; the latent heat of vaporization is 597.4 kcal/kg. From these data determine the differences in specific internal energy of water and ice and of steam and water at the triple point, and determine the latent heat of sublimation at the triple point. Ans: 79.7, 567.2, 677.1 kcal/kg.

2. On the basis of the data in the preceding problem, show that if heat is added to a vessel of constant volume containing ice, liquid water, and steam, almost all the heat is used to melt ice to form liquid and that very little of the heat goes into the formation of more vapor. Estimate roughly the percentages of heat used in these two processes.

3. At 1 atm pressure and the sublimation temperature, solid CO_2 has a density of 1.53 g/cm^3. Using the vapor density given by the ideal gas law, find the percentage of the latent heat of sublimation that goes into increase in internal energy and the percentage that goes into external work. Ans: 93.6%; 6.4%.

4. If a vessel contains only CO_2 at the following temperatures and pressures, what phase or phases (solid, liquid, gas) can be present? (a) −90° C, 1 atm; (b) −78.5° C, 1 atm; (c) 20° C, 1 atm; (d) 20° C, 56.5 atm; (e) 20° C, 75 atm; (f) −56.6° C, 5.11 atm; (g) −56.4° C, 15.11 atm.

5. What changes of phase take place in 100 g of water that is initially at 150° C and is cooled to −150° C under the constant pressure of 1 atm? What changes take place if the cooling is done at 2 mm pressure? Give the approximate temperatures at which the changes occur.

4. THE CRITICAL POINT

A study of Table I and Fig. 7 shows that the vaporization curve for water *ends* at a temperature of 374.15° C and a pressure of 218.4 atm. The end of the vaporization curve is called the *critical point.* As we follow up the vaporization curve, the difference in density between liquid and vapor becomes less and less, and the latent heat of vaporization less and less, until both vanish at the critical point. For other substances the vaporization curve ends similarly at a critical point. Values of pressure, temperature, and specific volume at the critical points of various substances are tabulated in Table II.

The critical point of CO_2 is easily demonstrated. The correct amount of CO_2 is contained in a sealed tube of strong glass. (The volume must be exactly 2.15 cm^3 per gram of CO_2.) At room temperature the CO_2 will be partly liquid, partly gaseous, with a distinct meniscus. As the temperature is gently raised, the temperature and pressure of the CO_2 will follow up the vaporization curve; as the critical temperature is approached, approximately half the volume will be liquid and half vapor. Then the meniscus gradually disappears where it stands, as the optical and all other properties of the liquid and the vapor approach identity. The disappearance is complete at 31.1° C. The reverse process occurs and separation into two phases takes place as the temperature is lowered back through 31.1° C.

TABLE II

CRITICAL CONSTANTS

Substance	Critical Temperature (°C)	Critical Pressure (atm)	Critical Volume (cm^3/g)
Water	374.15	218.4	3.1
Carbon dioxide	31.1	72.9	2.15
Oxygen	−119	50	2.33
Argon	−122	48	1.88
Nitrogen	−147	33.5	3.21
Neon	−229	27	2.07
Hydrogen	−240	13	3.23
Helium	−268	2.3	14.5

Above the critical pressure, no liquid-vapor phase transition occurs for any temperature. Above the critical temperature, no liquid-vapor phase transition occurs for any pressure. Above the critical pressure only two phases occur, instead of the usual three. These are the crystalline solid phase and the *amorphous* phase. As the temperature is raised with the pressure above critical, the amorphous phase changes *continuously* from a state which behaves much like a liquid to a state which behaves like a perfect gas, as we shall see in the next section. Similarly, above the critical temperature only *one* phase, the amorphous phase, exists. (There is some evidence, however, that if the pressure is raised to sufficiently enormous values, a crystalline solid can be obtained for any substance at any temperature, however high.)

We now see that we can make a *continuous* change from liquid (say water at 20° C and 1 atm) to vapor (say steam at 110° C and 1 atm) by so altering the pressure and temperature that we go *around the end* of the vaporization curve. This possibility emphasizes the fact that there is no *fundamental* difference between a liquid and a vapor, whereas each of these differ fundamentally from a crystalline solid, in that their structure is amorphous rather than crystalline, with a random rather than a regular molecular arrangement.

Gases for which a phase transition cannot be observed when the pressure is increased at room temperature, because their critical temperature lies below room temperature, are sometimes called *permanent* gases. All except the first two substances of Table II are of this type.

5. REAL GASES

Real gases do not obey the ideal gas law perfectly. This discrepancy is best illustrated by plotting *isothermals* on a p, ρ diagram as we do in Fig. 12 for dry air, Fig. 13 for CO_2, and Fig. 14 for H_2O. These are curves in which density is plotted against pressure at a constant temperature. Considering the triple-point diagram (see Figs. 10 and 11), these are

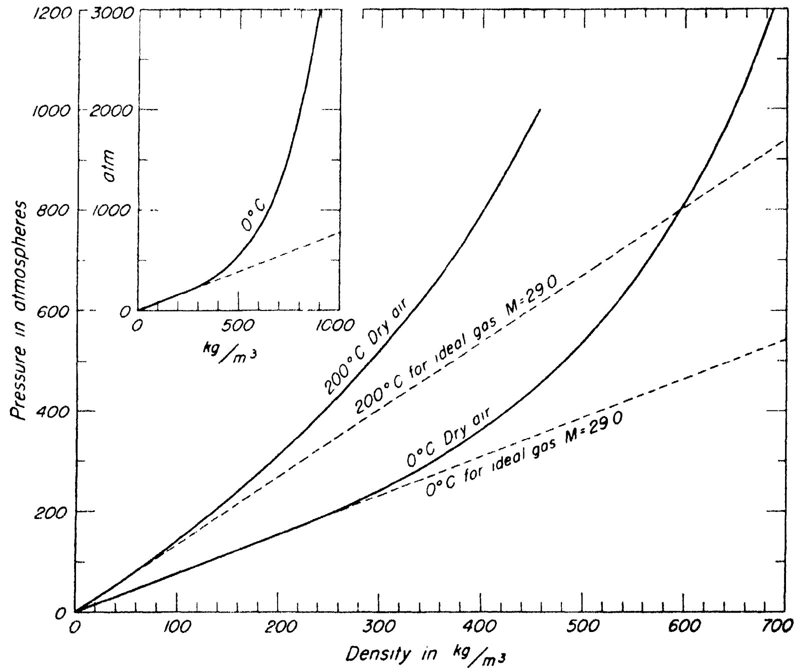

Fig. 12. Isothermals at 0° C and 200° C for dry air.

curves showing the increase of density with increase of pressure as we go up a straight vertical line in the triple-point diagram. If we are at a temperature above the critical temperature, the density increases along a smooth curve as the pressure is increased. If we are at a temperature below the critical temperature, the density increases smoothly until the vaporization (or sublimation) pressure is reached; then the density increases by a finite amount with no change in pressure as liquefaction (or solidification) takes place. Then the density of the liquid again increases smoothly with temperature. The process of liquefaction occurs along a horizontal portion of the isothermal, on which liquid and vapor coexist. As we move to the right along this horizontal portion, the density of the vapor and that of the liquid remain constant, but the fraction of the substance that is liquid increases from 0 to 100 per cent.

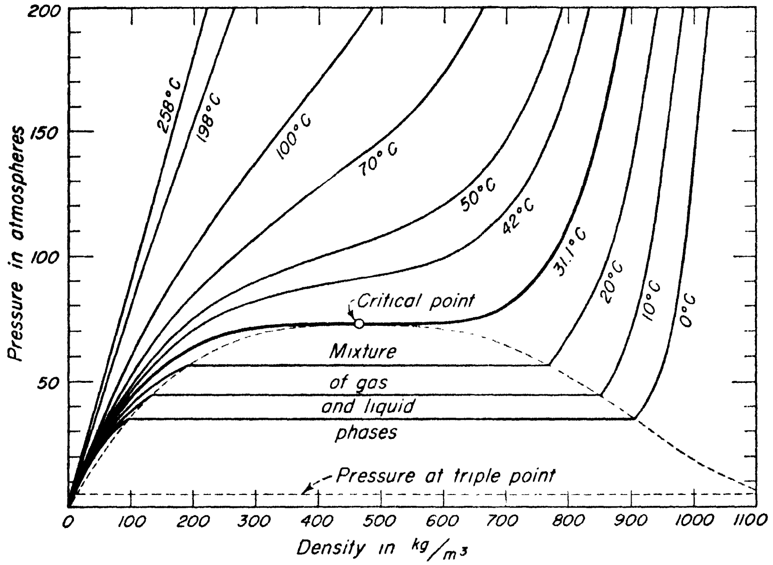

Fig. 13. Isothermals for CO_2, including the critical isothermal at 31.1° C, and showing the region in which vapor and liquid coexist. At the bottom of this region, below the 5-atm pressure of the triple point indicated by the horizontal broken line, is a region in which vapor and solid coexist.

On such a p, ρ diagram, the isothermals for an ideal gas are straight lines passing through the origin, of slope.

$$\frac{p}{\rho}=\frac{RT}{M}=0.08206\,\frac{T}{M}, \qquad (5)$$

as given by the general gas law when p is in atmospheres, ρ in kg/m³, and T in ° K, and M is the molecular weight.

The isothermals for a real gas always start out from the origin with the slope given by (5) *and behave like an ideal gas if the pressure and density are sufficiently low.**

---

* It is to be noted that if the temperature is very high, the gas molecules may be partially dissociated into atoms so that the average molecular weight to be inserted in (5) is less than the molecular weight at ordinary temperatures. Dissociation of the O_2 and N_2 atoms of air begins to be of significance at a temperature of about 2500° C.

Departures from the ideal gas law arise from forces between the molecules of a gas. These forces are of two types: (1) strong repulsive forces when the molecules come very close together, these forces giving the molecules a more or less well-defined size and making them appear to bounce when they collide; and (2) attractive forces of somewhat longer range which are the cause of the cohesion of liquids and solids.

The attractive forces cause the density to be greater than the ideal density and result in the curves of Figs. 13 and 14 swinging to the right of the ideal curve. Even the dry-air curves of Fig. 12 are slightly to the right of the ideal curve at pressures in the neighborhood of 1 atm, but by an amount too little to show on Fig. 12. We list below the ratio of actual density to ideal density for various gases at NTP:

Gas	$\frac{\textit{Actual density}}{\textit{Ideal density}}$
Air	1.0006
H_2	0.9994
He	0.9995
CO_2	1.0071

The density difference is seen to be negligibly small for most purposes.

The short-range repulsive force causes the isothermals to swing sharply upward as a density of the order of that of water (1000 kg/m^3) is approached. The density seems to refuse to increase beyond a certain limiting value no matter how much pressure is applied. This is the density at which the molecules are apparently packed tightly together.

As seen in Fig. 14, water vapor behaves very closely like a perfect gas at all pressures up to its saturation pressure. For temperatures below 60° C, the density of water vapor never differs from the ideal density by more than 0.5 per cent, even at saturation. This statement is of importance in connection with the discussion of humidity in Sec. 7.

We can now discuss the properties of the amorphous state, at a pressure above the critical pressure, in more detail. Let us look at Fig. 13 and consider the properties of CO_2 at 100 atm pressure. At 0° C and 100 atm, the substance has all the properties of a liquid. It will start to evaporate and split into two phases if the pressure is lowered to 35 atm isothermally. It is relatively incompressible; doubling the pressure (from 100 to 200 atm) isothermally only results in a 4 per cent density increase or a 4 per cent volume decrease. If now we start with the substance at 0° C and 100 atm and raise the temperature *isobarically* (that is, keeping the pressure constant), we move to the left in Fig. 13 along a horizontal isobar, crossing the isothermals in continuous fashion until we reach a temperature of, say, 258° C. At 258° C and 100 atm the substance behaves exactly like a perfect gas, obeying the ideal gas law very accurately. At this temperature, doubling the pressure isothermally doubles the density or halves the volume. The transition from a substance that is 'obviously' a liquid to one that is 'obviously' a gas has, however, been continuous, and there is no good criterion for deciding where we should stop calling the substance a liquid and start calling it a gas. The change in terminology is sometimes made at the critical isothermal (31.1° C), the material being called a liquid at temperatures below the critical temperature because it would split into two phases if the pressure were lowered isothermally, and being called a gas at temperatures above the critical temperature because it would not.

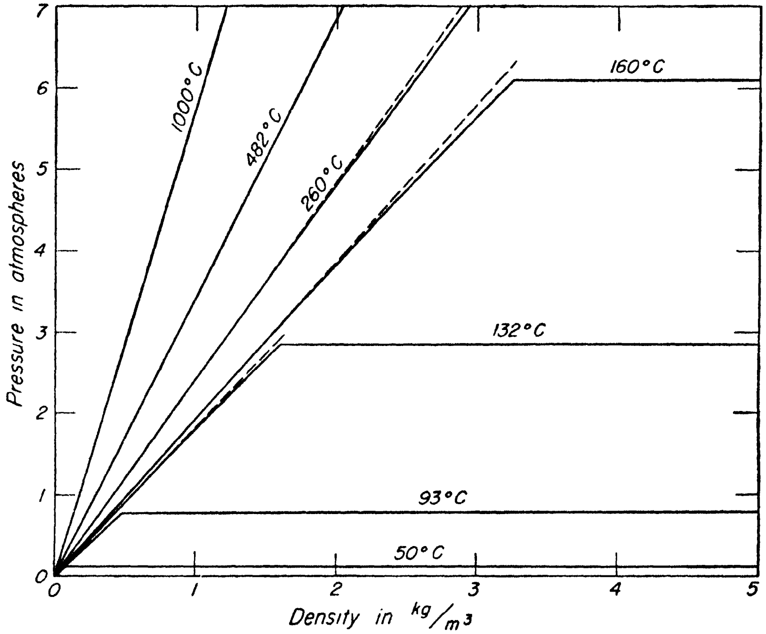

Fig. 14. Isothermals for H_2O in the region of importance in connection with superheated steam. Broken lines show the behavior of an ideal gas of the same molecular weight.

PROBLEMS

1. Verify the numerical factor given in equation (5).

2. Compute the ideal density of air at 0° C and 500 atm pressure. Which point does this give on the curves of Fig. 12?

3. Compute the ideal density of superheated steam at 482° C and 7 atm pressure. This ideal density is actually 0.7 per cent less than the observed density under these conditions—a difference too little to show on the plot of Fig. 14. Ans. 2.03 kg/m³.

6. BOILING

So far in this chapter we have confined our attention to a single pure substance in an enclosure. It is important to consider the effect of the presence of a foreign permanent gas such as air.

Let us start by considering the heating of water in an open vessel. Let us suppose that the barometer reads 76 cm. Then the pressure on the surface of the water is 76 cm and the pressure within the water is the same except for a slight addition arising from the weight of the water. When the temperature of the water is below 100° C, evaporation takes place only from the surface of the water, and some of the heat added to the water goes into latent heat in this evaporation. The evaporation takes place only at the free surface, since an incipient steam bubble within the water would be unable to support a pressure of 76 cm and would collapse. The amount of evaporation can be restricted by restricting the surface area.

If heat is added at a sufficient rate, the temperature of the water will increase to 100° C. At this temperature, for the first time, steam bubbles can form and grow within the body of the water, because now the vapor pressure within the bubble equals the hydrostatic pressure of the atmosphere. Ebullition will begin and continue until all the water has boiled away, with no further increase in temperature.

From this type of argument we see that the vaporization curve of Fig. 7 gives the boiling point of water in an open vessel as a function of atmospheric pressure applied to the surface. If the applied pressure is reduced, the boiling temperature goes down. Boiling or cooking in a vessel in which the pressure is kept down by vacuum pumps is utilized in many industrial processes where a controlled temperature less than 100° C is desired or where it is desired to evaporate water in dehydration of materials that would be harmed by a temperature as high as 100° C. The reverse is accomplished in a pressure cooker where the air and steam are allowed to build up to a pressure above normal so that the cooking temperature will be above 100° C.

The behavior of a geyser is interesting. Here a mass of underground water is connected to the surface by a narrow channel filled with water. The mass of water is heated from below. Because of the large hydrostatic head, the boiling point of the deep water is well above normal. It thus gets heated well above 100° C before ebullition starts. With ebullition, some of the water in the channel is forced out and replaced by steam. This process lowers the hydrostatic head and hence the boiling point. The water that now finds itself above its own boiling point immediately changes its own excess energy into the latent heat of additional steam, and the process rapidly culminates in a violent eruption that throws water and steam high into the air.

Although nothing analogous to ebullition takes place in the sublimation of CO_2, nevertheless the temperature of solid CO_2 in a container open to the atmosphere is in general the sublimation temperature corresponding to the pressure of the atmosphere, because the CO_2, in subliming, tends to surround itself with a layer of CO_2 vapor. If the container is open to the atmosphere, this vapor cannot have a pressure greater than one atmosphere, so if the solid CO_2 were warmed to a temperature slightly above its sublimation temperature, it would evaporate definitely faster than vapor could condense. Since the rate of sublimation increases rapidly with temperature, the cooling effect of sublimation would keep the temperature from rising much above the sublimation temperature, even if heat were added fairly rapidly. Ordinarily, solid CO_2 in a somewhat confined enclosure will be at a temperature close to its sublimation temperature (−78.5° C at 1 atm), whereas if it is in a strong current of air that carries the vapor away as fast as it is formed, the CO_2 will cool to still lower temperatures, just as water will cool by evaporation.

PROBLEMS

1. If it is desired to have water boil at 70° C, what pressure should be maintained in a vacuum cooker? Ans: 23.4 cm of Hg.

2. If it is desired to have water in a pressure cooker boil at 110° C when the ordinary boiling point is 100° C (the change of 10° about doubles the cooking speed), to what gauge pressure in lbf/in^2 should the valve be set?

3. If it is desired to have water in a pressure cooker boil at 110° C in a mountain location where the ordinary boiling point is only 90° C, to what gauge pressure in lbf/in^2 should the valve be set? Ans: 10.6 lbf/in^2.

4. Using the standard pressure vs. altitude curve (Fig. 3, p. 377), estimate the boiling points of water at altitudes of 5000, 10,000, and 15,000 ft above sea level.

5. From the standard pressure vs. altitude curve (Fig. 3, p. 377), estimate the boiling point of water in Mexico City, which is at 7,800 ft above sea level.
Ans: about 92° C.

6. A large vacuum pump continuously exhausts the vapor from a thermally insulated vessel containing water. Why does the water cool down and freeze?

7. If you had an ample supply of liquid oxygen at its boiling point under 1 atm pressure, how would you proceed to make solid oxygen? (Compare with Prob. 6. This procedure has been used to solidify all gases except He.)

8. To what approximate pressure would a vessel containing solid CO_2 have to be continuously pumped down if the CO_2 were to be cooled to −100° C?

9. What is the approximate boiling point of methyl alcohol at 0.5 atm pressure?
Ans: 48° C.

10. What is the approximate boiling point of ethyl ether at 1 atm pressure?

7. MIXTURES OF GASES AND VAPORS, HYGROMETRY

Let us now return to Fig. 5, p. 402, and suppose that the space above the liquid water contains air or some other permanent gas as well as water vapor. Most permanent gases are only slightly soluble in liquid water, so that only a comparatively few of the gas molecules will enter the water and be dissolved in it. Unlike water-vapor molecules, most of the air or gas molecules that strike the liquid surface will bounce off as if it were a solid surface.

The water seems to ignore the presence of the air. The rate of evaporation is still determined purely by the temperature of the water and is independent of the pressure of the air. The rate of condensation is the same function of the partial pressure of the water vapor and the temperature as if the air were not present. Hence, *in equilibrium the partial pressure of the water vapor is given by the vaporization curve of Fig. 7.* The total pressure in the space above the liquid water will be the sum of the partial pressure of the air, which is given by the general gas law in terms of the temperature, volume, and mass of air, and of the partial pressure of the water vapor, which is given by the vaporization curve.

Similar considerations apply to the case of any permanent gas confined in the space above any volatile liquid in which the gas is not very soluble.

The case we have discussed is that in which the air or other gas is *saturated* with the vapor of the liquid. Another important case is that in which there is no liquid present but only vapor, which may have less partial pressure than the saturation vapor pressure. A vapor cannot have *greater* partial pressure than the saturation pressure corresponding to the temperature of the vapor because then condensation of the vapor to liquid or solid would take place.* Thus, the water vapor in the atmosphere can have any partial pressure up to that of saturation. If the partial pressure exceeds that of saturation, condensation occurs, and formation of clouds, fog, dew, or frost results.

The term *humidity* is used to describe the water-vapor content of the atmosphere.

> The *absolute humidity* is the mass of water vapor per unit volume of atmosphere.
>
> The *relative humidity* is the ratio of the partial pressure of water vapor in the atmosphere to the partial pressure that would cause saturation at the temperature of the atmosphere.

The relative humidity is also, to a close approximation, the ratio of the actual density of water vapor to the saturation density at the same temperature, since water vapor below 60° C obeys the ideal gas law very closely, as we have pointed out in Sec. 5.

Thus, if the atmosphere is at 20° C and there are 9 g of water vapor per m^3, the absolute humidity is 9 g/m^3. Since saturated vapor has a density of 17.3 g/m³ at 20° C (see Table I), the relative humidity is 52 per cent. The saturation pressure of water vapor at 20° C is 17.5 mm. The actual pressure of the water vapor under these conditions would be 52 per cent of 17.5 mm, or 9.1 mm.

---

* Dust particles and electric charges serve as 'condensation nuclei' on which droplets of liquid or small ice crystals form. If no condensation nuclei are present, supersaturation may exist momentarily.

The measurement of humidity is called *hygrometry*. Absolute humidity is measured by drawing a known volume of air through a drying agent and measuring the increase in weight of the drying agent. Relative humidity is most accurately measured by determining the *dew point*. In the dew-point apparatus, a polished metal surface is slowly cooled, and the temperature is noted at which dew first begins to cloud the surface. In this experiment, the atmosphere is cooled locally in the vicinity of the metal surface. Since the atmosphere in the region near the metal surface is in pressure equilibrium with the balance of the atmosphere, the partial pressures of air and water vapor in this region will be the same as in the remainder of the atmosphere near the station where the measurement is being made. But when the temperature of the air near the metal surface drops so that the partial pressure of the water vapor equals the saturation pressure, liquid water will begin to condense on the metal surface.

> The dew point is that temperature at which water vapor begins to condense, and hence that temperature at which the partial pressure of the water vapor present in the atmosphere equals the saturation pressure.

Since the saturation pressure at the dew point is the actual pressure of water vapor in the atmosphere, we see from the definition of relative humidity that

$$\left\{\begin{matrix}\text{relative}\\ \text{humidity}\end{matrix}\right\} = \frac{\text{saturation pressure at dew point}}{\text{saturation pressure at actual atmospheric temperature}}. \qquad (6)$$

Thus, in the example of the text above, the dew point would be 9.8° C, since the actual pressure of 9.1 mm would cause saturation at this temperature.

Another method of measuring humidity is by means of wet- and dry-bulb thermometers. If the bulb of a thermometer is covered with a wet cloth, evaporation of the water will lower the temperature reading. The rate of evaporation, and hence the change in temperature reading, will depend on the relative humidity—no evaporation will take place at 100 per cent relative humidity.

The wet-bulb thermometer will give a reading somewhere between the actual temperature (the dry-bulb thermometer reading) and the dew point. The wet-bulb reading cannot be below the dew point. Such a situation would result in condensation on, rather than evaporation from, the wet bulb, and hence in a heating, rather than a cooling, effect. Within this possible range, the actual temperature to which the wet bulb comes depends principally on the air velocity past the wet bulb. Handbooks contain tables that give the relative humidity as a function of dry- and wet-bulb readings for a standard air velocity of 15 ft/sec. As an example of these empirically determined values we give below the relative humidities corresponding to various wet-bulb readings when the air temperature is 20° C. We also give the dew points corresponding to these relative humidities.

Relative Humidities and Dew Points as a Function of Wet-bulb Reading When the Dry-bulb Reading Is 20° C

Wet-bulb reading	*Relative humidity*	*Corresponding dew point*
20° C	100%	20.0° C
19	90	18.2
18	81	16.5
17	72	14.7
16	63	12.8
15	55	10.8
14	47	8.5
13	39	5.8
12	32	2.7
11	25	−0.8
10	18	−4.7

It is seen that the wet-bulb reading is, very roughly, halfway between the dry-bulb reading and the dew point.

Weather observers usually use a form of wet- and dry-bulb hygrometer called a *sling psychrometer*, in which the thermometers are whirled rapidly about the hand in order to establish the necessary air current past the wet bulb.

A less accurate form of hygrometer depends on the fact that human hair expands in length in approximate proportion to relative humidity. The change in length of a single hair mounted under tension may be read on a dial calibrated to give relative humidity.

The behavior of a wet-bulb thermometer has a direct relation to the dependence of human comfort on humidity, since the body is cooled by evaporation of perspiration. The higher the relative humidity, the warmer a heated room 'feels.' It has been experimentally determined that a room at 70° F and 100 per cent relative humidity feels about as warm as one at 75° F and 50 per cent relative humidity or one at 80° F and 10 per cent relative humidity. For comfort and health, a relative humidity between 30 per cent and 70 per cent is usually recommended.

PROBLEMS

1. The measured volume of a quantity of hydrogen collected over water is 490 cm^3, the temperature being 21° C and the barometer reading 750 mm. The volume is measured with the water level the same inside and outside the hydrogen bottle. Calculate the volume of dry hydrogen at NTP. Ans: 438 cm^3.

2. A sample of oxygen collected over water at 19° C occupies a volume of 804 cm^3 when the water level is the same inside and outside the bottle. Find the volume of dry oxygen at NTP if the barometer reads 749.5 mm at the time of collection.

3. A flask is partly filled with warm water at 70° C. The air above the water is allowed to become saturated and then the flask is tightly corked. If the barometer reads 770 mm at the time of corking, what will be the pressure in the flask the next day when it is at temperature 30° C and the barometer has dropped to 730 mm? Ans: 506 mm.

4. A vessel containing air and water vapor is sealed at 100° C with the total pressure 76.0 cm, partial pressure of water vapor 16.7 cm. It is then allowed to cool. Compute the total pressure in the vessel at 70° C and at 30° C.

5. In a sample of air collected in a hot, damp location when the temperature was 45° C, pressure 724.9 mm, it was found that there were 50.9 g of H_2O per 1000 g of dry air. Find the relative humidity and the dew point. Ans: 77.0%, 40.0° C.

6. If the dew point is 5° C when the temperature is 23° C, what is the relative humidity?

7. What is the dew point when the temperature is 30° C and the relative humidity 45 per cent? Ans: 16.8° C.

8. On a winter day the outdoor temperature is −5° C, and because of the snow the relative humidity is 100 per cent. Unless moisture is added to the air, the partial pressure of water vapor will be the same inside a house at 20° C. What will be the relative humidity in the house?

9. If the outdoor temperature is 15° C and the relative humidity 75 per cent, what will be the relative humidity inside a room at 25° C if no moisture is added to the inside air? Ans: 40.4%.

10. If a man enters a room at 25° C from an outdoor temperature of 18° C and his glasses steam up, what is a minimum value of the relative humidity in the room?

11. At what relative humidity in a room at 20° C will ice-water glasses collect a coating of moisture? Ans: 26.1% or higher.

12. At what relative humidity on a summer day at 30° C will mint-julep glasses at −10° C collect a coating of frost?

13. A vessel contains 129 g of air and 1.5 g of water vapor at 30° C. The total pressure is 70 cm of Hg.

(a) Find the partial pressure of the water vapor.

(b) Find the relative humidity in the vessel. Ans: (a) 12.9 mm; (b) 40.4%.

14. If a vessel contains 1 liter of air saturated with ethyl-ether vapor at 30° C and 1 atm, what would be the volume of dry air at NTP?

15. To what maximum height could water be lifted by a suction pump when the barometer reads 76 cm and the water is at 40° C? Ans: 9.62 m.

16. To what maximum height could liquid ethyl ether at 20° C be raised by a suction pump when the barometer reads 760 mm?

17. An air-conditioning unit works in the summertime by cooling the air below the desired temperature to condense out excess water vapor, and then reheating the air. If outside air is taken in at 95° F and 65 per cent relative humidity, and delivered at 74° F and 45 per cent relative humidity, to what temperature must it be cooled to lower the water-vapor content to the desired value? Ans: 51° F.

18. In the air-conditioning installation described in Prob. 17, if air is taken in at 100° F and 30 per cent relative humidity, and delivered at 70° F and 60 per cent relative humidity, to what temperature must it first be cooled to lower the water-vapor content to the desired value?

CHAPTER 19

THERMODYNAMICS

In this chapter we give a very brief introduction to a discipline called *thermodynamics*, which was developed during the nineteenth century by Carnot, Joule, Kelvin, Clausius, Gibbs, Rankine, and many others. This discipline is capable of drawing a surprisingly large body of important conclusions and formulas as logical deductions from just two simple postulates known as the first and second laws of thermodynamics. The first law is merely a statement of the conservation of energy for transformations involving mechanical and thermal energy. The second law asserts the inability of any prime mover to derive mechanical energy from thermal energy unless substances at two *different* temperatures are available and utilized, the hotter substance giving up heat and the cooler substance absorbing heat. These postulates are accepted because no conclusions at variance with experience have yet been deduced from them. The conclusions that can be drawn from these simple postulates are fundamental to many branches of physics and chemistry as well as to power, refrigeration, and aeronautical engineering. In this chapter we confine our attention to certain simple applications in the fields of power and refrigeration, and describe briefly the steam engine and various types of refrigerators.

We conclude the chapter with a statement of the principle of Le Châtelier-Braun, which expresses qualitatively the general type of conclusion that can be drawn from the second law of thermodynamics in various types of physical problems. We shall have frequent occasion to refer to this principle in later sections of this book.

1. THE FIRST LAW OF THERMODYNAMICS

As we have already noted, the first law of thermodynamics is a statement of the law of conservation of energy when both mechanical and thermal forms of energy are taken into account. In dynamics we do not have a rigorous law of conservation of energy. Macroscopic mechanical energy (kinetic plus potential) is conserved only when there is no friction, that is, when no heat is generated. But if we include thermal energy, the law of conservation of energy is much more general, and the sum of the mechanical energy and the thermal energy of an isolated system is conserved provided only that there are no electrical or chemical phenomena occurring which result in changes in the electrical or chemical state of the constituents of the system.

The first law of thermodynamics may be extended nowadays to include all forms of energy and stated as follows:

> *The total energy (mechanical kinetic + mechanical potential + thermal + chemical + nuclear + electromagnetic) of an isolated system, or of the universe, is constant in time.*

In this chapter, however, we shall be concerned only in transformations of energy between mechanical and thermal forms; for such processes the first law of thermodynamics can be stated in the following form:

> *When mechanical energy disappears, an equivalent quantity of thermal energy appears; and when thermal energy disappears, an equivalent quantity of mechanical energy appears; the equivalence being expressed by the relation*

$$1 \text{ kilocalorie} = 4186 \text{ joules.}$$

Thus, when a bullet is fired into a fixed block of wood and embeds itself there, it loses all its mechanical kinetic energy, but an equivalent quantity of heat appears in the bullet and the block. When a coasting automobile is brought to rest, the mechanical kinetic energy of the car and its passengers appears as heat in the brakes and bearings and at the road surface. Reciprocally, when steam in a cylinder expands and does mechanical work, the work done is accompanied by an equivalent decrease in the thermal energy of the steam.

In the useful processes for converting heat into work (engine) or work into heat (refrigerator), there is always a working substance (for example steam, gaseous products of combustion of gasoline, ammonia, or Freon) which undergoes certain changes of state accompanied by certain changes in internal thermal energy. In this case the first law of thermodynamics gives the relation we have already stated on p. 380:

$$\begin{Bmatrix}\text{heat added to}\\ \text{substance}\end{Bmatrix} + \begin{Bmatrix}\text{mechanical work}\\ \text{done on substance}\end{Bmatrix} = \begin{Bmatrix}\text{increase in internal}\\ \text{energy of substance}\end{Bmatrix}. \qquad (1)$$

This relation presupposes that changes in macroscopic mechanical kinetic and potential energy of the substance are of negligible importance, as is the case in actual engines and refrigerators.

PROBLEMS

1. An automobile of 3,000-lb mass, traveling at 30 mi/hr, is brought to rest by the action of its brakes. Assuming that all the energy is converted into heat in the brakes, which have a mass of 60 lb and a specific heat of 0.12 BTU/lb·F deg, what will be the rise in temperature of the brakes? Ans: 16.1 F deg.

2. A lead bullet at 20° C is fired into sand at a speed of 450 m/sec. Assuming that 50 per cent of the heat generated is distributed uniformly in the lead of the bullet, what will be the temperature rise of the lead? (Take the specific heat as 0.030 kcal/kg, the melting point as 327° C, the latent heat of fusion as 6.0 kcal/kg.)

3. During launching, a yacht of 600-ton mass slides down inclined ways through a vertical distance of 9 ft before striking the water. If the yacht is then moving at 16 ft/sec, how many BTU of heat were developed by friction on the ways?

Ans: 7720 BTU.

4. A locomotive exerts a drawbar pull of 30,000 lbf on a 400-ton train when it is going up an incline of 1° slope at 45 mi/hr. How many BTU per second are generated in the wheels and bearings of the cars of the train?

5. The electrical input to an electric motor is 750 kw, of which 85 per cent is delivered as useful mechanical work, the remainder being wasted as heat. Compute the mechanical horsepower output and the rate of heat generation.

Ans: 855 hp; 26.9 kcal/sec.

6. The mechanical input to an electric generator is 500 hp, of which 95 per cent is delivered as electrical energy, the balance being wasted as heat. Determine the electrical output in kilowatts and the rate of heat generation.

2. ADIABATIC EXPANSION AND COMPRESSION

Consider a substance (which might be a gas or might be partly liquid and partly vapor) in the cylinder of Fig. 1. The cylinder and the face of the piston are assumed to be perfectly insulated thermally so that no heat can enter or leave the working substance. The substance is at pressure p, which requires that a certain force $F=pA$ be applied to the piston.

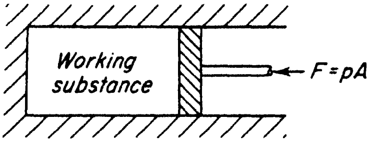

Fig. 1. A working substance in a cylinder with non-conducting walls and a non-conducting piston.

If the force F is gradually increased, the working substance will be gradually compressed. At each stage it will be compressed to such a volume that its pressure just balances the force of the piston. Work will be done on the substance by the piston. Since no heat enters or leaves the substance, this work will go entirely into increasing the internal energy of the substance, as we see from (1). This increase will result in a gradual increase in temperature of the substance. Here *gradual* denotes that the work is done sufficiently slowly to give the energy added by the work (in the first instance at the face of the piston) ample time to distribute itself throughout the whole of the substance so that at each instant the working substance can be considered to be in thermal equilibrium, with no more than an infinitesimal temperature gradient needed to carry the heat from the part of the substance near the face of the piston to the balance of the substance.

A compression carried out under the above ideal conditions is said to be *adiabatic*.

The reverse process, in which the force on the piston is gradually decreased, so that the substance gradually expands, doing work, and hence decreases in internal energy and temperature, is called an *adiabatic expansion.* In an adiabatic expansion the work done by the gas exactly equals the decrease in internal energy.

The adiabatic expansion is to be contrasted with the free expansion discussed on p. 381, in which the substance, an ideal gas, does no work and suffers no change in internal energy. One can approximate a free expansion by drawing the piston back so fast that the gas is 'left behind.' The gas then exerts no force and does no work. Hence the insistence, in defining an adiabatic process, that equilibrium conditions exist continuously, so that at each instant the substance exerts a pressure on the piston corresponding to its equilibrium pressure at a given temperature and volume.

> An adiabatic process is one that takes place with no addition or subtraction of heat, and with sufficient slowness so that the substance can be considered to be continuously in thermal and mechanical equilibrium.

No entirely adiabatic process occurs in practice, but the expansion of steam in the cylinder of a steam engine, of the hot gases in an internal-combustion engine, and the compression of the air in a Diesel engine or in an air compressor are all approximately adiabatic. These processes take place rapidly enough that only a small amount of heat has time to enter or leave the substance from the cylinder walls, and yet slowly enough so that thermal and mechanical equilibrium is approximately maintained throughout the gas in the cylinder. This furnishes one reason for studying adiabatic processes; however, the most important reason is that an ideal engine or refrigerator would employ processes that are exactly adiabatic. Study of such ideal devices sets theoretical limits to the capabilities of real engines and refrigerators which it is very important to understand thoroughly.

The laws governing adiabatic processes in an ideal gas are very easy to derive. Let a mass m of gas expand adiabatically from volume v_1 with pressure p_1, temperature T_1, and density ρ_1, to a greater volume v_2, with pressure p_2, temperature T_2, and density ρ_2, as in Fig. 2. Since no heat is added, the work done by the gas equals its decrease in internal energy. But by (21), p. 383, the decrease in internal energy is $mc_v(T_1-T_2)$. So in the *adiabatic expansion* of Fig. 2,

$$\text{work done by gas} = mc_v(T_1 - T_2). \qquad (2)$$

Similarly, in the reverse *adiabatic compression,* with notation also as in Fig. 2,

$$\text{work done on gas} = mc_v(T_1 - T_2). \qquad (3)$$

Equations (2) and (3) give the mechanical work done in an adiabatic process in terms of the temperature change. But we do not yet know how to relate the temperature change to the volume change. We can determine the relation between the temperature change and the volume change by considering an infinitesimal step in the expansion process. When the volume changes from v to $v+dv$, the gas does work $p\,dv$. At the same time let the temperature change from T to $T+dT$. Then the increase in internal energy of the gas is $mc_v\,dT$. Since dT is negative if dv is positive, it is better to say that the decrease in internal energy is $-mc_v\,dT$. Since the work done by the gas equals the decrease in internal energy, we have

$$p\,dv = -mc_v\,dT. \qquad (4)$$

The minus sign is associated with the fact that dT/dv is negative as indicated in Fig. 2.

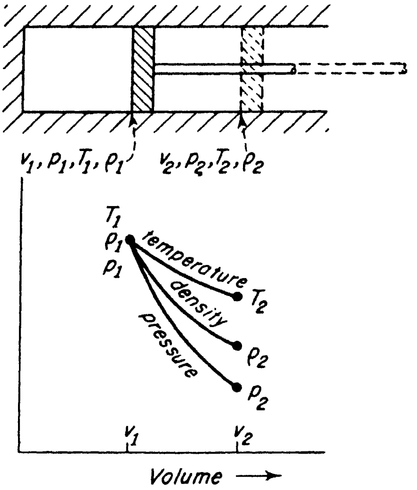

Fig. 2. Adiabatic expansion or compression of an ideal gas.

Now in (4) we can substitute the value of p given by the general gas law (4), p. 371:

$$p = \frac{R}{M}\,m\,\frac{T}{v},$$

and the value of c_v given by (28), p. 387:

$$c_v = \frac{R}{M}\,\frac{1}{\gamma-1},$$

where γ is the ratio c_p/c_v. When we make these substitutions in (4) and cancel common factors, we obtain

$$\frac{T}{v}\,dv = -\,\frac{dT}{\gamma-1},$$

which can be written as

$$dT+(\gamma-1)Tv^{-1}\,dv=0.$$

If we multiply this equation through by $v^{\gamma-1}$, we obtain

$$v^{\gamma-1}\,dT+(\gamma-1)Tv^{\gamma-2}\,dv=0.$$

The left side of this equation is just the differential $d(Tv^{\gamma-1})$, so

$$d(Tv^{\gamma-1})=0.$$

This result means that the change in the product $Tv^{\gamma-1}$ is zero in every step in the adiabatic expansion, so this product remains constant throughout the process:

$$Tv^{\gamma-1}=\text{const.} \tag{5}$$

in an adiabatic process. [Although we have discussed an adiabatic expansion explicitly, the compression is exactly the reverse process, and (4) and (5) are equally valid for an adiabatic compression.] In particular, this product has the same value at the beginning and the end of the process, so

$$T_2\, v_2{}^{\gamma-1}=T_1\, v_1{}^{\gamma-1} \tag{6}$$

in the notation of Fig. 2. This is the equation we need to find T_2 if T_1, v_1, and v_2 are given, so that we can compute the work from (2) or (3) if the volume change is given.

By combining (6) with the general gas law in the form (7), p. 372:

$$\frac{p_2v_2}{T_2}=\frac{p_1v_1}{T_1}, \tag{7}$$

we can get equations relating pressure and volume, or pressure and temperature, in an adiabatic expansion:

$$p_2\, v_2{}^{\gamma}=p_1\, v_1{}^{\gamma}, \tag{8}$$

$$\frac{p_2}{T_2{}^{\gamma/(\gamma-1)}}=\frac{p_1}{T_1{}^{\gamma/(\gamma-1)}}. \tag{9}$$

The proof of (8) and (9) is left as an exercise. By substituting $v_1=m/\rho_1$ and $v_2=m/\rho_2$ in (6) and (8), we can get relations that determine the gas densities in an adiabatic expansion:

$$\frac{T_2}{\rho_2{}^{\gamma-1}}=\frac{T_1}{\rho_1{}^{\gamma-1}}, \qquad \frac{p_2}{\rho_2{}^{\gamma}}=\frac{p_1}{\rho_1{}^{\gamma}}. \tag{10}$$

In an adiabatic expansion, the pressure drops more rapidly with volume than it does in an isothermal expansion, since in the adiabatic case not only does the volume increase but also the temperature decreases, both of which factors contribute to a pressure drop. This point, which is important for later discussion in this chapter, is illustrated in Fig. 3, which shows a family of p, v curves satisfying the isothermal equation

$$pv=\text{const.} \qquad \text{(isothermal)} \tag{11}$$

and also a family satisfying the adiabatic equation

$$pv^{1.67}=\text{const.} \qquad \text{(adiabatic)} \tag{12}$$

for the case $\gamma = 1.67$ appropriate to an ideal monatomic gas. The isothermals can be labeled by values of the temperature as at the top of Fig. 3. The adiabatics can be similarly labeled by values of a quantity called the *entropy*, which is defined in more advanced treatments of thermodynamics. We note, however, that adiabatic processes are sometimes called *isentropic* processes.

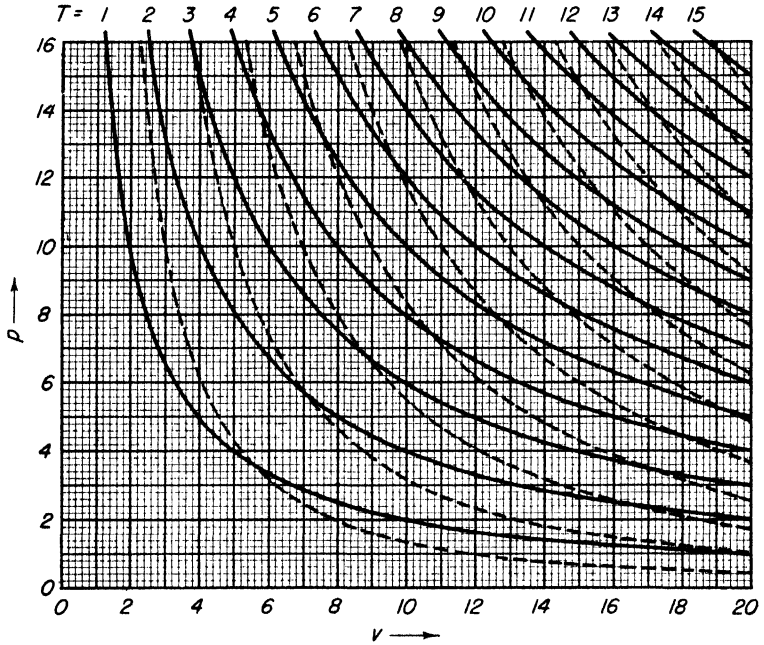

Fig. 3. Isothermals (solid lines) and adiabatics (broken lines) on a p, v diagram for an ideal gas with $\gamma = 1.67$. The units in which p, v, T are measured will vary with the quantity and kind of gas and must be chosen so that the general gas law is satisfied.

PROBLEMS

1. Show that at any point p, v of the diagram of Fig. 3, the isothermal passing through the point has slope $dp/dv = -p/v$, whereas the adiabatic passing through the same point has slope $dp/dv = -\gamma p/v$, which is γ times as great as the slope of the isothermal.

2. Derive equations (8) and (9) from (6) and (7).

3. The air in the cylinder of a Diesel engine is at 30° C and 1 atm before compression. It is compressed to $\frac{1}{18}$ of its original volume. Find the final temperature and pressure of the air if the compression is adiabatic and $\gamma = 1.4$.

Ans: 691° C; 57.3 atm.

4. The air in the cylinder of a Diesel engine is at a temperature of 200° F and a pressure of 18 lbf/in² at the beginning of compression. It is compressed to $\frac{1}{12}$ of its original volume. Find the temperature and pressure of the air at the end of the adiabatic compression. Take $\gamma = 1.4$.

5. Find the work necessary to compress 22.4 liters of helium at NTP adiabatically to one-quarter of this volume. Ans: 5180 joules.

6. A compressor takes in 10 m³/sec of CO_2 at 20° C and 1 atm, and compresses it adiabatically to $\frac{1}{10}$ its volume. The CO_2 then passes through a heat exchanger where it is cooled back to 20° C without change of volume. Assuming that the CO_2 behaves like an ideal gas, find the number of kcal/sec that the heat exchanger must extract.

NOTE: In the *Laval nozzle* of a steam turbine, the nozzle of a supersonic wind tunnel, or the nozzle at the rear of a rocket, the gas expands *adiabatically*. In this case the work that is done by a given parcel of gas in expanding can be somewhat crudely visualized as being done on the gas ahead of it, resulting in the macroscopic mechanical acceleration of the gas. In any case the adiabatic expansion through the nozzle converts thermal energy into kinetic energy of translation, the exact relation between temperature change and speed V acquired being

$$\tfrac{1}{2}V^2 = c_p(T_1 - T_2)$$

for a perfect gas. Here c_p is the specific heat at constant pressure, T_1 is temperature of the gas at rest in the reservoir, and T_2 is the temperature of the gas that has been accelerated to speed V. The objective in a turbine or rocket nozzle is to eject the gas with as much momentum as possible, hence with as high a speed as possible. There is seen to be a theoretical upper limit to V for a given T_1, which is obtained by setting $T_2 = 0$.

7. If a supersonic wind tunnel operates by adiabatic expansion of air from a reservoir at a pressure of 16 atm down to a pressure of 1 atm, and the reservoir temperature is 40° C, compute the temperature after the expansion and the speed acquired by the air. Ans: −131° C; 587 m/sec.

8. In the first stage of a steam turbine, the superheated steam at a reservoir temperature of 205° C and a pressure of 13 atm is expanded adiabatically through a Laval nozzle to a pressure of 10 atm. Treating the steam as a perfect gas with $\gamma = 1.3$, determine the discharge temperature and speed.

NOTE: In a *Wilson cloud chamber* (Fig. 4), air that is saturated with water vapor is rapidly expanded adiabatically to about 20 per cent greater volume. After the expansion the water vapor is at a partial pressure above that of saturation at the reduced temperature and begins to condense. The air is said to be *supersaturated*. The water vapor condenses on electrically charged ions and on dust particles, which serve as condensation nuclei. Electrically charged ions are particularly effective as condensation nuclei. Hence, if a high-speed charged particle or a high-energy light quantum (gamma ray) passes through the chamber close to the time of the expansion, leaving behind a trail of ionized air molecules, this trail will be shown up as a track of fog droplets which can be photographed. A number of such photographs will be found in Chap. 44.

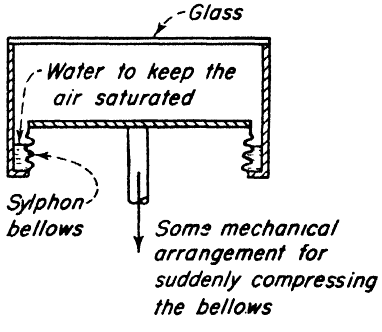
Glass
Water to keep the air saturated
Syphon bellows
Some mechanical arrangement for suddenly compressing the bellows

Fig. 4. Schematic drawing of one type of Wilson cloud chamber.

9. If air saturated with water vapor at 30° C and a total pressure of 1 atm is expanded adiabatically to 20 per cent greater volume, find the partial pressure of the water vapor in the supersaturated air just after the expansion, and compare with the pressure of saturated vapor. In computing the temperature after the expansion, neglect the difference between γ for water vapor and that for air and assume that the whole gas has $\gamma = 1.4$. This method will introduce little error because the water vapor, which has a slightly lower γ, constitutes only a small fraction of the whole gas. Ans: 24.6 mm as compared with 8.4 mm for saturation.

10. Make the same computation as in Prob. 9 for the case where the air is initially saturated at 25° C and expands to 15 per cent greater volume.

3. AN IDEAL HEAT ENGINE EMPLOYING THE CARNOT CYCLE

We shall now discuss a highly idealized heat engine known as the Carnot engine.* This discussion will show how it is possible to obtain mechanical work from heat by using a working substance that is carried through a cyclic process. The Carnot cycle is of fundamental importance because it gives an expression for engine efficiency which, as will be shown in Sec. 5, furnishes a theoretical limit to the efficiency that a real heat engine can have.

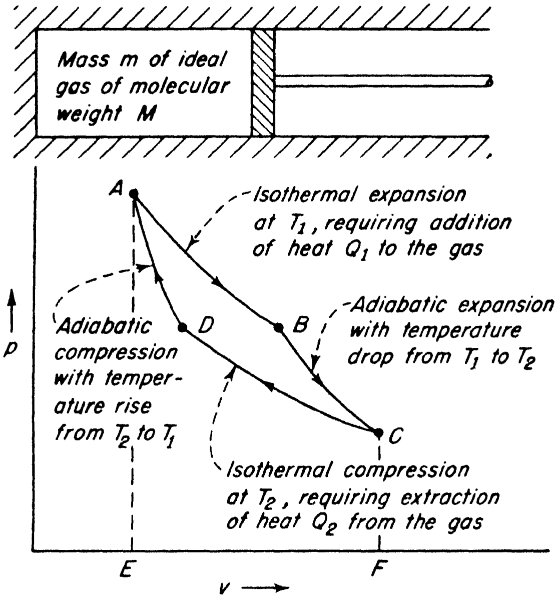
Mass m of ideal gas of molecular weight M
A
Isothermal expansion at T_1, requiring addition of heat Q_1 to the gas
Adiabatic expansion with temperature drop from T_1 to T_2
B
D
C
Adiabatic compression with temperature rise from T_2 to T_1
Isothermal compression at T_2, requiring extraction of heat Q_2 from the gas
p
E
F
v

Fig. 5. The Carnot engine.

In the Carnot engine, the working substance is an ideal gas contained in a cylinder equipped with a piston. By suitable manipulations, the gas is made to follow around a closed curve on the p, v diagram. The curve is a quadrilateral made up of pieces of two isothermals and two adiabatics from Fig. 3. Such a cycle is drawn schematically in Fig. 5.

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* Devised by Sadi Carnot, French engineer, in 1824, in connection with a study of methods for improvement of the efficiency of the steam engine.

We imagine that we have two large reservoirs with which the gas in the cylinder can somehow be put into thermal communication. We have a hot reservoir at temperature T_1 from which heat can be supplied to the gas, and a cold reservoir at temperature T_2 to which the gas can give up heat. We start with the gas at point A, with volume v_A, pressure p_A, and temperature T_1. The four steps in the cycle are then executed as follows:

$A \rightarrow B$:

The gas is placed in thermal communication with the hot reservoir. It expands slowly from volume v_A to volume v_B, remaining in thermal equilibrium with the reservoir, so that the gas maintains constant temperature T_1. In this isothermal expansion the work done by the gas is that given by equation (30) on p. 388, namely

$$(m/M)RT_1 \log_e(v_B/v_A).$$

Since the temperature, and hence the internal energy, of the gas does not change in going from A to B, the heat supplied by the reservoir must exactly equal the work done. *We call the heat supplied by the hot reservoir* Q_1. Then

$$Q_1 = (m/M)RT_1 \log_e(v_B/v_A). \tag{13}$$

$B \rightarrow C$:

The gas is next thermally insulated and is allowed to continue its expansion adiabatically until its temperature has fallen to T_2, the temperature of the cold reservoir. In this expansion the work done by the gas is, from (2),

$$mc_v(T_1 - T_2). \tag{14}$$

The ratio of volumes is given by (6) as

$$T_2 \, v_C^{\gamma-1} = T_1 \, v_B^{\gamma-1},$$

or

$$\frac{v_C}{v_B} = \left[\frac{T_1}{T_2}\right]^{\frac{1}{\gamma-1}}. \tag{15}$$

$C \rightarrow D$:

The gas is placed in thermal communication with the cold reservoir. It is compressed slowly from volume v_C to volume v_D, remaining in thermal equilibrium with the reservoir so the gas maintains constant temperature T_2. Since the temperature, and hence the internal energy, does not change, the work done on the gas must all be given up as heat to the cold reservoir. *We call the heat given to the cold reservoir* Q_2. Then

$$Q_2 = (m/M)RT_2 \log_e(v_C/v_D). \tag{16}$$

The volume v_D to which the isothermal compression is carried is chosen to be that volume lying on the adiabatic through A. This volume will be given by the relation

$$\frac{v_D}{v_A} = \left[\frac{T_1}{T_2}\right]^{\frac{1}{\gamma-1}}. \tag{17}$$

$D \rightarrow A$:

The gas is again insulated and compressed adiabatically back to its initial condition at A. The work done *on* the gas in this compression is

$$mc_v(T_1 - T_2). \tag{18}$$

We can see from an examination of Fig. 5 that in going once around the whole cycle the gas must have done a net amount of work. Since $\int p\,dv$ equals the area under the curve on a p, v diagram, the work done *by* the gas in expanding from A to B to C equals the area $ABCFEA$, and the work done *on* the gas during the compression from C to D to A equals the smaller area $ADCFEA$. The *net* work done *by* the gas equals the difference between these areas, which is just the area of the quadrilateral $ABCDA$.

This net work per cycle is given by

$$(13)+(14)-(16)-(18)=Q_1-Q_2. \tag{19}$$

Since the net work is positive, it must be that $Q_1>Q_2$. If we compare (15) and (17) we see that

$$\frac{v_C}{v_B}=\frac{v_D}{v_A}, \qquad \text{and hence} \qquad \frac{v_C}{v_D}=\frac{v_B}{v_A}.$$

Therefore the logarithms in (13) and (16) are equal, and we see from these equations that

$$\boxed{\frac{Q_1}{Q_2}=\frac{T_1}{T_2}.} \tag{20}$$

Hence $Q_1>Q_2$, since $T_1>T_2$ by assumption. That the net work per cycle is Q_1-Q_2 follows directly from the first law of thermodynamics, since in going once around the cycle we have added heat Q_1-Q_2 to the gas. But since the gas ends with the same internal energy as at the start, this heat must all have gone into doing external work.

> The efficiency of this engine is defined as the ratio of the work done per cycle to the heat extracted from the hot body per cycle. It has the value

$$\text{efficiency}=\frac{Q_1-Q_2}{Q_1}=1-\frac{Q_2}{Q_1}=1-\frac{T_2}{T_1}=\frac{T_1-T_2}{T_1}. \tag{21}$$

The efficiency depends only on the ratio T_1/T_2 of the absolute temperature of the hot reservoir from which heat is extracted to the absolute temperature of the cold reservoir to which heat is given. The efficiency is zero at $T_1/T_2=1$, and increases as T_1/T_2 increases. It is 0.5 at $T_1/T_2=2$, 0.75 at $T_1/T_2=4$, and so on.

PROBLEMS

1. Compute the efficiency of a Carnot engine operating with reservoir temperatures of 150° C and 30° C. Ans: 28.4%.

2. Compute the efficiency of a Carnot engine operating with reservoir temperatures of 250° C and 30° C.

3. If the Carnot engine of Fig. 5 contains 10 kg of gas of $M=29$ and $\gamma=1.4$, and if $p_A=10$ atm, $p_B=5$ atm, $T_1=273°$ C, and $T_2=0°$ C, make a table showing in joules (a) the heat added *to* the gas, (b) the work done *by* the gas, and (c) the *increase* in internal energy, in each of the four steps of the cycle and for the whole cycle.

Ans:	$A \rightarrow B$	$B \rightarrow C$	$C \rightarrow D$	$D \rightarrow A$	*Whole Cycle*
(a)	1,086,000	0	−543,000	0	543,000
(b)	1,086,000	196,000	−543,000	−196,000	543,000
(c)	0	−196,000	0	196,000	0

4. If the Carnot engine of Fig. 5 contains 2 kg of gas of $M=4$ and $\gamma=1.67$, and if $p_A=12$ atm, $p_B=8$ atm, $T_1=400°$ C, and $T_2=20°$ C, make a table showing in joules for each step of the cycle and for the whole cycle: (a) the heat added *to* the gas, (b) the work done *by* the gas, (c) the *increase* in internal energy.

4. AN IDEAL REFRIGERATOR EMPLOYING THE CARNOT CYCLE

We can run the heat engine of Fig. 5 'backwards,' meaning that we go around the cycle in the counterclockwise direction rather than in the clockwise, reversing all the arrowheads in Fig. 5. We can start at C with an adiabatic *compression* to B; then an isothermal compression to A, during which heat is *given* to the *hot* reservoir; then an adiabatic expansion to D; and finally an isothermal expansion back to C, during which heat is taken *from* the *cold* reservoir. The machine is now acting as a *refrigerator*, taking heat from a cold body and giving it to a hot body.

The magnitudes of Q_1, the heat given to the hot body, and Q_2, the heat taken from the cold body, are the same as computed in Sec. 3. Furthermore, the net work done in the cycle is again the area $ABCDA$, but now this is net work done *on* the gas, since more work is done on the gas during the compression than is done by the gas during the expansion. This net work is Q_1-Q_2.

To extract an amount Q_2 of heat from the cold body, we must do mechanical work of amount Q_1-Q_2. The sum, $Q_2+(Q_1-Q_2)=Q_1$, of the heat extracted from the cold body and the mechanical work done must be added to the hot body as heat. To accomplish the extraction of Q_2 units of heat requires mechanical work equal to

$$W=Q_1-Q_2=\left(\frac{Q_1}{Q_2}-1\right)Q_2=\left(\frac{T_1}{T_2}-1\right)Q_2=\frac{T_1-T_2}{T_2}\,Q_2. \qquad (22)$$

The greater the temperature ratio T_1/T_2, the more the work required to effect the transfer of a given quantity of heat from the cold to the hot body.

PROBLEMS

1. To extract 1 joule of heat from a body at 0° C and give it to a body at 100° C, how many joules of mechanical energy are required by a Carnot refrigerator, and how many joules are given to the hot body? Ans: 0.366; 1.366.

2. To extract 1 kcal of heat from a body at −100° C and give it to a body at 100° C, how many kcal of mechanical energy are required by a Carnot refrigerator, and how many kcal are given to the hot body?

3. If a Carnot refrigerator extracts enough heat from water at 32° F to make 1 ton of ice at 32° F per hour, giving this heat to the air at 90° F, how many mechanical horsepower are required to drive the refrigerator? Ans: 13.3.

4. If a Carnot refrigerator which extracts heat from a cold-storage plant at −20° F and gives it to the atmosphere at 100° F has 100 'ton'-per-day capacity (meaning that the amount of heat extracted in 24 hr is the latent heat of 100 tons of ice), what horsepower is needed to drive the refrigerator?

5. THE SECOND LAW OF THERMODYNAMICS

The first law of thermodynamics places no restrictions on our ability to change thermal energy into mechanical energy; it merely tells how much mechanical energy will appear if thermal energy disappears.

Experience, however, tells us that whereas it is very easy to turn mechanical energy completely into thermal energy (as in friction), there are severe restrictions on our ability to effect the reverse transformation. In particular, the oceans of the world are filled with water at 4° C which contains an enormous amount of thermal energy; yet experience teaches us that it is impossible to devise a machine that will change this energy into mechanical energy. The only way we seem to be able to turn thermal energy into mechanical energy is to have bodies at two *different* temperatures available and to introduce between them some kind of machine that interferes to a certain extent with the natural tendency of heat to flow from the hot body to the cold body and manages to turn some, *but only some*, of the heat flowing from the hot to the cold body into work. We shall see that the Carnot efficiency (21) is the maximum fraction of the heat that can be turned into work.

The *second law of thermodynamics* is an inference from experience that embodies the above ideas, for example, that of the nonutilizability of the heat of the oceans. From this simple law a great many detailed conclusions can be drawn, all of which are in agreement with experiment. This agreement gives us complete confidence in the universal applicability of the law. The second law states:

> *It is impossible even in principle to construct an engine that will deliver mechanical work derived purely from the cooling of a single heat reservoir, no heat being given out to a reservoir at lower temperature.*
>
> (SECOND LAW OF THERMODYNAMICS)

From this postulate we can proceed to derive a number of important theorems regarding engines and refrigerators.

THEOREM: *All reversible engines have the same efficiency when they operate between reservoirs at the same two temperatures.*

A reversible engine is one which, like the Carnot engine, will work equally well as a refrigerator, running through a cycle that accomplishes *exactly* the reverse of the engine cycle. All reversible engines are idealized concepts in that truly isothermal and truly adiabatic processes never occur, and friction is never entirely absent. An engine that works exactly like the Carnot engine but employs a real imperfect gas would still be reversible. The gas may even liquefy and evaporate during the cycle, as does the working substance in actual refrigerating cycles. One can imagine using isothermals like those of CO_2 in Fig. 13, p. 415 (replotted on a p, v diagram), and corresponding highly distorted adiabatics. Such a cycle would still be entirely reversible so long as the steps in the cycle were truly isothermal or adiabatic.

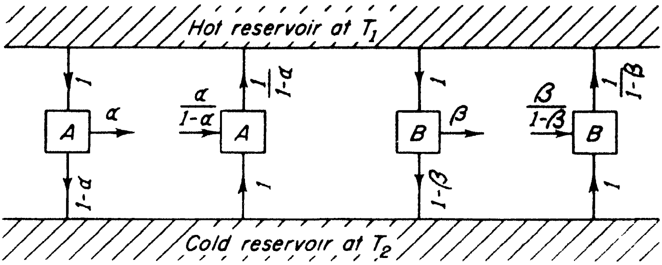

Fig. 6. Energy transformations effected by reversible engines. Left to right: ABLE as engine; reversed ABLE as refrigerator; BAKER as engine; reversed BAKER as refrigerator. The arrows at the center represent mechanical output of the engine or input to the refrigerator. Numbers indicate relative quantities of energy.

The above theorem is proved as follows: Suppose that we have two reversible engines ABLE and BAKER, of efficiencies α and β. As an engine, ABLE takes heat Q_1 from the hot reservoir, delivers mechanical energy αQ_1, and gives heat $Q_2 = (1-\alpha)Q_1$ to the cold reservoir. Note that $Q_1 = Q_2/(1-\alpha)$. As a refrigerator, it takes $(1-\alpha)Q_1 = Q_2$ from the cold body, is supplied with mechanical energy $\alpha Q_1 = \alpha Q_2/(1-\alpha)$, and delivers $Q_1 = Q_2/(1-\alpha)$ to the hot body. BAKER does similarly, with efficiency β. These relations are illustrated in Fig. 6 for each unit of heat taken from the hot body in the case of the engine or each unit of heat taken from the cold body in the case of the refrigerator.

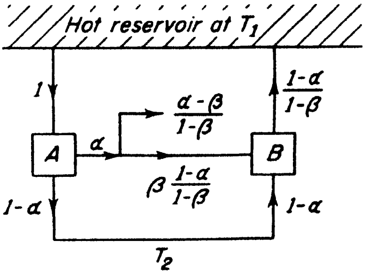
Hot reservoir at T_1
1
A
α
$\frac{\alpha-\beta}{1-\beta}$
$\beta\frac{1-\alpha}{1-\beta}$
$\frac{1-\alpha}{1-\beta}$
B
$1-\alpha$
$1-\alpha$
T_2

Fig. 7. If $\alpha > \beta$, we could use ABLE to drive BAKER as a refrigerator and get net work $(\alpha-\beta)/(1-\beta)$ for each unit of heat that ABLE took in. The symbolism in the above diagram is the same as in Fig. 6, and the quantities of energy can be verified by comparison with Fig. 6.

Now assume that ABLE is more efficient than BAKER, that is, that $\alpha > \beta$. Then, as indicated in Fig. 7, we can use ABLE as an engine to drive BAKER as a refrigerator. The heat given out by ABLE at the low temperature will be the heat input to BAKER. ABLE will be able to drive BAKER and still furnish *additional* mechanical energy. The combination would be a self-acting engine that takes thermal energy from a single reservoir and changes it into work, in contradiction to the second law. Therefore ABLE is *not* more efficient than BAKER. Similarly, we prove that BAKER is *not* more efficient than ABLE. Therefore, ABLE and BAKER must have the same efficiency. Therefore *every reversible engine has the Carnot efficiency* (21), since it has the same efficiency as the Carnot reversible engine which employs an ideal gas.

THEOREM: *No actual engine can have a greater efficiency than an ideal reversible engine working between the same two temperatures.*

An actual engine is nonreversible. It always has friction losses that cause nonreversible transformations of mechanical energy into heat, departures from thermal equilibrium that cause nonreversible transfers of heat, loss of thermal energy by conduction to colder bodies, etc.

Let ABLE, Fig. 7, be an actual engine of efficiency α; BAKER a reversible engine of efficiency β. Then if $\alpha > \beta$, we can use part of the mechanical output of ABLE to drive BAKER as a refrigerator to return all the heat given out by ABLE to the hot body and still have some mechanical output from ABLE left over. This result is in violation of the second law. Therefore, ABLE is *not* more efficient than BAKER. In this case we cannot turn the argument around, since ABLE is not reversible. We conclude that *any actual engine is less efficient than an ideal reversible engine and hence has less than the Carnot efficiency* (21).

We shall postpone discussion of this theorem until we consider actual engines in Sec. 7.

THEOREM: *For any refrigerator to take Q_2 units of heat from a cold body at T_2 and transfer it to a hot body at T_1, at least $W = Q_2(T_1 - T_2)/T_2$ units of mechanical work must be done on the working substance of the refrigerator.*

The mechanical work mentioned in the theorem is the value (22) required by a Carnot cycle, or, from arguments similar to those earlier in this section, by any ideal reversible cycle.

Let us suppose that we use the heat output at temperature T_1 of the refrigerator of Fig. 8 as the input to an ideal engine, which has the maximum possible efficiency $(T_1 - T_2)/T_1$. Then the situation depicted in principle in Fig. 8, in which the ideal engine drives the refrigerator and still has mechanical energy left over, *is in violation of the second law*, since this result would represent generation of mechanical energy purely by extraction of heat from the cold reservoir.

In Fig. 8, let the refrigerator take heat Q_2 from the cold reservoir. Let work W be required to drive the refrigerator, so that by energy conservation, heat $W+Q_2$ is given out at T_1. Let this be the input to the ideal engine. Since the efficiency of the ideal engine is $(T_1-T_2)/T_1$, the ideal engine does $[(T_1-T_2)/T_1](Q_2+W)$ units of work, returning the remainder of the heat to the cold reservoir. The situation depicted in Fig. 8, in which this amount of work is greater than W, is in violation of the second law, so this quantity of work must be less than or equal to W:

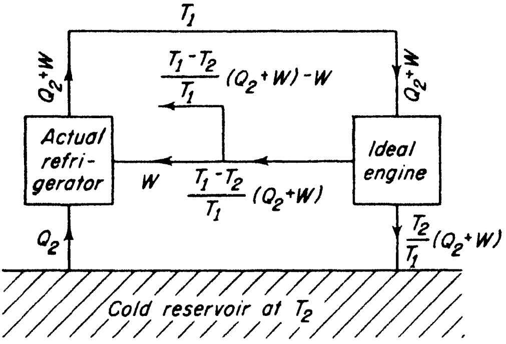

Fig. 8. The above system violates the second law if it is self-acting and a net amount of work is delivered. By 'ideal' engine we do not necessarily mean one employing an ideal gas, but any perfectly reversible engine such as is contemplated in Fig. 6.

$$\frac{T_1-T_2}{T_1}(Q_2+W)\leqq W,$$

or

$$WT_1\geqq(T_1-T_2)(Q_2+W),$$

$$WT_1\geqq(T_1-T_2)Q_2+WT_1-WT_2.$$

If we subtract WT_1 and add WT_2 on both sides of this inequality, we obtain

$$WT_2\geqq(T_1-T_2)Q_2,$$

or

$$W\geqq\frac{T_1-T_2}{T_2}Q_2,$$

which proves the theorem.

The equal sign in the above relation would hold for an *ideal* refrigerator. To extract heat Q_2, an *actual* refrigerator must be supplied with mechanical energy

$$W>\frac{T_1-T_2}{T_2}Q_2, \tag{23}$$

and will give to the hot body heat

$$Q_1 = Q_2 + W > Q_2 + \frac{T_1 - T_2}{T_2} Q_2,$$

or

$$Q_1 > \frac{T_1}{T_2} Q_2. \tag{24}$$

We shall discuss actual refrigerators in Sec. 8.

6. THE THERMODYNAMIC TEMPERATURE SCALE

So far we have been very loose about our definition of temperature. We are now prepared to define a temperature scale of fundamental significance that does not depend on the properties of any particular substance. This scale was defined by the English physicist William Thomson (Lord Kelvin) about 1850, and is called *the absolute thermodynamic temperature scale.* According to the size of units employed, this scale becomes either the *Kelvin scale* or the *Rankine scale* which we have been using throughout our discussion but which we have not yet defined rigorously.

We start with an *arbitrary* scale, say the mercury scale defined in Chap. 14, or preferably the hydrogen scale which employs a constant-volume hydrogen thermometer and assigns a temperature proportional to the pressure of the gas. The hydrogen scale is already close to the Kelvin scale over a wide temperature range and gives an approximate value for the absolute zero of temperature.

We have defined an *ideal gas* as one that rigorously obeys the relation

$$\rho = pM/RT. \tag{25}$$

If we had an ideal gas, this equation would serve as an excellent definition of temperature. But we do not have an ideal gas. However, we shall see that the theorems we have derived from the second law of thermodynamics will enable us to determine ideal-gas temperatures from actual substances. The thermodynamic scale is essentially an ideal-gas scale.

We have seen that for a Carnot cycle employing an ideal gas, the ratio of heat exchanged with the hot body to heat exchanged with the cold body is

$$\frac{Q_1}{Q_2} = \frac{T_1}{T_2}. \tag{26}$$

We have *proved* from the second law that (26) holds also for *any* reversible cycle employing an *actual* substance. Hence (26), for any reversible cycle working on adiabatics and isothermals, is taken as the *definition* of temperature ratio. We still need to specify the temperature *difference* between two fixed points to define the size of the units on our scale. The temperature *difference* between the *steam-point* and the *ice-point* is taken as 100 K deg or 180 R deg.

Suppose that we have a quantity of actual substance—say a gas—and that we have studied its properties in detail, employing our arbitrary temperature scale. In particular, we know the pressure and volume as a function of our arbitrary temperature, and we know c_p and c_v (say in joules/kg per arbitrary degree) for every pressure and temperature. It turns out that this information gives us enough data to draw adiabatics and isothermals on a p, v diagram as in Fig. 9. It also gives us enough data to *compute* the heat that would have to be added in an isothermal expansion or subtracted in an isothermal compression at any temperature on our arbitrary scale.

If the ice-point isothermal has temperature $T_{\text{i.p.}}$ on the Kelvin scale the size of the units is fixed so that the steam-point isothermal will have temperature $T_{\text{s.p.}} = T_{\text{i.p.}} + 100$ Then if we consider a cycle (crosshatched in Fig. 9) operating between the ice-point and the steam-point and make the computations of $Q_{\text{s.p.}}$ and $Q_{\text{i.p.}}$ mentioned above, we can write

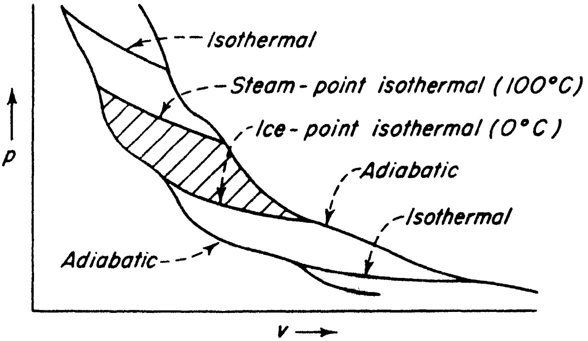

Fig. 9. Isothermals and adiabatics for a real substance (schematic).

$$\frac{T_{\text{i.p.}} + 100}{T_{\text{i.p.}}} = \frac{Q_{\text{s.p.}}}{Q_{\text{i.p.}}},$$

from which we can compute the absolute value of $T_{\text{i.p.}}$, the ice-point temperature on the Kelvin scale.

The second law of thermodynamics guarantees that we shall get the same value for $T_{\text{i.p.}}$ no matter what substance we use. It is in this way, by careful measurement of the properties of real gases, that the experimental value

$$T_{\text{i.p.}} = 273.16° \pm 0.01° \text{ K}$$

has been determined. Then

$$T_{\text{s.p.}} = 373.16° \text{ K}.$$

The absolute thermodynamic temperature at any other arbitrary fixed temperature point can be similarly determined by computations employing a cycle reaching to an isothermal at that temperature, as indicated in Fig. 9.

Although these ideas are simple in principle, the fixing of the absolute temperature scale has required a formidable amount of experimental work on the accurate determination of the properties of real substances, particularly of hydrogen and helium.

7. THE STEAM ENGINE

The steam engine, as improved by James Watt in 1769, was the first practical apparatus for transforming heat energy into mechanical energy. The development of the steam engine brought on the Industrial Revolution, and this type of prime mover is still the major source of mechanical and electrical power in our present mechanical and electrical age.

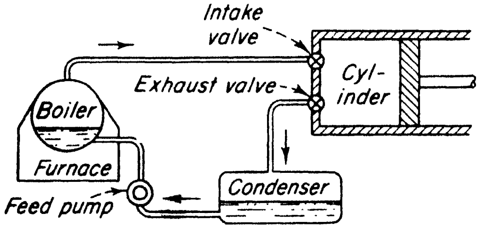

Fig. 10. A simple steam power plant. The circulating-water system that extracts the latent heat in the condenser is not shown.

A simple steam power plant is represented schematically in Fig. 10. Water is heated and evaporated in a boiler, usually at a pressure well above atmospheric and hence at a temperature well above 100° C. The pressure is controlled by a feed pump which feeds in water as needed to maintain the pressure. The temperature is the vaporization temperature at the fixed pressure. We consider the simplest case, in which the steam is not superheated after it has been formed. The steam enters the cylinder through an intake valve and is exhausted to a condenser through an exhaust valve. The condenser is pumped free from air so that the pressure in the condenser is close to the low vapor pressure at the temperature of the water that circulates through the condenser to condense the steam.

The conditions *in the cylinder* of the engine are shown by the *indicator diagram* of Fig. 11. On this diagram the actual pressure in the cylinder is plotted as a function of the varying volume of the cylinder. The closed curve on the indicator diagram is traversed counterclockwise during each cycle. The net mechanical work done per cycle of the engine, in ft·lbf, is seen to be just the area of the closed curve on the indicator diagram in $(\text{lbf/ft}^2)\cdot\text{ft}^3$. This plot is called an indicator diagram because a steam-engine cylinder can be readily equipped with a card that moves back and forth with the piston, and on which a recording pressure gauge connected to the cylinder will write the diagram automatically. This is the simplest way of determining the actual mechanical power developed by the engine.

If the back of the piston is open to the atmosphere, as in the *single-acting* engine of Fig. 10, the atmosphere does no net work during a complete cycle, so that the pressure of the atmosphere is irrelevant. A *double-acting* engine is equipped with intake and exhaust valves on both ends of a completely closed cylinder and does twice as much work per cycle as a single-acting engine with the same indicator diagram.

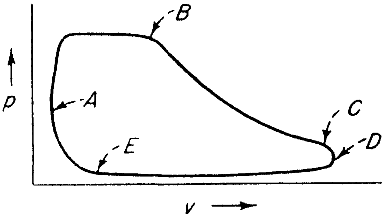

Fig. 11. Typical indicator diagram showing conditions in a steam-engine cylinder.

In Fig. 11, the intake valve is opened at A, and steam from the boiler is admitted during the first part of the stroke until the intake valve is closed at B. During the major portion of this part of the stroke the pressure is essentially constant at boiler pressure. From B to C, both valves are closed and the steam undergoes an essentially adiabatic expansion with resultant drop in pressure and temperature. Since the steam was saturated on admission, some of the steam condenses in the cylinder during this expansion. The exhaust valve is opened at C a short time before the end of the expansion stroke at D, and the pressure rapidly drops to the condenser pressure, which is maintained during the return stroke until the exhaust valve is closed at E. From E to A the remaining steam is compressed to cushion somewhat the sudden pressure rise occurring when the intake valve is opened at A.

One must not make the mistake of considering the indicator diagram of Fig. 11 as analogous to the Carnot-cycle diagram of Fig. 5. Figure 11 plots conditions in the cylinder for a variable quantity of working substance as steam enters and leaves. The diagram that is analogous to Fig. 5 is the Rankine diagram of Fig. 12, in which one takes a pound of H_2O and follows this same material around the closed cycle of Fig. 10—from water to steam in the boiler, to the cylinder, to the condenser, and back to water in the boiler.

At A on the Rankine diagram, the substance is in the form of water which has just been pumped into the boiler but is still at condenser temperature T_2. Between A and B the water has been heated to the boiling point T_1. B to C represents the volume increase (actually enormous in comparison with that shown schematically on Fig. 12) on evaporation. This is an isothermal process in which heat is added at T_1. At the condition C the gas enters the cylinder and accomplishes the first part of the work cycle in the cylinder before the intake valve is closed. CD represents the approximately adiabatic expansion from T_1 to T_2, partly in the cylinder and partly through the exhaust valve into the condenser; this last part is inherently irreversible. DE represents the volume decrease accompanying complete condensation to water, with latent heat given up at temperature T_2. Finally, EA represents the pressure increase on the liquid when it is pumped back into the boiler.

A great deal can be learned from a detailed analysis of the diagram of Fig. 12 and its implications, but this we must leave for a specialized course in the thermodynamics of heat engines. We shall merely consider the theoretical limit to the efficiency of the steam engine. Let heat Q_1 be added to our pound of H_2O. This heat is added in the boiler, partly at temperatures between T_2 and T_1 as the water is heated from A to B, mostly at T_1 as the water is evaporated. In principle, all this heat could have been derived from a body at temperature T_1. Heat is given up to the cooling water in the condenser at T_2. Our actual engine cannot be more efficient than an ideal reversible engine acting between a reservoir at T_1 and one at T_2. So of the heat added to the water in the boiler, at most only a fraction

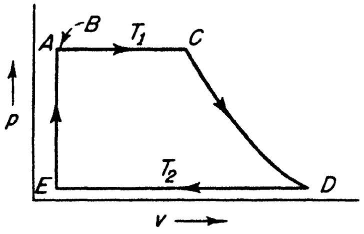

Fig. 12. Rankine diagram for a pound of H_2O passing once around the cycle of the power plant of Fig. 10 (schematic).

$$\frac{T_1 - T_2}{T_1} \qquad \begin{pmatrix} T_1 = \text{boiler temperature} \\ T_2 = \text{condenser temperature} \end{pmatrix} \qquad (27)$$

can be converted into work. If we define the *thermal efficiency* of our steam cycle as the ratio: (work done)/(heat added to the water in the boiler), then expression (27) furnishes a theoretical upper limit to this thermal efficiency. The value (27) is called the *Carnot efficiency.*

For example, if $T_1 = 350°\text{ F} = 810°\text{ R}$ and $T_2 = 100°\text{ F} = 560°\text{ R}$, the Carnot efficiency is

$$\text{Carnot efficiency} = \frac{T_1 - T_2}{T_1} = \frac{250}{810} = 31\%.$$

The thermal efficiency of an actual engine of 250-hp rating operating between these temperatures might be given by:

$$\text{thermal efficiency} = \frac{\text{work done}}{\text{heat added to water}} = 23\%;$$

whereas the over-all efficiency, which takes into account waste in heat value of the fuel in heating the water, might be given by

$$\text{over-all efficiency} = \frac{\text{work done}}{\text{heat value of fuel burned}} = 17\%.$$

The Carnot efficiency furnishes an upper limit to the efficiency of a steam engine. No changes in the steam plant described above could bring the efficiency above 31 per cent unless T_1 and T_2 were changed. The advantage of using a high steam temperature and a low condenser temperature is immediately apparent. This idea has led to increase from early boiler pressures of less than 50 lbf/in^2 (280° F) to modern pressures of 600 lbf/in^2 (490° F) or more, and to increasing use of condensers rather than steam discharge to the atmosphere at 212° F. Since steam turbines are capable of approaching the Carnot efficiency more closely than reciprocating engines, most large installations now employ turbines.

PROBLEMS

NOTE: These problems are in English units. Take the specific heat of liquid water as 1 BTU/lb·F deg over the whole range required. The following table will give the required temperatures and latent heats in English units:

p (lbf/in^2)	1.0	2.0	100	165
T (° F)	102	126	328	366
L (BTU/lb)	1035	1022	888	857

1. A steam engine which exhausts to a condenser at a pressure of 2.0 lbf/in^2 receives steam at 100 lbf/in^2 absolute pressure. The steam consumption is 14 lb per indicated horsepower-hour. What is the thermal efficiency of the engine? What is the Carnot efficiency? Ans: 16.7%; 25.6%.

2. Steam is supplied to an engine at 165 lbf/in^2 absolute. The condenser pressure is 1.0 lbf/in^2 absolute. The steam consumption is 10 lb per indicated horsepower-hour. What is the thermal efficiency? What is the Carnot efficiency?

3. If the area of the indicator diagram for a single-acting cylinder is 50 (lbf/in^2)·ft^3 and the engine operates at 400 cycles/min, what horsepower does it develop? Ans: 87.3 hp.

4. If the area of the indicator diagram on each side of a double-acting cylinder is 40 (lbf/in^2)·ft^3, at what speed must the engine run to develop 90 hp?

8. REFRIGERATION; THE HEAT PUMP

A refrigerator acts as a reversed heat engine in that it extracts heat from a cold body at temperature T_2 and gives it to a hot body at temperature T_1. We have seen that to effect this transfer requires mechanical work.* For a given pair of temperatures, the more heat the refrigerator extracts from the cold body per unit of work done, the better is the refrigerator, since the cost of operation is principally the cost of supplying the work by means of an electric motor or other type of engine. Hence, we define a performance coefficient as

---

* We note that there is one type of refrigerator in which no work is supplied—the gas-burning refrigerator, such as the 'Electrolux,' which has no moving parts. This refrigerator does not violate the second law of thermodynamics because it supplies heat from a still higher temperature level—that of the gas flame. In effect, an engine operating between flame temperature and T_1 furnishes the work necessary to raise heat from T_2 to T_1.

$$\text{performance coefficient} = \frac{\text{heat extracted}}{\text{work done}} = \frac{Q_2}{W}. \tag{28}$$

The argument of Sec. 5 proves that *the performance coefficient of an actual refrigerator is always less than that of an ideal reversible refrigerator,* which is

$$\text{Carnot performance coefficient} = \frac{T_2}{T_1 - T_2} = \frac{1}{(T_1/T_2) - 1}. \tag{29}$$

The Carnot performance coefficient depends only on the ratio T_1/T_2; it is very large when T_1/T_2 is close to 1; it is greater than unity when T_1/T_2 is less than 2, less than unity when T_1/T_2 is greater than 2.

The refrigeration cycle actually used is a vapor cycle, employing a gas that is easily condensed by pressure. Ammonia (NH_3), sulphur dioxide (SO_2), and Freon 12 (CCl_2F_2) are most commonly used. Figure 13 shows a diagram of the equipment and Fig. 14 shows a typical cycle on a p, v diagram. These two figures are lettered in corresponding fashion.

The diagram of Fig. 14 is like the p, ρ diagram of isothermals for CO_2 on p. 415 of Chap. 18, except that the abscissa is $v = 1/\rho$. We shall consider a unit mass of working substances and follow its course on this diagram as it moves around the circuit of Fig. 13. We shall use the typical numerical values of Fig. 14 and assume that the heat is being transferred from a refrigerated space at 0° F to the atmosphere at 80° F.

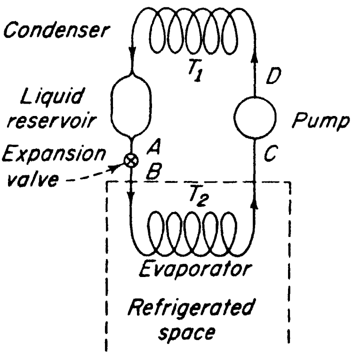

Fig. 13. Refrigeration system (schematic). Heat is extracted from air at the low temperature T_2 and given up to atmospheric air at the high temperature T_1.

We start with the pump. Its function is twofold—to keep the pressure in the evaporator down and to keep the pressure in the condenser up. We want the liquid *to evaporate at low temperature* and to *condense at high temperature* in order to extract heat at low temperature and give it out at high temperature. The two temperatures involved will be the vaporization temperatures at the high and low pressures maintained by the pump.

The pump takes saturated vapor at point C and compresses it adiabatically to vapor at D. The adiabatic curve CD in Fig. 14 has such a slope that this vapor will be superheated when it reaches the 100 lbf/in^2

pressure at D. It will be at about 95° F instead of the 80° F corresponding to saturated vapor at 100 lbf/in^2. This vapor will be cooled in the condenser back to 80° F, then liquefied to point A, the pump meantime maintaining the pressure constant. In this process the vapor gives up heat to the atmosphere at the high temperature $T_1 = 80°$ F.

Warm high-pressure liquid from the reservoir at condition A expands at a controlled rate through an expansion valve to the lower pressure maintained by the pump on the other side of the expansion valve. In this decidedly irreversible expansion to lower pressure, the liquid partially evaporates and cools to 0° F, ending up at B. The balance of the evaporation, with isothermal expansion from B to C, is accomplished by extraction of the latent heat from the refrigerated space.

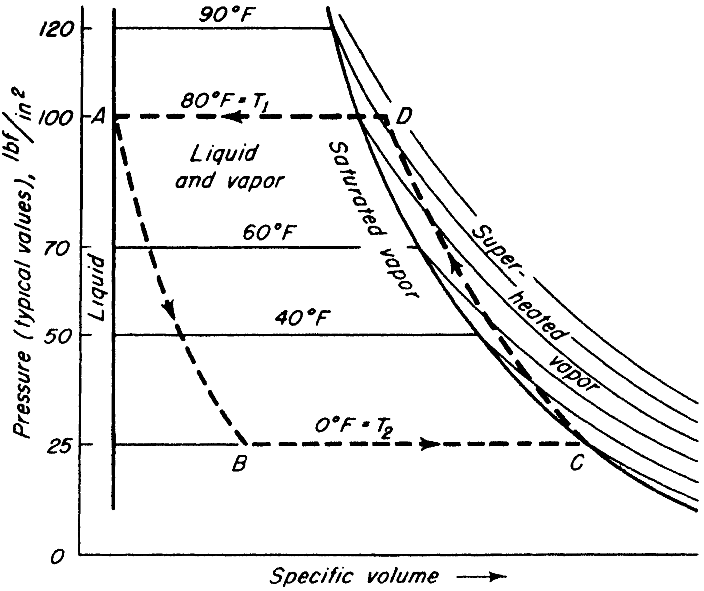

Fig. 14. Vapor cycle for refrigeration system of Fig. 13. Schematic isothermals are shown at typical pressures and temperatures. (This diagram is approximately correct for Freon except for the scale of specific volumes, which has been distorted for clarity.)

Since the pump is continuously pumping gas into the condenser, if the atmospheric temperature should rise above 80° F, so that heat was no longer being given off to the atmosphere and vapor no longer condensing, the pressure in the condenser would rise, and with it the temperature. In this way the condenser temperature will within limits adjust itself to a value slightly above ambient.

As an example, consider a large ice-making machine which employs ammonia as a refrigerant and maintains the evaporator at 34 lbf/in^2 corresponding to saturated vapor at 5° F, and the condenser at 169 lbf/in^2 corresponding to saturated vapor at 86° F. The evaporator is used to cool brine to 14° F, which in turn is circulated to cool and freeze water. The condenser is cooled by flowing water supplied at 70° F and heated to 80° F. The temperature differences between ammonia and brine and between ammonia and cooling water are necessary to get reasonably rapid heat transfer. The Carnot performance coefficient corresponding to actual ammonia temperatures is, from (29),

$$\frac{T_2}{T_1-T_2}=\frac{460+5}{86-5}=5.74.$$

The actual performance coefficient must be less than this value because the ammonia cycle is not a perfect reversible cycle—although except for the irreversible expansion through the valve in which the work done by the gas is not recovered, it comes close to being a reversible cycle. The actual performance coefficient of such an installation might be 80 per cent of the Carnot, or say

$$\text{performance coefficient}=4.6.$$

This equation means that for every BTU of mechanical work done, 4.6 BTU of heat is extracted from the brine, and, by conservation of energy, 5.6 BTU is supplied to the cooling water or otherwise transferred to the atmosphere.

The capacity of a refrigerating machine is usually given in *'tons' per day*, where

$$1\ \text{'ton'}=2000\times 143.5\ \text{BTU}=287{,}000\ \text{BTU},$$

the latent heat of 1 ton of ice. For each 'ton'-per-day capacity of the machine we have been discussing, 287,000 BTU per day of heat is extracted. To perform this refrigeration,

$$287{,}000/4.6=62{,}400\ \text{BTU/day}$$

of mechanical energy must be supplied. Since

$$62{,}400\ \frac{\text{BTU}}{\text{day}}=\frac{62{,}400\times 778\ \text{ft·lbf}}{24\times 60\ \text{min}}=\frac{62{,}400\times 778}{24\times 60\times 33{,}000}\ \text{hp}=1.02\ \text{hp},$$

the pump must be furnished with 1.02 hp of mechanical energy for each 'ton' per day of capacity.

We now see the possibility of using the refrigerator for another purpose, namely, as a *heat pump* to extract heat from the earth or from underground water and use it to heat a house. Although we cannot do this without supplying mechanical energy, the mechanical energy supplied can be small compared to the amount of heat transferred, provided the temperature differences involved are low.

For example, suppose that an ample supply of underground water is available at 50° F to circulate past the evaporator, and it is desired to heat air, circulating past the condenser, to 80° F to use in heating a house. To accomplish the heat transfer, suppose that the refrigerant is operated at 40° F in the evaporator and at 110° F in the condenser. The Carnot performance coefficient for a refrigerator operating between 40° F and 110° F is

$$\frac{460+40}{110-40}=\frac{500}{70}=7.1.$$

Suppose that the mechanical power is furnished electrically and that our installation will give an over-all performance coefficient of even half the Carnot. Then the heat extracted from the underground water will be 3.5 times the total electrical energy used. The heat given to the house will be 4.5 times the total electrical energy used. As compared with house heating by direct expenditure of electrical energy in an electric heater, we have gained a factor of 4.5 in heat, 3.5/4.5 = 78 per cent of the heat coming from the cold underground water. On this basis, heating by electrical energy begins to compete economically with heating by combustion of fuel, and heat pumps are beginning to be installed in increasing numbers for house heating. Although an ample supply of underground water is desirable, installations have been made in which the evaporator tubing has been buried in such a way as to thread through a large volume of earth, the heat being extracted directly from the earth.

PROBLEMS

1. How many pounds of ice can be frozen per hour by a refrigerating machine working between 0° F and 100° F if it has 50 per cent of the performance coefficient of a Carnot machine working between these same limits and is driven by a ¼-hp motor? The ice is frozen to a temperature of 15° F from water initially at 60° F. Ans: 8.14.

2. How many horsepower are required to freeze 100 tons per day of ice at 15° F from water initially at 60° F by a refrigerating machine working between 0° F and 90° F which has 70 per cent of the performance coefficient of a Carnot machine working between these same limits? What is the standard 'tons' per day rating of this machine?

3. What should be the capacity, in 'tons' per day, of the air-conditioning unit ordered for a public building in which it is desired to cool 100,000 ft^3/min of air at 95° F to 70° F at constant pressure? If the refrigerator operates between 50° F and 115° F and has 60 per cent of the Carnot performance coefficient, how many horsepower are required? Ans: 220 'tons' per day; 219.

4. Compare the cost of heating a house by gas of 1050 BTU/ft^3 at $0.90 per thousand cubic feet, with 70 per cent over-all combustion efficiency, and by electricity at 1.3 cents per kilowatt-hour if a heat pump of over-all performance coefficient 3.5 is used.

5. Compare the cost of heating a house by coal of 14,000 BTU/lb at $15.00 per ton, with 60 per cent over-all combustion efficiency, and by electricity at 1.5 cents per kilowatt-hour if a heat pump of over-all performance coefficient of 3.5 is used.
Ans: 11,200 BTU/cent for coal; 10,200 BTU/cent for the heat pump.

9. LIQUEFACTION OF GASES*

Gases that can be liquefied by pressure alone at normal temperature, that is, those whose critical temperature lies well above normal temperature, can easily be obtained in liquid form at their boiling point at normal atmospheric pressure. For example, suppose we want to obtain a quantity of liquid ammonia in an open flask at $-33.5°$ C, its boiling point at 1 atm pressure. We start with gaseous ammonia and compress it to about 10 atm pressure, at which point it will liquefy at room temperature. We extract the heat of compression and the latent heat by means of circulating water or circulating air until we have liquid ammonia at 10 atm pressure and room temperature. If we then release the pressure, with the liquid in a thermally insulated vessel, the liquid will first evaporate rapidly; but since the latent heat of evaporation must come from the liquid itself, the liquid will also cool rapidly so that soon we shall be left with a large fraction of the liquid at $-33.5°$ C. This liquid can be kept for a long time in an almost tightly closed, well insulated Dewar flask.

This procedure will not work for a permanent gas whose critical temperature is below room temperature because the gas cannot be liquefied by pressure at room temperature, and hence no latent heat can be made available for cooling.

A permanent gas (oxygen) was first liquefied in 1877 by Raoul Pictet (Swiss physicist), who used a vapor refrigeration cycle similar to the one we have described in the previous section except that the cycle was in two stages. The high-temperature stage employed SO_2, and the evaporator in the SO_2 cycle drew its heat from the condenser in a lower temperature CO_2 cycle. The evaporator in the CO_2 cycle cooled the oxygen gas to its liquefaction temperature. All gases except hydrogen and helium have been liquefied by such multistage vapor refrigeration cycles.

Another process of liquefaction, the Claude process widely used for liquefaction of air, employs simply an adiabatic expansion for cooling. In principle, by letting a gas expand and do work, the temperature can be lowered as much as we please. In practice, the process is applied to the liquefaction of air in the Claude apparatus sketched in Fig. 15. In this process air compressed to several hundred atmospheres is cooled first by air or water to remove the heat of compression; it is then cooled regeneratively by air already at a very low temperature; it then passes to an expansion engine, where it expands adiabatically to a pressure of a few atmospheres and very low temperature. The work done in the expansion engine furnishes part of the energy to drive the compressor. It then passes through a heat exchanger, where it cools high-pressure air below its condensation temperature and some of it condenses; it then passes through a second heat exchanger, where it regeneratively cools incoming air, and then returns to the compressor. After a period of operation, sufficient liquid air collects to be drawn off at the cock shown. This liquid air is likely to be warmer than its boiling point at atmospheric pressure, but the latent heat furnished to evaporate some of it will quickly reduce it to this boiling point.

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* This section may be omitted without loss of continuity.

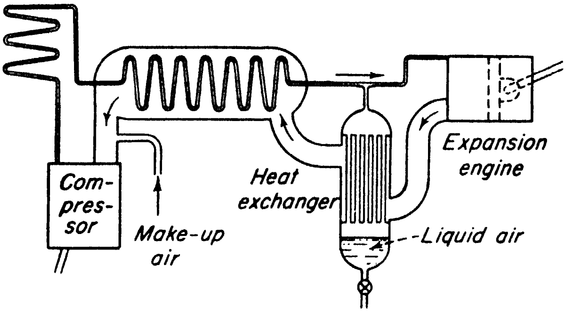

Fig. 15. Claude apparatus for liquefaction of air (schematic).

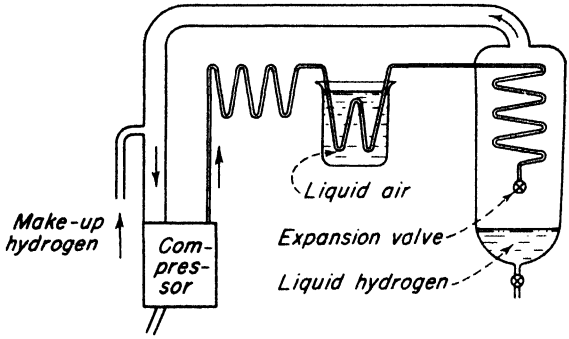

Fig. 16. Linde apparatus for liquefaction of hydrogen (schematic).

Finally, the *Linde process* has been most successful in liquefying gases such as hydrogen and helium, and is also widely used for liquefaction of air. This process, whose theory we shall not attempt to discuss rigorously, depends on the fact that actual gases are not perfect and do suffer some temperature change on *free expansion.* Although, as we have noted on p. 381, a perfect gas would not suffer a change on free expansion because its internal energy depends on temperature alone, the internal energy of a real gas depends somewhat on pressure as well as on temperature; and at temperatures not too far above the critical, the temperature suffers a substantial *decrease* on free expansion through a pressure range of several hundred atmospheres. The Linde apparatus of Fig. 16 is like the Claude apparatus of Fig. 15 except that a free expansion through a valve is substituted for the adiabatic expansion in the expansion engine to effect the cooling. By precooling hydrogen with liquid air before expansion, hydrogen is readily liquefied; and by precooling helium with liquid hydrogen before expansion, helium is readily liquefied.

10. THE PRINCIPLE OF LE CHATELIER-BRAUN

There is a general principle that can be used to predict the changes produced in a thermodynamic system when one or more of the 'external' conditions is changed. This principle was discovered by Le Châtelier and Braun as a result of the examination of numerous particular cases, that is, as a result of *inductive* reasoning rather than by *deduction* from the two laws of thermodynamics. The statement of the *general* principle cannot be made in such a way that it is completely rigorous in the light of the thermodynamic 'discipline' without considerable qualification and without the actual statement of *two* principles. However, the Le Châtelier-Braun principle* is so useful in making qualitative predictions as to the results to be expected when a given external condition is changed adiabatically that we shall include a qualitative statement of this principle:

> *If the external conditions of a thermodynamic system are altered adiabatically, the physical state of the system will alter in such a way as to* oppose *the changes resulting from the alteration of the external conditions.*
>
> (Principle of Le Châtelier-Braun)

In order to see how the principle can be applied in predicting the results of alterations of external conditions, let us consider two particular cases, one of which is already familiar to us.

Case 1. *Adiabatic compression of a gas.* We know that when the temperature of a gas increases, the volume of the gas tends to increase. With this fact in mind, let us predict from Le Châtelier's principle the alteration in temperature that occurs when the external pressure on a gas sample is adiabatically increased For example, consider a gas in a cylinder like that shown in Fig. 1. The gas is in thermodynamic equilibrium with its surroundings when the external pressure is 1 atm; that is, the temperature of the gas is the same as that of the walls of the cylinder and the pressure it exerts on the walls of the cylinder is 1 atm. Now by exerting an *external* force on the piston, let us increase the pressure adiabatically on the enclosed gas to 3 atm. This *increase in external pressure tends to decrease* the volume of the gas sample. From Le Châtelier's principle, we predict that the temperature of the gas will be altered in such a way as to *oppose* this change in volume; *the change in temperature that will tend to increase the volume, and hence to oppose the change produced by the external force, is a temperature rise.* From our previous discussion we know that this is the change in temperature that is actually observed; therefore, our argument based on the Le Châtelier-Braun principle leads to a qualitative prediction of the observed result.

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* In later chapters dealing with electricity we shall have occasion to make use of Lenz's law, which can be regarded as a special case of the Le Châtelier-Braun principle; the formulation of Lenz's law actually preceded the initial statement of the Le Châtelier-Braun principle.

Case 2. *Adiabatic stretch of a rubber band.* Let us now apply the principle to a problem to which we do not already know the answer and then check our answer by a simple experiment. First, we state as a known fact that *a rubber band contracts when it is heated.* Now consider a rubber band that is in thermodynamic equilibrium with its environment; for example, one at room temperature and not under tension. Suppose we stretch the rubber band to five or six times its initial length. What will be the effect on the temperature of the rubber? From the principle, we predict that the change will be one which will tend to oppose the lengthening of the rubber band. From the experimental fact that an increase in temperature tends to shorten the rubber band, we conclude that *the temperature of the rubber will increase.* This prediction can be verified by a simple experiment. One takes an unstretched rubber band and 'measures' its temperature by touching it to his lip. Then he suddenly stretches the rubber band and immediately touches it to his lip again. The band is found to be noticeably warmer. This result verifies the prediction we made on the basis of the Le Châtelier-Braun principle.

We shall in later chapters apply the principle to a number of other cases. The experimental verification of the predictions we make for these cases on the basis of the Le Châtelier-Braun principle might serve to convince us of the generality of this principle when it is properly stated and applied. Thermodynamic systems in general oppose the external imposition of changes. 'Nature' seldom offers us help in producing changes and usually offers determined opposition!*

PROBLEM

1. Discuss the expected temperature change when a wire with positive coefficient of thermal expansion is stretched.

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* This property of nature is well known to the experimentalist as one aspect of the *principle of maximum vexation.* The usual statement of this 'fourth law of thermodynamics' would be highly out of place here; any competent experimentalist is capable of discussing the subject.

APPENDIX

1. SYSTEMS OF MECHANICAL UNITS

Several systems of mechanical units are used in scientific and engineering work. Scientists and electrical engineers prefer the metric absolute system. Mechanical and civil engineers find it convenient to use the force units that occur in gravitational systems.

Once the length and time units have been selected, the key relationship involved in setting up a consistent system of units is the form of Newton's second law used to give the relationship between force units and mass units. Thus, if we express the second law in the form

$$\boldsymbol{F} = m\boldsymbol{a}, \tag{1}$$

the force unit is *defined* by this equation in terms of the mass unit, or conversely. In the MKS system, the acceleration is given in meters per second per second, the mass unit is the kilogram, and the force unit to be used is the *newton*, which is defined by (1) as the resultant force that will give a mass of 1 kg an acceleration of 1 m/sec^2. In the CGS system, the acceleration is given in centimeters per second per second, the mass unit is the gram, and (1) defines another force unit, the *dyne;* the dyne is the resultant force required to give a mass of 1 g an acceleration of 1 cm/sec^2.* In the FPS system, the foot per second per second is the acceleration unit, the pound is the mass unit, and (1) defines a force unit called the *poundal,* which is the resultant force required to give a mass of 1 lb an acceleration of 1 ft/sec^2. These are the *absolute* systems in current use; other units in these systems are listed in Table I. There is no reference to the strength of the earth's gravitational field in the definition of any of the units in these absolute systems.

Equation (1) will also apply when systems of gravitational units of one type (which we call Type I) are used. In systems of this type, a gravitational force unit is chosen and then (1) is used to *define* a gravitational mass unit. In the gravitational system we have used in this book (called British Type I in Table I), the force unit in (1) is taken as the pound-force, defined as the force of gravity on a one-pound mass at a location where the acceleration of gravity has its standard value

$$g_0 = 980.665 \text{ cm/sec}^2 = 32.17398 \text{ ft/sec}^2.$$

With this force unit, and ft/sec^2 for the acceleration unit, (1) is used to define a mass unit, the slug, as the mass to which 1 lbf will give an acceleration of 1 ft/sec^2. This mass unit has the value 1 slug $=32.174$ lb. There is also a metric gravitational system of Type I in use on the European continent. In this system, which we do not list in Table I, the force unit is the kilogram-force, the acceleration unit is the m/sec^2, and the mass unit defined by (1) has the magnitude 9.8 kg.

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* In the CGS system, the unit of energy, the cm·dyne, is called the *erg.*

TABLE I

SYSTEMS OF MECHANICAL UNITS

	Absolute systems			Gravitational systems		
	MKS	CGS	FPS	British Type I	British Type II	Metric Type II
Statement of Newton's second law:	$F = ma$	$F = ma$	$F = ma$	$F = ma$	$g_0F = ma$ ($g_0 = 32.174$)	$g_0F = ma$ ($g_0 = 9.80665$)
Length.	meter	centimeter	foot	foot	foot	meter
Mass.	kilogram	gram	pound	slug	pound	kilogram
Time.	second	second	second	second	second	second
Force.	newton	dyne	poundal	pound-force	pound-force	kilogram-force
Velocity.	m/sec	cm/sec	ft/sec	ft/sec	ft/sec	m/sec
Acceleration.	m/sec^2	cm/sec^2	ft/sec^2	ft/sec^2	ft/sec^2	m/sec^2
Torque.	nt·m	dyne·cm	pdl·ft	lbf·ft	lbf·ft	kgf·m
Moment of inertia.	kg·m^2	g·cm^2	lb·ft^2	slug·ft^2	lb·ft^2	kg·m^2
Pressure.	nt/m^2	dyne/cm^2	pdl/ft^2	lbf/ft^2	lbf/ft^2	kgf/m^2
Energy.	joule	erg	ft·pdl	ft·lbf	ft·lbf	m·kgf
Power.	watt	erg/sec	ft·pdl/sec	ft·lbf/sec	ft·lbf/sec	m·kgf/sec
Momentum.	kg·m/sec	g·cm/sec	lb·ft/sec	slug·ft/sec	lb·ft/sec	kg·m/sec
Impulse.	nt·sec	dyne·sec	pdl·sec	lbf·sec	lbf·sec	kgf·sec

Since Newton's second law has been used in the form (1) throughout this book, the equations derived in this book will in general apply only when one of the systems of units discussed above is used. But they will apply when *any* of these systems is used; that is, *any of the first four systems given in Table I may be used in any of the equations of this volume.* In these four systems, the quantities appearing in Newton's second law have the following units:

System	F	$= m$	a
MKS:	newton	kg	m/sec^2
CGS:	dyne	g	cm/sec^2
FPS:	poundal	lb	ft/sec^2
British Type I:	lbf	slug	ft/sec^2.

Some engineers prefer to use the pound (lb) as the mass unit and the pound-force (lbf) as the force unit. This can be done, provided Newtons' second law is written in the form

$$g_0 F = ma, \tag{2}$$

where g_0 is a *dimensionless* constant having the value 32.174 when F is in lbf, m is in lb, and a is in ft/sec². Similarly, continental engineers sometimes use the kilogram (kg) as the mass unit and the kilogram-force (kgf) as the force unit; they do this by writing Newton's second law in the form (2) where the constant g_0 has the value 9.80665 when F is in kgf, m in kg, and a in m/sec². These gravitational systems are designated as Type II in the last two columns of Table I. *The relations derived in this book do not in general hold in these systems of Type II.*

2. FUNDAMENTAL PHYSICAL CONSTANTS*

Gravitation constant (G)	$(6.670 \pm 0.005) \times 10^{-11}$ nt·m²/kg²
Volume of mole of ideal gas at NTP	$(22{,}420.7 \pm 0.6)$ cm³
Standard atmosphere	$(101{,}324.6 \pm 0.4)$ nt/m²
Ice-point	$273.16° \pm 0.01°$ K
Mechanical equivalent of heat (number of joules equal to one kilocalorie)	4185.5 ± 0.4
Avogadro's number (number of molecules in one mole)	$(6.0251 \pm 0.0004) \times 10^{23}$
Atomic weight of natural oxygen on physical scale	16.00436 ± 0.00009
Ratio of atomic weights on physical scale to atomic weights on chemical scale (16.00436/16)	1.000272 ± 0.000005
Atomic mass unit (amu)	$(1.6597 \pm 0.0001) \times 10^{-27}$ kg
Density of mercury at NTP	$(13{,}595.04 \pm 0.06)$ kg/m³
Universal gas constant (R)	(8316.6 ± 0.4) joules/kg·K deg $= (1.9870 \pm 0.0002)$ kcal/kg·K deg
Boltzmann's constant (k)	$(1.3803 \pm 0.0001) \times 10^{-23}$ joule/K deg

Atomic weights of selected elements on physical scale:

argon (A)	39.955	helium (He)	4.0039	neon (Ne)	20.188
bromine (Br)	79.938	hydrogen (H)	1.00828	nitrogen (N)	14.0112
carbon (C)	12.0146	iodine (I)	126.950	oxygen (O)	16.0044
chlorine (Cl)	35.467	krypton (Kr)	83.7	sulfur (S)	32.075
fluorine (F)	19.01	mercury (Hg)	200.66	xenon (Xe)	131.3

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* These data are based on R. T. Birge, REVIEWS OF MODERN PHYSICS, **13**, 233 (1941) and J. W. M. DuMond and E. R. Cohen, *ibid.*, **20**, 82 (1948). All data in this table are on the *physical* scale of atomic weights (see p. 370).

3. TABLES OF CONVERSION FACTORS

Plane Angle

	°	′	″	rad	rev
1 degree =	1	60	3600	1.745×10^{-2}	2.778×10^{-3}
1 minute =	1.667×10^{-2}	1	60	2.909×10^{-4}	4.630×10^{-5}
1 second =	2.778×10^{-4}	1.667×10^{-2}	1	4.848×10^{-6}	7.716×10^{-7}
1 radian =	57.30	3438	2.063×10^{5}	1	0.1592
1 revolution =	360	2.16×10^{4}	1.296×10^{6}	6.283	1

$1\text{ rev} = 2\pi\text{ rad} = 360^\circ \qquad 1^\circ = 60' = 3600''$

$1\text{ artillery mil} = \tfrac{1}{6400}\text{ rev} = 0.0009817\text{ rad} = 0.^\circ05625$

Solid Angle

1 sphere $= 4\pi$ steradians $= 12.57$ steradians

Length

	cm	m	km	in	ft	mi
1 centimeter =	1	10^{-2}	10^{-5}	0.3937	3.281×10^{-2}	6.214×10^{-6}
1 meter =	100	1	10^{-3}	39.37	3.281	6.214×10^{-4}
1 kilometer =	10^{5}	1000	1	3.937×10^{4}	3281	0.6214
1 inch =	2.540	2.540×10^{-2}	2.540×10^{-5}	1	8.333×10^{-2}	1.578×10^{-5}
1 foot =	30.48	0.3048	3.048×10^{-4}	12	1	1.894×10^{-4}
1 statute mile =	1.609×10^{5}	1609	1.609	6.336×10^{4}	5280	1

1 foot $= \tfrac{1200}{3937}$ meter

1 meter $= \tfrac{3937}{1200}$ feet

1 angstrom (A) $= 10^{-10}$ m

1 X-unit $= 10^{-13}$ m

1 micron (μ) $= 10^{-6}$ m

1 millimicron ($m\mu$) $= 10^{-9}$ m

1 light-year $= 9.4600\times10^{12}$ km

1 parsec $= 3.084\times10^{13}$ km

1 fathom = 6 ft

1 yard = 3 ft

1 rod = 16.5 ft

1 mil $= 10^{-3}$ in

1 nautical mile = 1.15157 statute miles = 6080.27 ft

Area

	m^2	cm^2	ft^2	in^2	circ mil
1 square meter =	1	10^{4}	10.76	1550	1.974×10^{9}
1 square centimeter =	10^{-4}	1	1.076×10^{-3}	0.1550	1.974×10^{5}
1 square foot =	9.290×10^{-2}	929.0	1	144	1.833×10^{8}
1 square inch =	6.452×10^{-4}	6.452	6.944×10^{-3}	1	1.273×10^{6}
1 circular mil =	5.067×10^{-10}	5.067×10^{-6}	5.454×10^{-9}	7.854×10^{-7}	1

1 square mile = 27,878,400 ft^2 = 640 acres 1 acre = 43,560 ft^2

Volume

	m³	cm³	l	ft³	in³
1 cubic meter =	1	10^6	1000	35.31	6.102×10^4
1 cubic centimeter =	10^{-6}	1	1.000×10^{-3}	3.531×10^{-5}	6.102×10^{-2}
1 liter =	1.000×10^{-3}	1000	1	3.531×10^{-2}	61.02
1 cubic foot =	2.832×10^{-2}	2.832×10^4	28.32	1	1728
1 cubic inch =	1.639×10^{-5}	16.39	1.639×10^{-2}	5.787×10^{-4}	1

1 U.S. fluid gallon = 4 U.S. fluid quarts = 8 U.S. fluid pints = 128 U.S. fluid ounces = 231 in³.

1 British Imperial gallon = the volume of 10 lb of water at 62° F = 277.42 in³.

1 liter = the volume of 1 kg of water at its maximum density = 1000.028 cm³.

Mass

	g	kg	oz	lb	slug	ton
1 gram =	1	0.001	3.527×10^{-2}	2.205×10^{-3}	6.852×10^{-5}	1.102×10^{-6}
1 kilogram =	1000	1	35.27	2.205	6.852×10^{-2}	1.102×10^{-3}
1 ounce (Avoirdupois) =	28.35	2.835×10^{-2}	1	6.250×10^{-2}	1.943×10^{-3}	3.125×10^{-5}
1 pound (Avoirdupois) =	453.6	0.4536	16	1	3.108×10^{-2}	0.0005
1 slug =	1.459×10^4	14.59	514.8	32.17	1	1.609×10^{-2}
1 ton =	9.072×10^5	907.2	3.2×10^4	2000	62.16	1

1 Avoirdupois pound = 7000 grains

1 Troy or Apothecaries' pound = 12 Troy or Apothecaries' ounces = 5760 grains = 0.8229 Avoirdupois pound

1 long ton = 2240 lb
1 metric ton = 1000 kg = 2205 lb
1 lb = 453.5924277 g
1 stone = 14 lb
1 carat = 0.2 g
1 kg = 2.2046223 lb
1 hundredweight (cwt) = 112 lb
1 pennyweight (dwt) = 24 grains
1 slug = 32.17398 lb

1 atomic mass unit (amu) = 1.6597×10^{-27} kg

Time

	yr	day	hr	min	sec
1 year =	1	365.2	8.766×10^{3}	5.259×10^{5}	3.156×10^{7}
1 day =	2.738×10^{-3}	1	24	1440	8.640×10^{4}
1 hour =	1.141×10^{-4}	4.167×10^{-2}	1	60	3600
1 minute =	1.901×10^{-6}	6.944×10^{-4}	1.667×10^{-2}	1	60
1 second =	3.169×10^{-8}	1.157×10^{-5}	2.778×10^{-4}	1.667×10^{-2}	1

1 year = 365.24219879 days

Density

	slug/ft³	lb/ft³	lb/in³	kg/m³	g/cm³
1 slug per ft³ =	1	32.17	1.862×10^{-2}	515.4	0.5154
1 pound per ft³ =	3.108×10^{-2}	1	5.787×10^{-4}	16.02	1.602×10^{-2}
1 pound per in³ =	53.71	1728	1	2.768×10^{4}	27.68
1 kilogram per m³ =	1.940×10^{-3}	6.243×10^{-2}	3.613×10^{-5}	1	0.001
1 gram per cm³ =	1.940	62.43	3.613×10^{-2}	1000	1

Speed

	ft/sec	km/hr	m/sec	mi/hr	knot
1 foot per second =	1	1.097	0.3048	0.6818	0.5921
1 kilometer per hour =	0.9113	1	0.2778	0.6214	0.5396
1 meter per second =	3.281	3.6	1	2.237	1.943
1 mile per hour =	1.467	1.609	0.4470	1	0.8684
1 knot =	1.689	1.853	0.5148	1.152	1

1 knot = 1 nautical mile/hr   1 mi/min = 88 ft/sec = 60 mi/hr

Force

	dyne	gf	kgf	nt	lbf	pdl
1 dyne =	1	1.020×10^{-3}	1.020×10^{-6}	10^{-5}	2.248×10^{-6}	7.233×10^{-5}
1 gram-force =	980.7	1	0.001	9.807×10^{-3}	2.205×10^{-3}	7.093×10^{-2}
1 kilogram-force =	9.807×10^{5}	1000	1	9.807	2.205	70.93
1 newton =	10^{5}	102.0	0.1020	1	0.2248	7.233
1 pound-force =	4.448×10^{5}	453.6	0.4536	4.448	1	32.17
1 poundal =	1.383×10^{4}	14.10	1.410×10^{-2}	0.1383	3.108×10^{-2}	1

1 kgf = 9.80665 nt   1 lbf = 32.17398 pdl

Pressure

	atm	dyne/cm^2	inch of water	cm Hg	kgf/m^2	nt/m^2	lbf/in^2	lbf/ft^2
1 atmosphere =	1	1.013×10^{6}	406.8	76	1.033×10^{4}	1.013×10^{5}	14.70	2116
1 dyne per cm^2 =	9.869×10^{-7}	1	4.015×10^{-4}	7.501×10^{-5}	1.020×10^{-2}	0.1	1.450×10^{-5}	2.089×10^{-3}
1 inch of water at 4° C* =	2.458×10^{-3}	2491	1	0.1868	25.40	249.1	3.613×10^{-2}	5.202
1 centimeter of mercury at 0° C* =	1.316×10^{-2}	1.333×10^{4}	5.353	1	136.0	1333	0.1934	27.85
1 kilogram-force per m^2 =	9.678×10^{-5}	98.07	3.937×10^{-2}	7.356×10^{-3}	1	9.807	1.422×10^{-3}	0.2048
1 newton per m^2 =	9.869×10^{-6}	10	4.015×10^{-3}	7.501×10^{-4}	0.1020	1	1.450×10^{-4}	2.089×10^{-2}
1 pound-force per in^2 =	6.805×10^{-2}	6.895×10^{4}	27.68	5.171	703.1	6.895×10^{3}	1	144
1 pound-force per ft^2 =	4.725×10^{-4}	478.8	0.1922	3.591×10^{-2}	4.882	47.88	6.944×10^{-3}	1

* Where the acceleration of gravity has the standard value 9.80665 m/sec^2.

1 bar = 1 dyne/cm^2 1 millibar = 10^{-3} dyne/cm^2

Energy, Work, Heat

	BTU	erg	ft·lbf	hp·hr	joule	kcal	kwh
1 British thermal unit =	1	1.055×10^{10}	777.9	3.929×10^{-4}	1055	0.2520	2.930×10^{-4}
1 erg =	9.481×10^{-11}	1	7.376×10^{-8}	3.725×10^{-14}	10^{-7}	2.389×10^{-11}	2.778×10^{-14}
1 foot·pound-force =	1.285×10^{-3}	1.356×10^{7}	1	5.051×10^{-7}	1.356	3.239×10^{-4}	3.766×10^{-7}
1 horsepower·hour =	2545	2.685×10^{13}	1.980×10^{6}	1	2.685×10^{6}	641.4	0.7457
1 joule =	9.481×10^{-4}	10^{7}	0.7376	3.725×10^{-7}	1	2.389×10^{-4}	2.778×10^{-7}
1 kilocalorie =	3.968	4.186×10^{10}	3087	1.559×10^{-3}	4186	1	1.163×10^{-3}
1 kilowatt·hour =	3413	3.6×10^{13}	2.655×10^{6}	1.341	3.6×10^{6}	860.1	1

1 electron-volt (ev) = 1.6020×10^{-19} joules
1 m·kgf = 9.807 joules
1 watt·sec = 1 joule = 1 m·nt
1 cm·dyne = 1 erg

Relativistic energy equivalents:
1 kg = 8.9866×10^{16} joules
1 amu = 1.4915×10^{-10} joules

Power

	BTU/hr	ft·lbf/ min	ft·lbf/ sec	hp	kcal/sec	kw	w
1 British thermal unit per hour =	1	12.97	0.2161	3.929×10^{-4}	7.000×10^{-5}	2.930×10^{-4}	0.2930
1 foot·pound-force per minute =	7.713×10^{-2}	1	1.667×10^{-2}	3.030×10^{-5}	5.399×10^{-6}	2.260×10^{-5}	2.260×10^{-2}
1 foot·pound-force per second =	4.628	60	1	1.818×10^{-3}	3.239×10^{-4}	1.356×10^{-3}	1.356
1 horsepower =	2545	3.3×10^{4}	550	1	0.1782	0.7457	745.7
1 kilocalorie per second =	1.429×10^{4}	1.852×10^{5}	3087	5.613	1	4.186	4186
1 kilowatt =	3413	4.425×10^{4}	737.6	1.341	0.2389	1	1000
1 watt =	3.413	44.25	0.7376	1.341×10^{-3}	2.389×10^{-4}	0.001	1

1 watt = 1 joule/sec

MATHEMATICAL TABLES

4. NATURAL TRIGONOMETRIC FUNCTIONS

sin

	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9		
0°	.0000	.0017	.0035	.0052	.0070	.0087	.0105	.0122	.0140	.0157	.0175	89°
1°	.0175	.0192	.0209	.0227	.0244	.0262	.0279	.0297	.0314	.0332	.0349	88°
2°	.0349	.0366	.0384	.0401	.0419	.0436	.0454	.0471	.0488	.0506	.0523	87°
3°	.0523	.0541	.0558	.0576	.0593	.0610	.0628	.0645	.0663	.0680	.0698	86°
4°	.0698	.0715	.0732	.0750	.0767	.0785	.0802	.0819	.0837	.0854	.0872	85°
5°	.0872	.0889	.0906	.0924	.0941	.0958	.0976	.0993	.1011	.1028	.1045	84°
6°	.1045	.1063	.1080	.1097	.1115	.1132	.1149	.1167	.1184	.1201	.1219	83°
7°	.1219	.1236	.1253	.1271	.1288	.1305	.1323	.1340	.1357	.1374	.1392	82°
8°	.1392	.1409	.1426	.1444	.1461	.1478	.1495	.1513	.1530	.1547	.1564	81°
9°	.1564	.1582	.1599	.1616	.1633	.1650	.1668	.1685	.1702	.1719	.1736	80°
10°	.1736	.1754	.1771	.1788	.1805	.1822	.1840	.1857	.1874	.1891	.1908	79°
11°	.1908	.1925	.1942	.1959	.1977	.1994	.2011	.2028	.2045	.2062	.2079	78°
12°	.2079	.2096	.2113	.2130	.2147	.2164	.2181	.2198	.2115	.2233	.2250	77°
13°	.2250	.2267	.2284	.2300	.2317	.2334	.2351	.2368	.2385	.2402	.2419	76°
14°	.2419	.2436	.2453	.2470	.2487	.2504	.2521	.2538	.2554	.2571	.2588	75°
15°	.2588	.2605	.2622	.2639	.2656	.2672	.2689	.2706	.2723	.2740	.2756	74°
16°	.2756	.2773	.2790	.2807	.2823	.2840	.2857	.2874	.2890	.2907	.2924	73°
17°	.2924	.2940	.2957	.2974	.2990	.3007	.3024	.3040	.3057	.3074	.3090	72°
18°	.3090	.3107	.3123	.3140	.3156	.3173	.3190	.3206	.3223	.3239	.3256	71°
19°	.3256	.3272	.3289	.3305	.3322	.3338	.3355	.3371	.3387	.3404	.3420	70°
20°	.3420	.3437	.3453	.3469	.3486	.3502	.3518	.3535	.3551	.3567	.3584	69°
21°	.3584	.3600	.3616	.3633	.3649	.3665	.3681	.3697	.3714	.3730	.3746	68°
22°	.3746	.3762	.3778	.3795	.3811	.3827	.3843	.3859	.3875	.3891	.3907	67°
23°	.3907	.3923	.3939	.3955	.3971	.3987	.4003	.4019	.4035	.4051	.4067	66°
24°	.4067	.4083	.4099	.4115	.4131	.4147	.4163	.4179	.4195	.4210	.4226	65°
25°	.4226	.4242	.4258	.4274	.4289	.4305	.4321	.4337	.4352	.4368	.4384	64°
26°	.4384	.4399	.4415	.4431	.4446	.4462	.4478	.4493	.4509	.4524	.4540	63°
27°	.4540	.4555	.4571	.4586	.4602	.4617	.4633	.4648	.4664	.4679	.4695	62°
28°	.4695	.4710	.4726	.4741	.4756	.4772	.4787	.4802	.4818	.4833	.4848	61°
29°	.4848	.4863	.4879	.4894	.4909	.4924	.4939	.4955	.4970	.4985	.5000	60°
30°	.5000	.5015	.5030	.5045	.5060	.5075	.5090	.5105	.5120	.5135	.5150	59°
31°	.5150	.5165	.5180	.5195	.5210	.5225	.5240	.5255	.5270	.5284	.5299	58°
32°	.5299	.5314	.5329	.5344	.5358	.5373	.5388	.5402	.5417	.5432	.5446	57°
33°	.5446	.5461	.5476	.5490	.5505	.5519	.5534	.5548	.5563	.5577	.5592	56°
34°	.5592	.5606	.5621	.5635	.5650	.5664	.5678	.5693	.5707	.5721	.5736	55°
35°	.5736	.5750	.5764	.5779	.5793	.5807	.5821	.5835	.5850	.5864	.5878	54°
36°	.5878	.5892	.5906	.5920	.5934	.5948	.5962	.5976	.5990	.6004	.6018	53°
37°	.6018	.6032	.6046	.6060	.6074	.6088	.6101	.6115	.6129	.6143	.6157	52°
38°	.6157	.6170	.6184	.6198	.6211	.6225	.6239	.6252	.6266	.6280	.6293	51°
39°	.6293	.6307	.6320	.6334	.6347	.6361	.6374	.6388	.6401	.6414	.6428	50°
40°	.6428	.6441	.6455	.6468	.6481	.6494	.6508	.6521	.6534	.6547	.6561	49°
41°	.6561	.6574	.6587	.6600	.6613	.6626	.6639	.6652	.6665	.6678	.6691	48°
42°	.6691	.6704	.6717	.6730	.6743	.6756	.6769	.6782	.6794	.6807	.6820	47°
43°	.6820	.6833	.6845	.6858	.6871	.6884	.6896	.6909	.6921	.6934	.6947	46°
44°	.6947	.6959	.6972	.6984	.6997	.7009	.7022	.7034	.7046	.7059	.7071	45°
		.9	.8	.7	.6	.5	.4	.3	.2	.1	.0	

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	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9		
45°	.7071	.7083	.7096	.7108	.7120	.7133	.7145	.7157	.7169	.7181	.7193	44°
46°	.7193	.7206	.7218	.7230	.7242	.7254	.7266	.7278	.7290	.7302	.7314	43°
47°	.7314	.7325	.7337	.7349	.7361	.7373	.7385	.7396	.7408	.7420	.7431	42°
48°	.7431	.7443	.7455	.7466	.7478	.7490	.7501	.7513	.7524	.7536	.7547	41°
49°	.7547	.7559	.7570	.7581	.7593	.7604	.7615	.7627	.7638	.7649	.7660	40°
50°	.7660	.7672	.7683	.7694	.7705	.7716	.7727	.7738	.7749	.7760	.7771	39°
51°	.7771	.7782	.7793	.7804	.7815	.7826	.7837	.7848	.7859	.7869	.7880	38°
52°	.7880	.7891	.7902	.7912	.7923	.7934	.7944	.7955	.7965	.7976	.7986	37°
53°	.7986	.7997	.8007	.8018	.8028	.8039	.8049	.8059	.8070	.8080	.8090	36°
54°	.8090	.8100	.8111	.8121	.8131	.8141	.8151	.8161	.8171	.8181	.8192	35°
55°	.8192	.8202	.8211	.8221	.8231	.8241	.8251	.8261	.8271	.8281	.8290	34°
56°	.8290	.8300	.8310	.8320	.8329	.8339	.8348	.8358	.8368	.8377	.8387	33°
57°	.8387	.8396	.8406	.8415	.8425	.8434	.8443	.8453	.8462	.8471	.8480	32°
58°	.8480	.8490	.8499	.8508	.8517	.8526	.8536	.8545	.8554	.8563	.8572	31°
59°	.8572	.8581	.8590	.8599	.8607	.8616	.8625	.8634	.8643	.8652	.8660	30°
60°	.8660	.8669	.8678	.8686	.8695	.8704	.8712	.8721	.8729	.8738	.8746	29°
61°	.8746	.8755	.8763	.8771	.8780	.8788	.8796	.8805	.8813	.8821	.8829	28°
62°	.8829	.8838	.8846	.8854	.8862	.8870	.8878	.8886	.8894	.8902	.8910	27°
63°	.8910	.8918	.8926	.8934	.8942	.8949	.8957	.8965	.8973	.8980	.8988	26°
64°	.8988	.8996	.9003	.9011	.9018	.9026	.9033	.9041	.9048	.9056	.9063	25°
65°	.9063	.9070	.9078	.9085	.9092	.9100	.9107	.9114	.9121	.9128	.9135	24°
66°	.9135	.9143	.9150	.9157	.9164	.9171	.9178	.9184	9191	.9198	.9205	23°
67°	.9205	.9212	.9219	.9225	.9232	.9239	.9245	.9252	.9259	.9265	.9272	22°
68°	.9272	.9278	.9285	.9291	.9298	.9304	.9311	.9317	.9323	.9330	.9336	21°
69°	.9336	.9342	.9348	.9354	.9361	.9367	.9373	.9379	.9385	.9391	.9397	20°
70°	.9397	.9403	.9409	.9415	.9421	.9426	.9432	.9438	.9444	.9449	.9455	19°
71°	.9455	.9461	.9466	.9472	.9478	.9483	.9489	.9494	.9500	.9505	.9511	18°
72°	.9511	.9516	.9521	.9527	.9532	.9537	.9542	.9548	.9553	.9558	.9563	17°
73°	.9563	.9568	.9573	.9578	.9583	.9588	.9593	.9598	.9603	.9608	.9613	16°
74°	.9613	.9617	.9622	.9627	.9632	.9636	.9641	.9646	.9650	.9655	.9659	15°
75°	.9659	.9664	.9668	.9673	.9677	.9681	.9686	.9690	.9694	.9699	.9703	14°
76°	.9703	.9707	.9711	.9715	.9720	.9724	.9728	.9732	.9736	.9740	.9744	13°
77°	.9744	.9748	.9751	.9755	.9759	.9763	9767	.9770	.9774	.9778	.9781	12°
78°	.9781	.9785	.9789	.9792	.9796	.9799	.9803	.9806	.9810	.9813	.9816	11°
79°	.9816	.9820	.9823	.9826	.9829	.9833	.9836	.9839	.9842	.9845	.9848	10°
80°	.9848	.9851	.9854	.9857	.9860	.9863	.9866	.9869	.9871	9874	.9877	9°
81°	.9877	.9880	.9882	.9885	.9888	.9890	.9893	.9895	.9898	.9900	.9903	8°
82°	.9903	.9905	.9907	.9910	.9912	.9914	.9917	.9919	.9921	.9923	.9925	7°
83°	.9925	.9928	.9930	.9932	.9934	.9936	.9938	.9940	.9942	.9943	.9945	6°
84°	.9945	.9947	.9949	.9951	.9952	.9954	.9956	.9957	.9959	.9960	.9962	5°
85°	.9962	.9963	.9965	.9966	.9968	.9969	.9971	.9972	.9973	.9974	.9976	4°
86°	.9976	.9977	.9978	.9979	.9980	.9981	.9982	.9983	.9984	.9985	.9986	3°
87°	.9986	.9987	.9988	.9989	.9990	.9990	.9991	.9992	.9993	.9993	.9994	2°
88°	.9994	.9995	.9995	.9996	.9996	.9997	.9997	.9997	.9998	.9998	.9998	1°
89°	.9998	.9999	.9999	.9999	.9999	1.000	1.000	1.000	1.000	1.000	1.000	0°
		.9	.8	.7	.6	.5	.4	.3	.2	.1	.0	

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	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9		
0°	.0000	.0017	.0035	.0052	.0070	.0087	.0105	.0122	.0140	.0157	.0175	89°
1°	.0175	.0192	.0209	.0227	.0244	.0262	.0279	.0297	.0314	.0332	.0349	88°
2°	.0349	.0367	.0384	.0402	.0419	.0437	.0454	.0472	.0489	.0507	.0524	87°
3°	.0524	.0542	.0559	.0577	.0594	.0612	.0629	.0647	.0664	.0682	.0699	86°
4°	.0699	.0717	.0734	.0752	.0769	.0787	.0805	.0822	.0840	.0857	.0875	85°
5°	.0875	.0892	.0910	.0928	.0945	.0963	.0981	.0998	.1016	.1033	.1051	84°
6°	.1051	.1069	.1086	.1104	.1122	.1139	.1157	.1175	.1192	.1210	.1228	83°
7°	.1228	.1246	.1263	.1281	.1299	.1317	.1334	.1352	.1370	.1388	.1405	82°
8°	.1405	.1423	.1441	.1459	.1477	.1495	.1512	.1530	.1548	.1566	.1584	81°
9°	.1584	.1602	.1620	.1638	.1655	.1673	.1691	.1709	.1727	.1745	.1763	80°
10°	.1763	.1781	.1799	.1817	.1835	.1853	.1871	.1890	.1908	.1926	.1944	79°
11°	.1944	.1962	.1980	.1998	.2016	.2035	.2053	.2071	.2089	.2107	.2126	78°
12°	.2126	.2144	.2162	.2180	.2199	.2217	.2235	.2254	.2272	.2290	.2309	77°
13°	.2309	.2327	.2345	.2364	.2382	.2401	.2419	.2438	.2456	.2475	.2493	76°
14°	.2493	.2512	.2530	.2549	.2568	.2586	.2605	.2623	.2642	.2661	.2679	75°
15°	.2679	.2698	.2717	.2736	.2754	.2773	.2792	.2811	.2830	.2849	.2867	74°
16°	.2867	.2886	.2905	.2924	.2943	.2962	.2981	.3000	.3019	.3038	.3057	73°
17°	.3057	.3076	.3096	.3115	.3134	.3153	.3172	.3191	.3211	.3230	.3249	72°
18°	.3249	.3269	.3288	.3307	.3327	.3346	.3365	.3385	.3404	.3424	.3443	71°
19°	.3443	.3463	.3482	.3502	.3522	.3541	.3561	.3581	.3600	.3620	.3640	70°
20°	.3640	.3659	.3679	.3699	.3719	.3739	.3759	.3779	.3799	.3819	.3839	69°
21°	.3839	.3859	.3879	.3899	.3919	.3939	.3959	.3979	.4000	.4020	.4040	68°
22°	.4040	.4061	.4081	.4101	.4122	.4142	.4163	.4183	.4204	.4224	.4245	67°
23°	.4245	.4265	.4286	.4307	.4327	.4348	.4369	.4390	.4411	.4431	.4452	66°
24°	.4452	.4473	.4494	.4515	.4536	.4557	.4578	.4599	.4621	.4642	.4663	65°
25°	.4663	.4684	.4706	.4727	.4748	.4770	.4791	.4813	.4834	.4856	.4877	64°
26°	.4877	.4899	.4921	.4942	.4964	.4986	.5008	.5029	.5051	.5073	.5095	63°
27°	.5095	.5117	.5139	.5161	.5184	.5206	.5228	.5250	.5272	.5295	.5317	62°
28°	.5317	.5340	.5362	.5384	.5407	.5430	.5452	.5475	.5498	.5520	.5543	61°
29°	.5543	.5566	.5589	.5612	.5635	.5658	.5681	.5704	.5727	.5750	.5774	60°
30°	.5774	.5797	.5820	.5844	.5867	.5890	.5914	.5938	.5961	.5985	.6009	59°
31°	.6009	.6032	.6056	.6080	.6104	.6128	.6152	.6176	.6200	.6224	.6249	58°
32°	.6249	.6273	.6297	.6322	.6346	.6371	.6395	.6420	.6445	.6469	.6494	57°
33°	.6494	.6519	.6544	.6569	.6594	.6619	.6644	.6669	.6694	.6720	.6745	56°
34°	.6745	.6771	.6796	.6822	.6847	.6873	.6899	.6924	.6950	.6976	.7002	55°
35°	.7002	.7028	.7054	.7080	.7107	.7133	.7159	.7186	.7212	.7239	.7265	54°
36°	.7265	.7292	.7319	.7346	.7373	.7400	.7427	.7454	.7481	.7508	.7536	53°
37°	.7536	.7563	.7590	.7618	.7646	.7673	.7701	.7729	.7757	.7785	.7813	52°
38°	.7813	.7841	.7869	.7898	.7926	.7954	.7983	.8012	.8040	.8069	.8098	51°
39°	.8098	.8127	.8156	.8185	.8214	.8243	.8273	.8302	.8332	.8361	.8391	50°
40°	.8391	.8421	.8451	.8481	.8511	.8541	.8571	.8601	.8632	.8662	.8693	49°
41°	.8693	.8724	.8754	.8785	.8816	.8847	.8878	.8910	.8941	.8972	.9004	48°
42°	.9004	.9036	.9067	.9099	.9131	.9163	.9195	.9228	.9260	.9293	.9325	47°
43°	.9325	.9358	.9391	.9424	.9457	.9490	.9523	.9556	.9590	.9623	.9657	46°
44°	.9657	.9691	.9725	.9759	.9793	.9827	.9861	.9896	.9930	.9965	1.000	45°
		.9	.8	.7	.6	.5	.4	.3	.2	.1	.0	

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	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9		
45°	1 000	1.003	1.007	1.011	1.014	1.018	1.021	1.025	1.028	1.032	1.036	44°
46°	1.036	1.039	1.043	1.046	1.050	1.054	1.057	1.061	1.065	1.069	1.072	43°
47°	1.072	1.076	1.080	1.084	1.087	1.091	1.095	1.099	1.103	1.107	1.111	42°
48°	1.111	1.115	1.118	1.122	1.126	1.130	1.134	1.138	1.142	1.146	1.150	41°
49°	1.150	1.154	1.159	1.163	1.167	1.171	1.175	1.179	1.183	1.188	1.192	40°
50°	1.192	1.196	1.200	1.205	1.209	1.213	1.217	1.222	1.226	1.230	1.235	39°
51°	1.235	1.239	1.244	1.248	1.253	1 257	1.262	1.266	1.271	1.275	1.280	38°
52°	1.280	1.285	1.289	1.294	1.299	1.303	1.308	1.313	1.317	1.322	1.327	37°
53°	1.327	1.332	1.337	1.342	1.347	1.351	1.356	1.361	1.366	1.371	1.376	36°
54°	1.376	1.381	1.387	1.392	1.397	1.402	1.407	1.412	1.418	1.423	1.428	35°
55°	1.428	1.433	1.439	1.444	1.450	1.455	1.460	1.466	1.471	1.477	1.483	34°
56°	1.483	1.488	1.494	1.499	1.505	1.511	1.517	1.522	1.528	1.534	1.540	33°
57°	1.540	1.546	1.552	1.558	1.564	1.570	1.576	1.582	1.588	1.594	1.600	32°
58°	1.600	1.607	1.613	1.619	1.625	1.632	1.638	1.645	1.651	1.658	1.664	31°
59°	1.664	1.671	1.678	1.684	1.691	1.698	1.704	1.711	1.718	1.725	1.732	30°
60°	1.732	1.739	1.746	1 753	1.760	1.767	1.775	1.782	1.789	1.797	1.804	29°
61°	1.804	1.811	1.819	1 827	1.834	1.842	1.849	1.857	1.865	1.873	1.881	28°
62°	1.881	1.889	1.897	1.905	1.913	1.921	1.929	1.937	1.946	1.954	1.963	27°
63°	1.963	1.971	1.980	1.988	1.997	2.006	2.014	2.023	2.032	2.041	2.050	26°
64°	2.050	2.059	2.069	2.078	2.087	2.097	2.106	2.116	2.125	2.135	2.145	25°
65°	2.145	2.154	2.164	2.174	2.184	2.194	2.204	2.215	2.225	2.236	2.246	24°
66°	2.246	2.257	2.267	2.278	2.289	2.300	2.311	2.322	2.333	2.344	2.356	23°
67°	2.356	2.367	2.379	2.391	2.402	2.414	2.426	2.438	2.450	2.463	2.475	22°
68°	2.475	2.488	2.500	2.513	2.526	2.539	2.552	2.565	2.578	2.592	2.605	21°
69°	2.605	2.619	2.633	2.646	2.660	2.675	2.689	2.703	2.718	2.733	2.747	20°
70°	2.747	2.762	2.778	2.793	2.808	2.824	2.840	2.856	2.872	2.888	2.904	19°
71°	2.904	2.921	2.937	2.954	2.971	2.989	3.006	3.024	3.042	3.060	3.078	18°
72°	3.078	3.096	3.115	3.133	3.152	3.172	3.191	3.211	3.230	3.251	3.271	17°
73°	3.271	3.291	3.312	3.333	3.354	3.376	3.398	3.420	3.442	3.465	3.487	16°
74°	3.487	3.511	3.534	3.558	3.582	3.606	3.630	3.655	3.681	3.706	3.732	15°
75°	3 732	3.758	3.785	3.812	3.839	3.867	3.895	3.923	3.952	3.981	4.011	14°
76°	4.011	4.041	4.071	4.102	4.134	4.165	4.198	4.230	4.264	4.297	4.331	13°
77°	4.331	4.366	4.402	4.437	4.474	4.511	4.548	4.586	4.625	4.665	4.705	12°
78°	4.705	4.745	4.787	4.829	4.872	4.915	4.959	5.005	5.050	5.097	5.145	11°
79°	5.145	5.193	5.242	5.292	5.343	5.396	5.449	5.503	5.558	5.614	5.671	10°
80°	5.671	5.730	5.789	5.850	5.912	5.976	6.041	6.107	6.174	6.243	6.314	9°
81°	6.314	6.386	6.460	6.535	6.612	6.691	6.772	6.855	6.940	7.026	7.115	8°
82°	7.115	7.207	7.300	7.396	7.495	7.596	7.700	7.806	7.916	8.028	8.144	7°
83°	8.144	8.264	8.386	8.513	8.643	8.777	8.915	9.058	9.205	9.357	9.514	6°
84°	9.514	9.677	9.845	10.02	10.20	10.39	10.58	10.78	10.99	11.20	11.43	5°
85°	11.43	11.66	11.91	12.16	12.43	12.71	13.00	13.30	13.62	13.95	14.30	4°
86°	14.30	14.67	15.06	15.46	15.89	16.35	16.83	17.34	17.89	18.46	19.08	3°
87°	19.08	19.74	20.45	21.20	22.02	22.90	23.86	24.90	26.03	27.27	28.64	2°
88°	28.64	30.14	31.82	33.69	35.80	38.19	40.92	44.07	47.74	52.08	57.29	1°
89°	57.29	63.66	71.62	81.85	95.49	114.6	143.2	191.0	286.5	573.0	∞	0°
		.9	.8	.7	.6	.5	.4	.3	.2	.1	.0	

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5. TABLE OF LOGARITHMS TO BASE 10

N	0	1	2	3	4	5	6	7	8	9	P. P.				
											1	2	3	4	5
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	12	17	21
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	5	7	9
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	6
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4

NOTE: $\log_e N = \log_e 10 \log_{10} N = 2.3026 \log_{10} N$

$\log_{10} e^x = x \log_{10} e = 0.43429\ x$

N	0	1	2	3	4	5	6	7	8	9	P. P. 1	2	3	4	5
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	3	3
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2

6. TABLE OF EXPONENTIALS

x	0	1	2	3	4	5	6	7	8	9
0.0	1.000	1.010	1.020	1.031	1.041	1.051	1.062	1.073	1.083	1.094
0.1	1.105	1.116	1.127	1.139	1.150	1.162	1.174	1.185	1.197	1.209
0.2	1.221	1.234	1.246	1.259	1.271	1.284	1.297	1.310	1.323	1.336
0.3	1.350	1.363	1.377	1.391	1.405	1.419	1.433	1.448	1.462	1.477
0.4	1.492	1.507	1.522	1.537	1.553	1.568	1.584	1.600	1.616	1.632
0.5	1.649	1.665	1.682	1.699	1.716	1.733	1.751	1.768	1.786	1.804
0.6	1.822	1.840	1.859	1.878	1.896	1.916	1.935	1.954	1.974	1.994
0.7	2.014	2.034	2.054	2.075	2.096	2.117	2.138	2.160	2.181	2.203
0.8	2.226	2.248	2.270	2.293	2.316	2.340	2.363	2.387	2.411	2.435
0.9	2.460	2.484	2.509	2.535	2.560	2.586	2.612	2.638	2.664	2.691
1.0	2.718	2.746	2.773	2.801	2.829	2.858	2.886	2.915	2.945	2.974
1.1	3.004	3.034	3.065	3.096	3.127	3.158	3.190	3.222	3.254	3.287
1.2	3.320	3.353	3.387	3.421	3.456	3.490	3.525	3.561	3.597	3.633
1.3	3.669	3.706	3.743	3.781	3.819	3.857	3.896	3.935	3.975	4.015
1.4	4.055	4.096	4.137	4.179	4.221	4.263	4.306	4.349	4.393	4.437
1.5	4.482	4.527	4.572	4.618	4.665	4.712	4.759	4.807	4.855	4.904
1.6	4.953	5.003	5.053	5.104	5.155	5.207	5.259	5.312	5.366	5.419
1.7	5.474	5.529	5.585	5.641	5.697	5.755	5.812	5.871	5.930	5.989
1.8	6.050	6.110	6.172	6.234	6.297	6.360	6.424	6.488	6.554	6.619
1.9	6.686	6.753	6.821	6.890	6.959	7.029	7.099	7.171	7.243	7.316
2.0	7.389	7.463	7.538	7.614	7.691	7.768	7.846	7.925	8.004	8.085
2.1	8.166	8.248	8.331	8.415	8.499	8.585	8.671	8.758	8.846	8.935
2.2	9.025	9.116	9.207	9.300	9.393	9.488	9.583	9.679	9.777	9.875
2.3	9.974	10.07	10.18	10.28	10.38	10.49	10.59	10.70	10.80	10.91
2.4	11.02	11.13	11.25	11.36	11.47	11.59	11.70	11.82	11.94	12.06
2.5	12.18	12.30	12.43	12.55	12.68	12.81	12.94	13.07	13.20	13.33
2.6	13.46	13.60	13.74	13.87	14.01	14.15	14.30	14.44	14.59	14.73
2.7	14.88	15.03	15.18	15.33	15.49	15.64	15.80	15.96	16.12	16.28
2.8	16.44	16.61	16.78	16.95	17.12	17.29	17.46	17.64	17.81	17.99
2.9	18.17	18.36	18.54	18.73	18.92	19.11	19.30	19.49	19.69	19.89
3.0	20.09	20.29	20.49	20.70	20.91	21.12	21.33	21.54	21.76	21.98
3 -	22.20	22.42	22.65	22.87	23.10	23.34	23.57	23.81	24.05	24.29
3.2	24.53	24.78	25.03	25.28	25.53	25.79	26.05	26.31	26.58	26.84
3.3	27.11	27.39	27.66	27.94	28.22	28.50	28.79	29.08	29.37	29.67
3.4	29.96	30.27	30.57	30.88	31.19	31.50	31.82	32.14	32.46	32.79

x	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
3	20.09	22.20	24.53	27.11	29.96	33.12	36.60	40.45	44.70	49.40
4	54.60	60.34	66.69	73.70	81.45	90.02	99.48	109.9	121.5	134.3
5	148.4	164.0	181.3	200.3	221.4	244.7	270.4	298.9	330.3	365.0
6	403.4	445.9	492.7	544.6	601.8	665.1	735.1	812.4	897.8	992.3
7	1097	1212	1339	1480	1636	1808	1998	2208	2441	2697
8	2981	3295	3641	4024	4447	4915	5432	6003	6634	7332
9	8103	8955	9897	10938	12088	13360	14765	16318	18034	19930

$\log_{10}e^x = x \log_{10}e = 0.43429\,x$

e^{-x}

x		0	1	2	3	4	5	6	7	8	9
0.0		1.000	.9900	.9802	.9704	.9608	.9512	.9418	.9324	.9231	.9139
0.1		.9048	.8958	.8869	.8781	.8694	.8607	.8521	.8437	.8353	.8270
0.2		.8187	.8106	.8025	.7945	.7866	.7788	.7711	.7634	.7558	.7483
0.3		.7408	.7334	.7261	.7189	.7118	.7047	.6977	.6907	.6839	.6771
0.4		.6703	.6637	.6570	.6505	.6440	.6376	.6313	.6250	.6188	.6126
0.5		.6065	.6005	.5945	.5886	.5827	.5769	.5712	.5655	.5599	.5543
0.6		.5488	.5434	.5379	.5326	.5273	.5220	.5169	.5117	.5066	.5016
0.7		.4966	.4916	.4868	.4819	.4771	.4724	.4677	.4630	.4584	.4538
0.8		.4493	.4449	.4404	.4360	.4317	.4274	.4232	.4190	.4148	.4107
0.9		.4066	.4025	.3985	.3946	.3906	.3867	.3829	.3791	.3753	.3716
1.0		.3679	.3642	.3606	.3570	.3535	.3499	.3465	.3430	.3396	.3362
1.1		.3329	.3296	.3263	.3230	.3198	.3166	.3135	.3104	.3073	.3042
1.2		.3012	.2982	.2952	.2923	.2894	.2865	.2837	.2808	.2780	.2753
1.3		.2725	.2698	.2671	.2645	.2618	.2592	.2567	.2541	.2516	.2491
1.4		.2466	.2441	.2417	.2393	.2369	.2346	.2322	.2299	.2276	.2254
1.5		.2231	.2209	.2187	.2165	.2144	.2122	.2101	.2080	.2060	.2039
1.6		.2019	.1999	.1979	.1959	.1940	.1920	.1901	.1882	.1864	.1845
1.7		.1827	.1809	.1791	.1773	.1755	.1738	.1720	.1703	.1686	.1670
1.8		.1653	.1637	.1620	.1604	.1588	.1572	.1557	.1541	.1526	.1511
1.9		.1496	.1481	.1466	.1451	.1437	.1423	.1409	.1395	.1381	.1367
2.0		.1353	.1340	.1327	.1313	.1300	.1287	.1275	.1262	.1249	.1237
2.1		.1225	.1212	.1200	.1188	.1177	.1165	.1153	.1142	.1130	.1119
2.2		.1108	.1097	.1086	.1075	.1065	.1054	.1043	.1033	.1023	.1013
2.3		.1003	*9926	*9827	*9730	*9633	*9537	*9442	*9348	*9255	*9163
2.4	0.0	9072	8982	8892	8804	8716	8629	8544	8458	8374	8291
2.5	0.0	8208	8127	8046	7966	7887	7808	7730	7654	7577	7502
2.6	0.0	7427	7353	7280	7208	7136	7065	6995	6925	6856	6788
2.7	0.0	6721	6654	6587	6522	6457	6393	6329	6266	6204	6142
2.8	0.0	6081	6020	5961	5901	5843	5784	5727	5670	5613	5558
2.9	0.0	5502	5448	5393	5340	5287	5234	5182	5130	5079	5029
3.0	0.0	4979	4929	4880	4832	4783	4736	4689	4642	4596	4550
3.1	0.0	4505	4460	4416	4372	4328	4285	4243	4200	4159	4117
3.2	0.0	4076	4036	3996	3956	3916	3877	3839	3801	3763	3725
3.3	0.0	3688	3652	3615	3579	3544	3508	3474	3439	3405	3371
3.4	0.0	3337	3304	3271	3239	3206	3175	3143	3112	3081	3050

x		.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
3	0.0	4979	4505	4076	3688	3337	3020	2732	2472	2237	2024
4	0.0	1832	1657	1500	1357	1228	1111	1005	*9095	*8230	*7447
5	0.00	6738	6097	5517	4992	4517	4087	3698	3346	3028	2739
6	0.00	2479	2243	2029	1836	1662	1503	1360	1231	1114	1008
7	0.000	9119	8251	7466	6755	6112	5531	5004	4528	4097	3707
8	0.000	3355	3035	2747	2485	2249	2035	1841	1666	1507	1364
9	0.000	1234	1117	1010	*9142	*8272	*7485	*6773	*6128	*5545	*5017
10	0.0000	4540	4108	3717	3363	3043	2754	2492	2254	2040	1846

$$\log_{10} e^{-x} = -x \log_{10} e = -0.43429\, x$$

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